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WENTWORTH-SMITH MATHEMATICAL SERIES

PLANE AND SOLID GEOMETRY

TEACHERS' EDITION

BY

GEORGE WENTWORTH

AND

DAVID EUGENE SMITH

GINN AND COMPANY
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PREFACE

This book is intended for teachers, *and for teachers only.* The publishers will sell it to no persons except teachers of the Wentworth-Smith Geometry. Every teacher must consider himself in honor bound not to leave his copy where pupils can have access to it, and not to sell his copy except to the publishers, Messrs. Ginn and Company.

It is hoped that young teachers will derive great help from studying the systematic solutions of the exercises, and that every teacher who is pressed for time will find great relief by not being obliged to work out every problem himself.

Criticisms and corrections will be thankfully received.

GEORGE WENTWORTH
DAVID EUGENE SMITH

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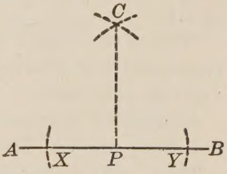
PLANE GEOMETRY

TEACHERS' EDITION

INTRODUCTION

Exercise 1. Page 8

1. *From a given point on a given straight line required to draw a perpendicular to the line.*



Let AB be the given line and P be the given point.

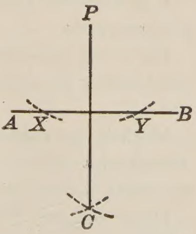
It is required to draw from P a line perpendicular to AB.

With P as a center and any convenient radius draw arcs cutting AB at X and Y.

With X as a center and XY as a radius draw a circle, and with Y as a center and the same radius draw another circle, and call one intersection of the circles C.

With a straightedge draw a line from P to C, and this will be the perpendicular required.

2. *From a given point outside a given straight line required to let fall a perpendicular to the line.*



Let AB be the given straight line and P be the given point.

It is required to draw from P a line perpendicular to AB.

With P as a center and any convenient radius draw an arc cutting AB at X and Y.

With X as a center and any convenient radius draw a circle, and with Y as a center and the same radius draw another circle, and call one intersection of the circles C.

With a straightedge draw a straight line from P to C, and this will be the perpendicular required.

3. *Required to draw a triangle having two sides each equal to a given line.*

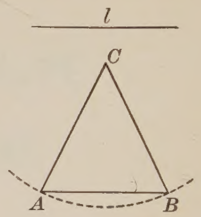
Let l be the given line.

It is required to draw a triangle having two sides each equal to l.

With any center, as C, and a radius equal to l draw an arc.

Join any two points on the arc, as A and B, with each other and with C by straight lines.

Then ABC is the triangle required.



4. *Required to draw a triangle having its three sides each equal to a given line.*

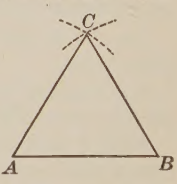
Let AB be the given line.

It is required to draw a triangle having its three sides each equal to AB.

With A as a center and AB as a radius draw a circle, and with B as a center and the same radius draw another circle.

Join either intersection of the circles with A and B by straight lines.

Then ABC is the triangle required.



In such cases draw the arcs only long enough to show the point of intersection.

5. *Required to draw a triangle having its sides equal respectively to three given lines.*

Let the three lines be l, m, and n.

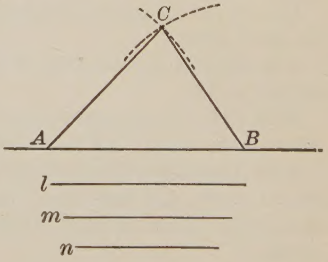
It is required to draw a triangle having its sides equal to l, m, and n respectively.

Upon any line mark off with the compasses a line-segment AB equal to l.

With A as a center and m as a radius draw a circle; with B as a center and n as a radius draw a circle.

Draw AC and BC.

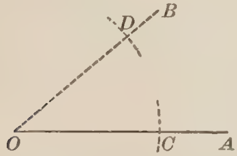
Then ABC is the required triangle.

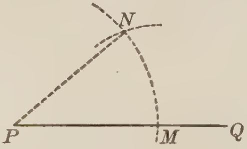


6. *From a given point on a given line required to draw a line making an angle equal to a given angle.*

Let P be the given point on the given line PQ, and let angle AOB be the given angle.

It is required to draw from P a line making with PQ an angle equal to the angle AOB.





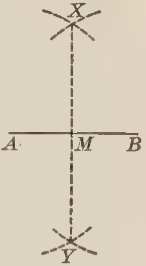
With O as a center and any radius draw an arc cutting AO at C and BO at D.

With P as a center and OC as a radius draw an arc cutting PQ at M.

With M as a center and the straight line joining C and D as a radius draw an arc cutting the one just drawn at N, and draw PN.

Then angle MPN is the required angle.

7. *Required to bisect a given straight line.*



Let AB be the given line.

It is required to bisect AB.

With A as a center and any convenient radius draw a circle, and with B as a center and the same radius draw a circle.

Call the two intersections of the circles X and Y.

Draw the straight line XY.

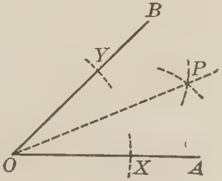
Then XY bisects the line AB at the point of intersection M.

8. *Required to bisect a given angle.*

Let AOB be the given angle.

It is required to bisect angle AOB.

With O as a center and any convenient radius draw an arc cutting OA at X and OB at Y.



With X as a center and any convenient radius draw a circle, and with Y as a center and the same radius draw a circle, and call one point of intersection of the circles P.

Draw the straight line OP.

Then OP is the required bisector.

9. *By the use of compasses and ruler draw the following figures:*

FIG. 1. Draw a circle with any point as a center and $\frac{3}{8}$ in. as a radius.

Through the center draw a horizontal diameter.

From each end of the diameter as a center and $\frac{3}{8}$ in. as a radius draw arcs cutting the circle in four points.

Connect by straight lines the six consecutive points thus obtained on the circle.

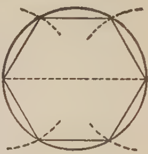


FIG. 1

FIG. 2. Draw a circle with any point as a center and $\frac{3}{8}$ in. as a radius.

With any point of the circle as a center and $\frac{3}{8}$ in. as a radius draw arcs cutting the circle in two points.

With each of these points as a center and $\frac{3}{8}$ in. as a radius draw an arc cutting the circle.

Connect by a straight line the two points last found, and each of these and the first point chosen.

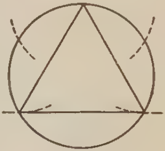


FIG. 2

FIG. 3. Draw a circle with any point as a center and $\frac{3}{8}$ in. as a radius.

Through the center draw a horizontal diameter.

With each end of this diameter as a center and a convenient radius draw an arc.

Draw a straight line from the intersection of these arcs through the center of the circle, intersecting the circle in two points.

Draw straight lines connecting the four consecutive points found on the circle.

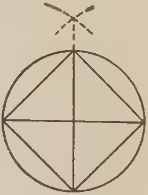


FIG. 3

10. *By the use of compasses and ruler draw the following figures:*

FIG. 1. Draw a circle with any point as a center and $\frac{1}{2}$ in. as a radius.

Through the center draw a horizontal diameter.

With each end of this diameter as a center and a convenient radius draw an arc.

Draw a straight line through the intersection of these arcs and through the center of the circle.

With each end of the horizontal diameter as a center and $\frac{1}{2}$ in. as a center draw two arcs intersecting the circle.

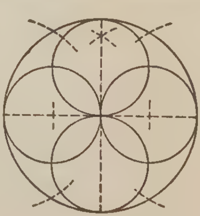


FIG. 1

Place a straightedge on each of these two pairs of points and mark a vertical line intersecting the horizontal diameter.

With each of these two points as a center and $\frac{1}{4}$ in. as a radius draw a circle.

In a similar manner bisect each half of the vertical diameter and draw the other two circles shown.

FIG. 2. With any point as a center and $\frac{1}{2}$ in. as a radius draw a circle.

With any point of the circle, as the highest point, as a center and $\frac{1}{2}$ in. as a radius draw an arc cutting the circle in two points.

With each of these two points as a center and the same radius draw an arc cutting the circle in the point first chosen and in a new point.

With each of these two new points as a center and the same radius draw an arc cutting the circle in the points first found and in a new point opposite the point first chosen.

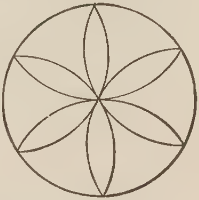


FIG. 2

With this new point as a center and the same radius draw an arc cutting the circle in the two points found in drawing the first pair of arcs.

11. *By the use of compasses and ruler draw the following figures:*

FIG. 1. Draw a circle with any point as a center and $\frac{1}{2}$ in. as a radius.

From the center draw any radius, as the radius to the highest point of the circle.

With this point of the circle as a center and $\frac{1}{2}$ in. as a radius draw an arc intersecting the circle.

With each of these points as a center and the same radius draw an arc intersecting the circle.

With one of the points last found as a center and the same radius draw an arc intersecting the circle.

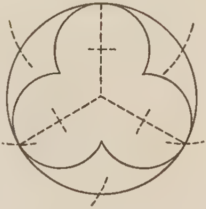


FIG. 1

For convenience call the point on the circle first chosen the first point and the other points, in rotation clockwise, the second, third, and so on.

From the center draw a radius to the third point and another radius to the fifth point.

Lay a straightedge on the second and fourth points and mark with a short line the point where it cuts the radius to the third point.

With this point as a center and $\frac{1}{4}$ in. as a radius draw a circle.

Similarly, lay a straightedge on the second and sixth points and on the fourth and sixth points, and draw the two circles.

Erase that part of each of the three small circles that is not shown in the figure.

FIG. 2. Draw a circle with any point as a center and $\frac{1}{2}$ in. as a radius.

With any point of the circle, as the highest point, as a center and $\frac{1}{2}$ in. as a radius draw an arc cutting the circle in two points.

With each of these two points as a center and the same radius draw an arc cutting the circle in the point first chosen and in a new point.

With each of these two new points as a center and the same radius draw an arc cutting the circle in the points first found and in a new point.

With this new point as a center and the same radius draw an arc cutting the circle in the two points found in drawing the first pair of arcs.

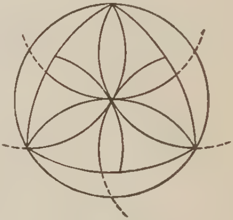


FIG. 2

Name the point on the circle first chosen the first point and the other points, in rotation clockwise, the second, third, fourth, fifth, and sixth.

With the first point as a center and the distance from the first to the third point as a radius draw an arc cutting the circle in the third and fifth points.

With the third point as a center and the same radius draw an arc cutting the circle in the fifth and first points.

With the fifth point as a center and the same radius draw an arc cutting the circle in the first and third points.

Erase that part of each arc that is not shown in the figure.

12. *By the use of compasses and ruler draw the following figures:*

FIG. 1. Draw a circle with any point as a center and $\frac{1}{2}$ in. as a radius.

Through the center of the circle draw a horizontal diameter.

With each end of this diameter as a center and a convenient radius draw an arc.

Draw a straight line through the intersection of these arcs and through the center of the circle.

Draw a straight line from the first to the second of these points on the circle, another from the second to the third, another from the third to the fourth, and another from the fourth to the first.

With the highest of these points as a center and a convenient radius draw an arc.

With the right-hand point of these four points as a center and the same radius draw an arc intersecting the arc first drawn.

Draw a straight line from the point of intersection of these arcs to the center, cutting the circle.

This point of the circle is the mid-point of that part of the circle between the first and second points.

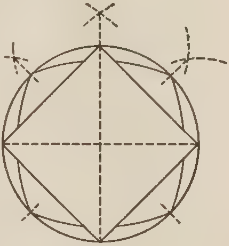


FIG. 1

In a similar manner find the mid-points of that part of the circle between the second and third points; the mid-point of that part of the circle between the third and fourth points; and of that part of the circle between the fourth and first points.

With the first point of the circle as a center and a radius equal to the distance from that point to the second mid-point draw an arc cutting the circle in the second and third mid-points.

With the same radius draw an arc with the second point of the circle as a center, cutting the circle in the third and fourth mid-points. With the same radius draw an arc with the third point of the circle as a center, cutting the circle in the fourth and first mid-points. With the same radius draw an arc with the fourth point of the circle as a radius, cutting the circle in the first and second mid-points.

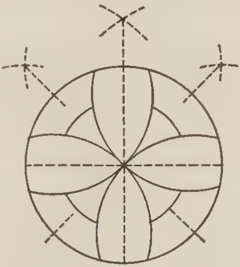


FIG. 2

FIG. 2. With any point as a center and $\frac{1}{2}$ in. as a radius draw a circle.

Through the center of the circle draw a horizontal diameter.

With the right-hand end of this diameter as a center and a convenient radius draw an arc.

With the left-hand end of the diameter as a center and the same radius draw an arc intersecting the first arc.

Through the intersection of these arcs and the center of the circle draw a straight line intersecting the circle in two points.

With the upper point of this diameter as a center and a convenient radius draw an arc.

With the right-hand end of the horizontal diameter as a center and the same radius draw an arc intersecting the first arc.

Draw a straight line from the intersection of these arcs to the center of the circle, intersecting the circle in two points.

In a similar manner draw a straight line intersecting the circle between the upper end of the vertical diameter and the left-hand end of the horizontal diameter, and between this and the lower end of the vertical diameter.

Beginning with the upper end of the vertical diameter, call these eight points of the circle, in order clockwise, the first, second, third, . . ., eighth point.

With the second point as a center and $\frac{1}{2}$ in. as a radius draw an arc intersecting the circle in two points.

With the fourth point as a center and $\frac{1}{2}$ in. as a radius draw an arc intersecting the circle in two points.

With the sixth point as a center and $\frac{1}{2}$ in. as a radius draw an arc intersecting the circle in two points.

With the eighth point as a center and $\frac{1}{2}$ in. as a radius draw an arc intersecting the circle in two points.

With the center of the circle as a center and $\frac{5}{16}$ in. as a radius draw a circle and erase such portions of this circle as are not shown in the figure.

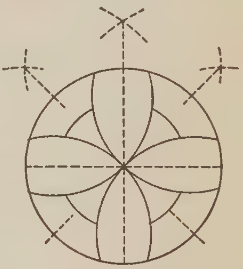


FIG. 2

13. *By the use of compasses and ruler draw the following figures:*

FIG. 1. With any point as a center and $\frac{1}{2}$ in. as a radius draw a circle.

With any point of the circle as a center and $\frac{1}{2}$ in. as a radius draw an arc cutting the circle.

With that point of intersection as a center and the same radius draw an arc cutting the circle again, and so on for six times in succession.

The circle is thus divided into six equal parts.

With the center of the circle as a center and $\frac{5}{16}$ in. as a radius draw a circle.

With each of the six points found on the larger circle as a center and $\frac{1}{2}$ in. as a radius draw an arc lying wholly within the smaller circle.

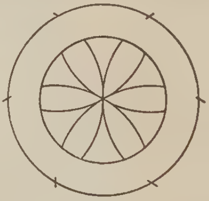


FIG. 1

FIG. 2. With any point as a center and $\frac{1}{2}$ in. as a radius draw a circle.

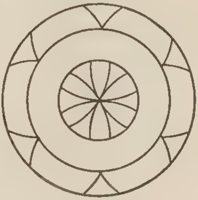


FIG. 2

With the same point as a center and $\frac{3}{8}$ in. as a radius draw a circle.

With the same point as a center and $\frac{3}{16}$ in. as a radius draw a circle.

With any point of the middle circle as a center and $\frac{3}{8}$ in. as a radius draw an arc cutting each circle in two points.

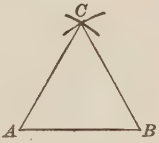
With each of the two points of intersection of this arc with the middle circle as a center and the same radius draw an arc cutting each circle in two points.

With each of the two points of intersection of these arcs with the middle circle as a center and the same radius draw an arc cutting each circle in two points. These two arcs intersect on the middle circle.

With that point as a center and the same radius draw an arc cutting each circle in two points.

Erase that part of each arc that lies between the middle circle and the smallest circle.

14. *Draw a triangle of which each side is* $1\frac{1}{4}$ *in.*



Draw a straight line AB $1\frac{1}{4}$ in. long.

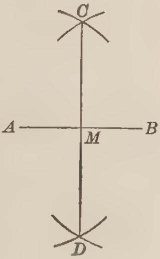
With A as a center and AB as a radius draw a circle, and with B as a center and AB as a radius draw another circle.

Call one point of intersection of these circles C.

Draw BC and CA.

Then ABC is the triangle required.

15. *Draw two lines bisecting each other at right angles.*



Draw a straight line AB of any convenient length.

With A as a center and a convenient radius draw a circle, and with B as a center and the same radius draw another circle.

Call the points of intersection of the two circles C and D.

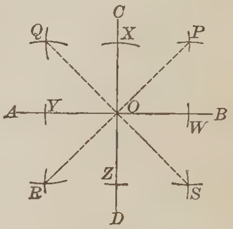
Draw CD intersecting AB at M.

Then AB and CD are two lines bisecting each other at right angles at the point M.

16. *Bisect each of the four right angles formed by two lines bisecting each other at right angles.*

Let AB and CD be two straight lines bisecting each other at right angles at the point O.

It is required to bisect each of the four right angles BOC, COA, AOD, and DOB.



With O as a center and any convenient radius draw a circle cutting OB, OC, OA, and OD in the points W, X, Y, and Z respectively.

With W and X as centers and the same radius draw circles and call one point of intersection P.

Similarly, with X and Y as centers, with Y and Z as centers, and with Z and W as centers. Call these points of intersection Q, R, and S respectively.

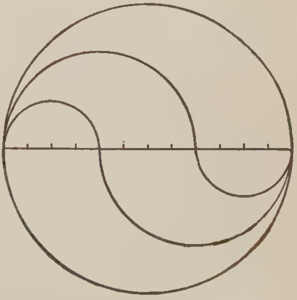
Draw OP, OQ, OR, and OS.

Then OP, OQ, OR, and OS are the required bisectors of the angles BOC, COA, AOD, and DOB respectively.

17. *Draw a line 1½ in. long and divide it into eighths of an inch, using the ruler. Then with the compasses draw this figure.*

Draw a line 1½ in. long and divide it into twelve eighths of an inch.

With the second point of division, counting from the left, as a center and ¼ in. as a radius draw half a circle above the line.



With the fourth point of division, counting from the left, as a center and ½ in. as a radius draw half a circle above the line.

With the sixth point of division, counting from the left, as a center and ¾ in. as a radius draw a circle.

With the eighth point of division, counting from the left, as a center and ½ in. as a radius draw half a circle below the line.

With the tenth point of division, counting from the left, as a center and ¼ in. as a radius draw half a circle below the line.

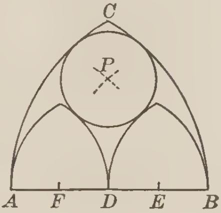
18. *In planning a Gothic window this drawing is needed. The arc BC is drawn with A as a center and AB as a radius. The small arches are described with A, D, and B as centers and AD as a radius. The center P is found by taking A and B as centers and AE as a radius. How may the points D, E, and F be found? Draw the figure.*

Draw the straight line AB 1 in. long.

Bisect AB at D, following the method of Ex. 7.

Bisect AD at F and DB at E, following the method of Ex. 7.

With A as a center and AB as a radius draw the arc BC, and with B as a center and the same radius draw the arc AC, intersecting the first arc at C.



With A as a center and AD as a radius draw an arc, with B as a center and the same radius draw an arc, and with D as a center and the same radius draw two arcs intersecting the two arcs first drawn.

With A as a center and AE as a radius draw an arc, and with B as a center and the same radius draw an arc intersecting the arc first drawn at P.

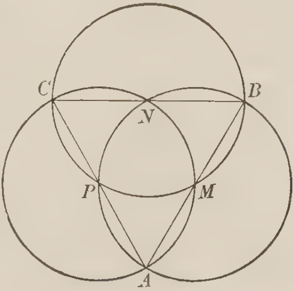
With P as a center and AF as a radius draw a circle.

19. *Draw a triangle of which each side is* 1 *in. Bisect each side, and with the points of bisection as centers and with radii* $\frac{1}{2}$ *in. long draw three circles.*

Using the method of Ex. 4, draw a triangle ABC having its three sides each equal to 1 in.

Using the method of Ex. 7, bisect the lines AB, BC, and CA in the points M, N, and P respectively.

With the points of bisection, M, N, and P, as centers and radii equal to $\frac{1}{2}$ in. draw three circles.



20. *A baseball diamond is a square* 90 *ft. on a side. Draw the plan, using a scale of* $\frac{1}{16}$ *in. to a foot. Locate the pitcher* 60 *ft. from the home plate.*

On the plan each side of the square will be represented by a line $90 \times \frac{1}{16}$ in., or $5\frac{5}{8}$ in. long.

Draw a line AB $5\frac{5}{8}$ in. long.

Prolong BA through A to a point E, making AE equal to BA.

Using the method of Ex. 1, draw AF perpendicular to EB at the point A.

With A as a center and a radius equal to AB draw a circle cutting AF at D.

Then $AD = AB$.

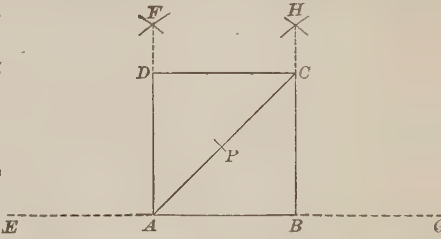
Similarly, prolong AB through B to a point G, making BG equa to AB; draw BH perpendicular to AB at the point B, and with B as a center and a radius equal to BA draw a circle cutting BH at C.

Then $BC = AB$.

Draw CD.

Then $ABCD$ is the square required.

Draw AC.



With A as a center and a radius equal to $\frac{2}{3}$ of AB, or $\frac{2}{3}$ of $5\frac{5}{8}$ in. or $3\frac{3}{4}$ in., draw a circle cutting AC at P.

Then P represents the pitcher 60 ft. from A.

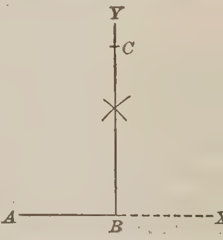
21. *A man travels from A directly east* 1 *mi. to B. He then turns and travels directly north* $1\frac{3}{4}$ *mi. to C. Draw the plan and find by measurement the distance AC to the nearest quarter of a mile. Use a scal of* $\frac{1}{2}$ *in. to a mile.*

On the scale the distance AB will be represented by a line $\frac{1}{2}$ in. long.

Draw a straight line AB $\frac{1}{2}$ in. long and prolong AB to a convenient point X.

Using the method of Ex. 1, draw a line BY perpendicular to AB at the point B.

The distance BC will be represented by a line $\frac{7}{8}$ in. long.



With B as a center and a radius equal to $\frac{7}{8}$ in. draw a circle intersecting BY at a point C.

Draw AC.

By measurement we find AC to be 1 in. long.

Therefore the distance AC is $1 \div \frac{1}{2}$, or 2 mi.

22. *A double tennis court is* 78 *ft. long and* 36 *ft. wide. The net is placed* 39 *ft. from each end and the service lines* 18 *ft. from each end. Draw the plan, using a scale of* $\frac{1}{16}$ *in. to a foot, making the right angles as shown in Ex.* 1. *The accuracy of the construction may be tested by measuring the diagonals, which should be equal.*

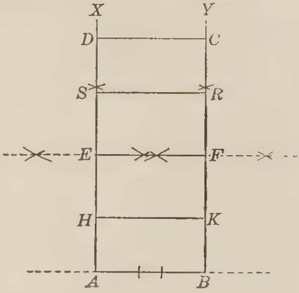
On the scale the length of the tennis court, 78 ft., will be represented by a line $4\frac{7}{8}$ in. long, and the width of the tennis court will be represented by a line $2\frac{1}{4}$ in. long.

Draw a straight line AB $2\frac{1}{4}$ in. long.

Using the method of Ex. 1, draw the lines BY and AX perpendicular to AB at the points B and A respectively.

With A as a center and a radius equal to $4\frac{7}{8}$ in. draw a circle cutting AX at a point D; and with B as a center and the same radius draw a circle cutting BY at a point C.

Draw CD.



Then $ABCD$ represents the whole tennis court, 78 ft. long and 36 ft. wide.

Bisect AD and BC in the points E and F, using the method of Ex. 7.

Since the service lines are 18 ft. from each end, they will be represented by lines $1\frac{1}{8}$ in. from each end.

With A and D as centers and radii equal to $1\frac{1}{8}$ in. draw circles cutting AD at H and S respectively; and with B and C as centers and radii equal to $1\frac{1}{8}$ in. draw circles cutting BC at K and R respectively.

Then $ABCD$ with the lines HK, EF, and SR is the plan required.

Since the distances AC and BD are equal, that test shows the plan to be accurately drawn.

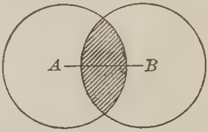
23. *At the entrance to New York harbor is a gun having a range of* 12 *mi. Draw a line inclosing the range of fire, using a scale of* $\frac{1}{16}$ *in. to a mile.*

Let G represent the position of the gun.

Since the gun will fire in every possible direction and 12 mi. is represented on the plan by a distance of $12 \times \frac{1}{16}$ in., or $\frac{3}{4}$ in., the line required is the circle drawn with G as a center and a radius of $\frac{3}{4}$ in.

24. *Two forts are placed on opposite sides of a harbor entrance,* 13 *mi. apart. Each has a gun having a range of* 10 *mi. Draw a plan showing the area exposed to the fire of both guns, using a scale of* $\frac{1}{16}$ *in. to a mile.*

On the plan a distance of 13 mi. is represented by a line $\frac{13}{16}$ in. long, and a distance of 10 mi. is represented by a length of $\frac{5}{8}$ in.



Draw a line AB $\frac{13}{16}$ in. long.

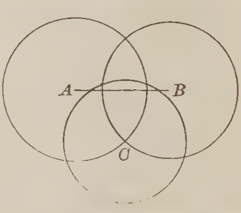
Then A represents the position of one gun and B the position of the other.

With A and B as centers and radii equal to $\frac{5}{8}$ in. draw circles which represent the area exposed to the fire of each gun.

The area exposed to the fire of both guns is the area common to the two circles and is shown in the plan by the shaded part.

25. *Two forts, A and B, are placed on opposite sides of a harbor entrance,* 16 *mi. apart. On an island in the harbor,* 12 *mi. from A and* 11 *mi. from B, is a fort C. The fort A has a gun with a range of* 12 *mi., fort B one with a range of* 11 *mi., and fort C one with a range of* 10 *mi. Draw a plan of the entrance to the harbor, showing the area exposed to the fire of each gun.*

Using a scale of $\frac{1}{16}$ in. to a mile, 16 mi. will be represented on the plan by 1 in., 12 mi. by $\frac{3}{4}$ in., 11 mi. by $\frac{11}{16}$ in., and 10 mi. by $\frac{5}{8}$ in.



Draw a straight line AB 1 in. long.

Then A represents the fort A and B represents the fort B.

With A as a center and a radius equal to $\frac{3}{4}$ in. draw a circle, and with B as a center and $\frac{11}{16}$ in. as a radius draw another circle.

Then these circles represent the area exposed to the fire of the guns at fort A and fort B, and the point of intersection, C, of these circles represents the fort C.

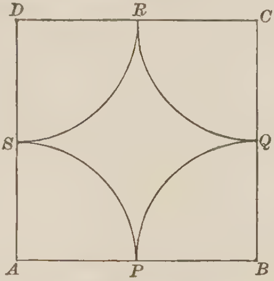
With C as a center and a radius equal to $\frac{5}{8}$ in. draw a circle.

This circle represents the area exposed to the fire of the gun at fort C.

26. *A horse, tied by a rope* 25 *ft. long at the corner of a lot* 50 *ft. square, grazes over as much of the lot as possible. The next day he is tied at the next corner, the third day at the third corner, and the fourth day at the fourth corner. Draw a plan showing the area over which he has grazed during the four days, using a scale of* $\frac{1}{8}$ *in. to* 5 *ft.*

On the plan a distance of 50 ft. will be represented by a line $1\frac{1}{4}$ in. long, and a distance of 25 ft. by a line $\frac{5}{8}$ in. long.

Using the method shown in Ex. 20, draw the square $ABCD$, $1\frac{1}{4}$ in. on a side.



With A as a center and a radius equal to $\frac{5}{8}$ in. draw a circle cutting AB at a point P and AD at a point S.

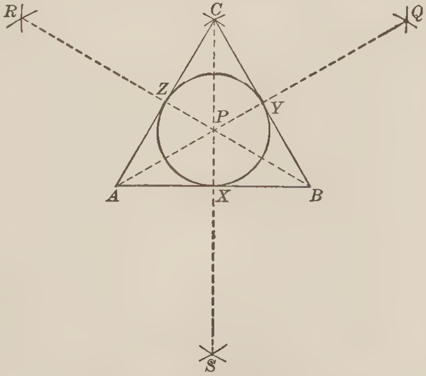
With B as a center and the same radius draw a circle cutting AB at the point P and BC at a point Q.

With C as a center and the same radius draw a circle cutting BC at the point Q and CD at a point R.

With D as a center and the same radius draw a circle cutting CD at the point R and AD at the point S.

Then the figures APS, BPQ, CRQ, and DSR represent the area over which the horse has grazed.

27. *A gardener laid out a flower bed on the following plan: He made a triangle ABC, 16 ft. on a side, and then bisected two of the angles. From the point of intersection of the bisectors, P, he drew perpendiculars to the three sides of the triangle, PX, PY, and PZ. Then he drew a circle with P as a center and PX as a radius, and found that it just fitted in the triangle. Draw the plan, using a scale of $\frac{1}{4}$ in. to a foot.*



On the plan a distance of 16 ft. will be represented by a line 4 in. long.

Draw a triangle ABC with each side 4 in., using the method of Ex. 4.

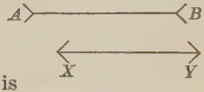
Bisect the angles A, B, and C by the lines AQ, BR, and CS, using the method of Ex. 8.

Let X, Y, and Z be the points in which the bisectors meet AB, BC, and CA respectively.

With P, the point of intersection of the bisectors, as a center and a radius equal to PX draw a circle.

Exercise 2. Page 15

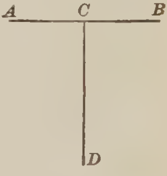
1. *Estimate which is the longer line, AB or XY, and how much longer. Then test your estimate by measuring with the compasses or with a piece of paper carefully marked.*



The line AB appears to be considerably the longer.

On measuring the lines AB and XY it is found that the two lines are exactly the same length.

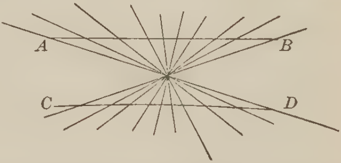
2. *Estimate which is the longer line, AB or CD, and how much longer. Then test your estimate by measuring as in Ex.* 1.



The line CD appears to be considerably the longer.

On measuring the lines AB and CD it is found that the two lines are exactly the same length.

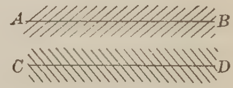
3. *Look at this figure and state whether AB and CD are both straight lines. If one is not straight, which one is it? Test your answer by using a ruler or the folded edge of a piece of paper.*



The line AB appears to be a curve line and the line CD appears to be a straight line.

On placing a straight ruler on AB and CD it is found that AB is straight and that CD is curved with the concave side toward the bottom of the page.

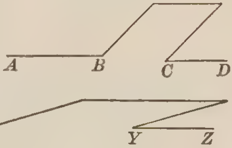
4. *Look at this figure and state whether AB and CD are the same distance apart at A and C as at B and D. Then test your answer as in Ex.* 1.



The straight lines AB and CD appear to be nearer together at B and D than at A and C.

On measuring the distances AC and BD it is found that the lines AB and CD are the same distance apart at A and C as at B and D.

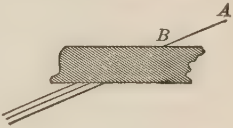
5. *Look at this figure and state whether AB will, if prolonged, lie on CD. Also state whether WX will, if prolonged, lie on YZ. Then test your answer by laying a ruler along the lines.*



On looking at the figures it appears that *AB*, if prolonged, will lie on *CD*, and that *WX*, if prolonged, will not lie on *YZ*.

On laying a straight ruler on *AB* and *WX* it is found that *AB*, if prolonged, will not lie on *CD*, and that *WX*, if prolonged, will lie on *YZ*.

6. *Look at this figure and state which of the three lower lines is AB prolonged. Then test your answer by laying a ruler along AB.*



On looking at the figure it appears that the middle line of the three lower lines is *AB* prolonged.

On placing a straight ruler on *AB* it is found that the uppermost of the three lower lines is *AB* prolonged.

Exercise 3. Page 19

1. *Find the complement of* 72°; *of* $65^\circ\,30'$; *of* $22^\circ\,20'\,15''$.

The complement of $72^\circ = 90^\circ - 72^\circ = 18^\circ$. *Ans.*
The complement of $65^\circ\,30' = 90^\circ - 65^\circ\,30' = 24^\circ\,30'$. *Ans.*
The complement of $22^\circ\,20'\,15'' = 90^\circ - 22^\circ\,20'\,15'' = 67^\circ\,39'\,45''$. *Ans.*

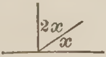
2. *What is the supplement of* 45°? *of* 120°? *of* $145^\circ\,5'$? *of* $22^\circ\,20'\,15''$?

The supplement of $45^\circ = 180^\circ - 45^\circ = 135^\circ$. *Ans.*
The supplement of $120^\circ = 180^\circ - 120^\circ = 60^\circ$. *Ans.*
The supplement of $145^\circ\,5' = 180^\circ - 145^\circ\,5' = 34^\circ\,55'$. *Ans.*
The supplement of $22^\circ\,20'\,15'' = 180^\circ - 22^\circ\,20'\,15'' = 157^\circ\,39'\,45''$. *Ans.*

3. *What is the conjugate of* 240°? *of* 280°? *of* $312^\circ\,10'\,40''$?

The conjugate of $240^\circ = 360^\circ - 240^\circ = 120^\circ$. *Ans.*
The conjugate of $280^\circ = 360^\circ - 280^\circ = 80^\circ$. *Ans.*
The conjugate of $312^\circ\,10'\,40'' = 360^\circ - 312^\circ\,10'\,40'' = 47^\circ\,49'\,20''$. *Ans.*

4. *The complement of a certain angle x is $2x$. How many degrees are there in x?*



$$x + 2x = 90^\circ.$$
$$3x = 90^\circ.$$
$$\therefore x = 30^\circ. \quad \textit{Ans.}$$

5. *The complement of a certain angle x is $3x$. How many degrees are there in x?*

$$x + 3x = 90°.$$
$$4x = 90°.$$
$$\therefore\ x = 22\tfrac{1}{2}° = 22°\,30'.\ \textit{Ans.}$$

6. *What is the angle of which the complement is four times the angle itself?*

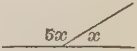
Let $x =$ the number of degrees in the angle.

$$x + 4x = 90.$$
$$5x = 90.$$
$$\therefore\ x = 18.$$

18°. *Ans.*

7. *The supplement of a certain angle x is $5x$. How many degrees are there in x?*

$$x + 5x = 180°.$$
$$6x = 180°.$$
$$\therefore\ x = 30°.\ \textit{Ans.}$$



8. *The supplement of a certain angle x is $14x$. How many degrees are there in x?*

$$x + 14x = 180°.$$
$$15x = 180°.$$
$$\therefore\ x = 12°.\ \textit{Ans.}$$

9. *What is the angle of which the supplement equals half of the angle itself?*

Let $x =$ the number of degrees in the required angle.

$$x + \frac{x}{2} = 180.$$
$$2x + x = 360.$$
$$3x = 360.$$
$$\therefore\ x = 120.$$

120°. *Ans.*

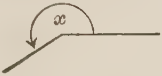
10. *How many degrees in an angle that equals its own complement? in one that equals its own supplement?*

Let $x =$ the number of degrees in the required angle.

$$x + x = 90. \qquad x + x = 180.$$
$$2x = 90. \qquad 2x = 180.$$
$$\therefore\ x = 45. \qquad \therefore\ x = 90.$$

45°; 90°. *Ans.*

11. *The conjugate of a certain angle x is $\frac{2}{3}x$. How many degrees are there in x?*



$$x + \tfrac{2}{3}x = 360^\circ.$$
$$3x + 2x = 1080^\circ.$$
$$5x = 1080^\circ.$$
$$\therefore\ x = 216^\circ.\ \textit{Ans.}$$

12. *The conjugate of a certain angle x is $\frac{1}{2}x$. How many degrees are there in x?*

$$x + \tfrac{1}{2}x = 360^\circ.$$
$$2x + x = 720^\circ.$$
$$3x = 720^\circ.$$
$$\therefore\ x = 240^\circ.\ \textit{Ans.}$$

13. *How many degrees in an angle that equals a third of its own conjugate? in one that equals its own conjugate?*

Let x = the number of degrees in the required angle.

$360 - x$ = the number of degrees in the conjugate.

$$x = \tfrac{1}{3}(360 - x). \qquad x = 360 - x.$$
$$3x = 360 - x. \qquad 2x = 360.$$
$$\therefore\ 4x = 360. \qquad \therefore\ x = 180.$$
$$\therefore\ x = 90.$$

90°; 180°. *Ans.*

14. *Find two angles, x and y, such that their sum is 90° and their difference is 10°.*

$$x + y = 90^\circ. \qquad (1)$$
$$x - y = 10^\circ. \qquad (2)$$

Add (1) and (2), $2x = 100^\circ.$

$$\therefore\ x = 50^\circ.$$

Subtract (2) from (1), $2y = 80^\circ.$

$$\therefore\ y = 40^\circ.$$

50°; 40°. *Ans.*

15. *Find two complementary angles such that their difference is 30°.*

Let x = the number of degrees in one angle,

and y = the number of degrees in the other angle.

$$x + y = 90. \qquad (1)$$
$$x - y = 30. \qquad (2)$$

Add (1) and (2), $2x = 120.$

$$\therefore\ x = 60.$$

Subtract (2) from (1), $2y = 60.$

$$\therefore\ y = 30.$$

60°; 30°. *Ans.*

16. *Find two supplementary angles such that one is 20° greater than the other.*

Let $x =$ the number of degrees in the smaller angle.

Then $x + 20 =$ the number of degrees in the larger angle.

$$x + (x + 20) = 180.$$
$$2x = 160.$$
$$\therefore x = 80.$$
$$x + 20 = 100.$$

80°; 100°. *Ans.*

17. *The angles x and y are conjugate angles, and their difference is a straight angle. How many degrees are there in each?*

$$x + y = 360°. \quad (1)$$
$$x - y = 180°. \quad (2)$$

Add (1) and (2), $2x = 540°.$

$$\therefore x = 270°.$$

Subtract (2) from (1), $2y = 180°.$

$$\therefore y = 90°.$$

270°; 90°. *Ans.*

18. *The angles x and y are conjugate angles, and their difference is zero. How many degrees are there in each?*

$$x + y = 360°. \quad (1)$$
$$x - y = 0. \quad (2)$$

Add (1) and (2), $2x = 360°.$

$$\therefore x = 180°.$$
$$\therefore y = 180°.$$

180°; 180°. *Ans.*

19. *Of two complementary angles one is four fifths of the other. How many degrees are there in each?*

Let $\frac{4}{5}x =$ the number of degrees in one angle.

Then $x =$ the number of degrees in the other angle.

$$\tfrac{4}{5}x + x = 90.$$
$$4x + 5x = 450.$$
$$9x = 450.$$
$$\therefore x = 50.$$
$$\tfrac{4}{5}x = 40.$$

40°; 50°. *Ans.*

20. *Of two supplementary angles one is five times the other. How many degrees are there in each?*

Let $5x =$ the number of degrees in one angle.

Then $x =$ the number of degrees in the other angle.

$$5x + x = 180.$$
$$6x = 180.$$

$\therefore x = 30$, and $5x = 150.$ 150°; 30°. *Ans*

21. *How many degrees are there in the smaller angle formed by the hands of a clock at 5 o'clock?*

At 5 o'clock the hour hand points at 5 and the minute hand points at 12.

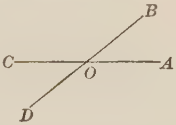
Hence the number of degrees in the smaller angle is $\frac{5}{12}$ of 360, or 150. 150°. *Ans.*

22. *How many degrees are there in the smaller angle formed by the hands of a clock at 10 o'clock?*

At 10 o'clock the hour hand points at 10 and the minute hand points at 12.

Hence the number of degrees in the smaller angle is $\frac{2}{12}$ of 360, or 60. 60°. *Ans.*

23. *In this figure, if angle AOB is 38°, how many degrees in angle BOC? How many in angle COD? How many in angle DOA?*

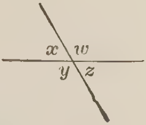


Angle $BOC = 180° - 38° = 142°$. *Ans.*
Angle $COD = 180° - 142° = 38°$. *Ans.*
Angle $DOA = 180° - 38° = 142°$. *Ans.*

24. *In the same figure, if angle AOB is equal to a third of angle BOC, how many degrees in each of the four angles?*

$$\text{Angle } AOB + \text{angle } BOC = 180°.$$
$$\text{Angle } AOB + 3 \times \text{angle } AOB = 180°.$$
$$4 \times \text{angle } AOB = 180°.$$
$$\therefore \text{angle } AOB = 45°. \textit{ Ans.}$$
$$\text{Angle } BOC = 180° - 45° = 135°. \textit{ Ans.}$$
$$\text{Angle } COD = 180° - 135° = 45°. \textit{ Ans.}$$
$$\text{Angle } DOA = 180° - 45° = 135°. \textit{ Ans.}$$

25. *In the angles of this figure, if $w = 2x$, how many degrees in each? How many degrees in y? How many degrees in z?*



$$w + x = 180°.$$
$$2x + x = 180°.$$
$$3x = 180°.$$
$$\therefore x = 60°. \textit{ Ans.}$$
$$w = 180° - x = 180° - 60° = 120°. \textit{ Ans.}$$
$$z = 180° - w = 180° - 120° = 60°. \textit{ Ans.}$$
$$y = 180° - x = 180° - 60° = 120°. \textit{ Ans.}$$

26. *Find the angle whose complement decreased by 30° equals the angle itself.*

Let $x =$ the number of degrees in the required angle.
Then $90 - x =$ its complement.

$$(90 - x) - 30 = x.$$
$$2x = 60.$$
$$\therefore x = 30.$$

30°. *Ans.*

27. *Find the angle whose complement divided by 2 equals the angle itself.*

Let $x =$ the number of degrees in the required angle.
Then $90 - x =$ its complement.

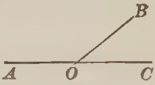
$$\frac{90 - x}{2} = x.$$
$$90 - x = 2x.$$
$$3x = 90.$$
$$\therefore x = 30.$$

30°. *Ans.*

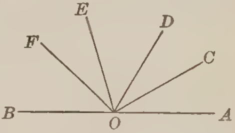
28. *Draw a figure to show that if two adjacent angles have their exterior sides in the same straight line, their sum is a straight angle.*

Let BOA and COB be two adjacent angles whose exterior sides AO and OC are in the same straight line.

Then angle BOA + angle $COB = 180°$.



29. *Draw a figure to show that the sum of all the angles on the same side of a straight line, at a given point, is equal to two right angles.*



Let O be a point on the straight line AB, and OC, OD, OE, OF be lines drawn from the point O and all on the same side of AB.

Now angle AOC + angle $COB = 180°$.

But angle $COB =$ angle $COD +$ angle $DOE +$ angle $EOF +$ angle FOB.

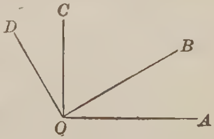
Hence the sum of all the angles is 180°, or 2 right angles.

30. *Draw a figure to show that the complements of equal angles are equal.*

Let angle AOC be a right angle, and let angle $COD =$ angle AOB.

If accurately drawn, it will be seen that angle BOD is a right angle.

Then angle BOC is the complement of angle AOB and also the complement of angle COD.



Exercise 4. Page 24

1. *If* $10^\circ + \angle x = 27^\circ\,30'$, *find the value of* $\angle x$.

$$\begin{aligned} 10^\circ + \angle x &= 27^\circ\,30'. \\ \therefore\ \angle x &= 17^\circ\,30'. \end{aligned}$$

2. *If* $\angle x + 37^\circ = \frac{1}{2}\angle x + 40^\circ$, *find the value of* $\angle x$.

$$\begin{aligned} \angle x + 37^\circ &= \tfrac{1}{2}\angle x + 40^\circ. \\ \angle x - \tfrac{1}{2}\angle x &= 40^\circ - 37^\circ. \\ \tfrac{1}{2}\angle x &= 3^\circ. \\ \therefore\ \angle x &= 6^\circ. \end{aligned}$$

3. *If* $\frac{2}{3}\angle x + \angle b = 5\angle b$, *find the value of* $\angle x$.

$$\begin{aligned} \tfrac{2}{3}\angle x + \angle b &= 5\angle b. \\ \tfrac{2}{3}\angle x &= 4\angle b. \\ \therefore\ \angle x &= 6\angle b. \end{aligned}$$

4. *If* $\angle x + \angle a = 4\angle a - \angle x$, *find the value of* $\angle x$.

$$\begin{aligned} \angle x + \angle a &= 4\angle a - \angle x. \\ 2\angle x &= 3\angle a. \\ \therefore\ \angle x &= \tfrac{3}{2}\angle a. \end{aligned}$$

5. *Find the value of* $\angle x$ *in the equation* $\angle x + 13^\circ = 39^\circ$.

$$\begin{aligned} \angle x + 13^\circ &= 39^\circ. \\ \angle x &= 39^\circ - 13^\circ. \\ \therefore\ \angle x &= 26^\circ. \end{aligned}$$

6. *Find the value of* $\angle x$ *in the equation* $\angle x - 17^\circ = 46^\circ$.

$$\begin{aligned} \angle x - 17^\circ &= 46^\circ. \\ \angle x &= 46^\circ + 17^\circ. \\ \therefore\ \angle x &= 63^\circ. \end{aligned}$$

7. *Find the value of* $\angle x$ *in the equation* $2\angle x = \angle x + 23^\circ$.

$$\begin{aligned} 2\angle x &= \angle x + 23^\circ. \\ 2\angle x - \angle x &= 23^\circ. \\ \therefore\ \angle x &= 23^\circ. \end{aligned}$$

8. *Find the value of* $\angle x$ *in the equation* $5\angle x = 2\angle x + 21^\circ$.

$$\begin{aligned} 5\angle x &= 2\angle x + 21^\circ. \\ 5\angle x - 2\angle x &= 21^\circ. \\ 3\angle x &= 21^\circ. \\ \therefore\ \angle x &= 7^\circ. \end{aligned}$$

9. *Find the value of* $\angle x$ *in the equation* $4\angle x = \frac{1}{2}\angle x + 70^\circ$.

$$\begin{aligned} 4\angle x &= \tfrac{1}{2}\angle x + 70^\circ. \\ 4\angle x - \tfrac{1}{2}\angle x &= 70^\circ. \\ \tfrac{7}{2}\angle x &= 70^\circ. \\ \tfrac{1}{2}\angle x &= 10^\circ. \\ \therefore\ \angle x &= 20^\circ. \end{aligned}$$

10. *Find the value of* $\angle x$ *in the equation* $\angle x = 0.7\angle x + 33^\circ$.

$$\begin{aligned} \angle x &= 0.7\angle x + 33^\circ. \\ \angle x - 0.7\angle x &= 33^\circ. \\ 0.3\angle x &= 33^\circ. \\ \therefore\ \angle x &= 110^\circ. \end{aligned}$$

11. *Find the value of* $\angle x$ *in the equation* $\angle x = 0.1\angle x + 18^\circ$.

$$\begin{aligned} \angle x &= 0.1\angle x + 18^\circ. \\ \angle x - 0.1\angle x &= 18^\circ. \\ 0.9\angle x &= 18^\circ. \\ \therefore\ \angle x &= 20^\circ. \end{aligned}$$

12. *Find the value of* $\angle x$ *in the equation* $\frac{3}{4}\angle x = \frac{1}{2}\angle x + 2\frac{1}{2}^\circ$.

$$\begin{aligned} \tfrac{3}{4}\angle x &= \tfrac{1}{2}\angle x + 2\tfrac{1}{2}^\circ. \\ 3\angle x &= 2\angle x + 10^\circ. \\ 3\angle x - 2\angle x &= 10^\circ. \\ \therefore\ \angle x &= 10^\circ. \end{aligned}$$

13. *Find the value of* $\angle x$ *in the equation* $\frac{4}{5}\angle x = 0.1\angle x + 14°$.

$$\begin{aligned}
\tfrac{4}{5}\angle x &= 0.1\angle x + 14°.\\
0.8\angle x &= 0.1\angle x + 14°.\\
0.8\angle x - 0.1\angle x &= 14°.\\
0.7\angle x &= 14°.\\
\therefore\ \angle x &= 20°.
\end{aligned}$$

14. *Find the value of* $\angle x$ *in the equation* $\frac{2}{3}\angle x = \frac{1}{2}\angle x + 2°$.

$$\begin{aligned}
\tfrac{2}{3}\angle x &= \tfrac{1}{2}\angle x + 2°.\\
\tfrac{2}{3}\angle x - \tfrac{1}{2}\angle x &= 2°.\\
\tfrac{1}{6}\angle x &= 2°.\\
\therefore\ \angle x &= 12°.
\end{aligned}$$

15. *Find the value of* $\angle x$ *in the equation* $12\angle x + 17° = 9\angle x + 32°$

$$\begin{aligned}
12\angle x + 17° &= 9\angle x + 32°.\\
12\angle x - 9\angle x &= 32° - 17°.\\
3\angle x &= 15°.\\
\therefore\ \angle x &= 5°.
\end{aligned}$$

16. *Find the value of* $\angle x$ *in the equation* $5\angle x - 22°\,30' = 2\angle x + 11°$.

$$\begin{aligned}
5\angle x - 22°\,30' &= 2\angle x + 11°.\\
5\angle x - 2\angle x &= 22°\,30' + 11°.\\
3\angle x &= 33°\,30'.\\
\therefore\ \angle x &= 11°\,10'.
\end{aligned}$$

17. *Find the value of* $\angle x$ *in the equation* $51°\,20' - \frac{2}{3}\angle x = 5°\,1' + 3\angle x.$

$$\begin{aligned}
51°\,20' - \tfrac{2}{3}\angle x &= 5°\,1' + 3\angle x.\\
3\angle x + \tfrac{2}{3}\angle x &= 51°\,20' - 5°\,1'\\
\tfrac{11}{3}\angle x &= 46°\,19'.\\
11\angle x &= 138°\,57'.\\
\therefore\ \angle x &= 12°\,37'\,54\tfrac{6}{11}''.
\end{aligned}$$

18. *Find the value of* $\angle x$ *in the equation*

$$73°\,21'\,4'' - \angle x = 3°\,3'\,12'' + 4\angle x.$$

$$\begin{aligned}
73°\,21'\,4'' - \angle x &= 3°\,3'\,12'' + 4\angle x.\\
4\angle x + \angle x &= 73°\,21'\,4'' - 3°\,3'\,12''.\\
5\angle x &= 70°\,17'\,52''.\\
\therefore\ \angle x &= 14°\,3'\,34\tfrac{2}{5}''.
\end{aligned}$$

19. *If* $x + 20^\circ = y$ *and* $y - 5^\circ = 2x$, *what is the value of* x *and of* y?

$$x + 20^\circ = y, \tag{1}$$
$$y - 5^\circ = 2x. \tag{2}$$

From (2), $$y = 2x + 5^\circ. \tag{3}$$

Substitute in (1), $$x + 20^\circ = 2x + 5^\circ.$$
$$\therefore x = 15^\circ.$$

Substitute in (3), $$y = 30^\circ + 5^\circ$$
$$\therefore y = 35^\circ.$$

20. *Find the value of* x *and of* y *in this set of equations:*

$$x + y = 45^\circ, \tag{1}$$
$$x - y = 35^\circ. \tag{2}$$

Add (1) and (2), $$2x = 80^\circ.$$
$$\therefore x = 40^\circ.$$

Subtract (2) from (1), $$2y = 10^\circ.$$
$$\therefore y = 5^\circ.$$

21. *Find the value of* x *and of* y *in this set of equations:*

$$x - 8y = 0^\circ, \tag{1}$$
$$x + 8y = 80^\circ. \tag{2}$$

Add (1) and (2), $$2x = 80^\circ.$$
$$\therefore x = 40^\circ.$$

Subtract (1) from (2), $$16y = 80^\circ.$$
$$\therefore y = 5^\circ.$$

22. *Find the value of* x *and of* y *in this set of equations:*

$$2x + y = 64^\circ, \tag{1}$$
$$3x - y = 88^\circ. \tag{2}$$

Add (1) and (2), $$5x = 152^\circ.$$
$$\therefore x = 30^\circ\ 24'.$$

Multiply (1) by 3, $$6x + 3y = 192^\circ. \tag{3}$$

Multiply (2) by 2, $$6x - 2y = 176^\circ. \tag{4}$$

Subtract (4) from (3), $$5y = 16^\circ.$$
$$\therefore y = 3^\circ\ 12'.$$

23. *Find the value of* x *and of* y *in this set of equations:*

$$x + 2y = 21^\circ, \tag{1}$$
$$x + 3y = 26^\circ\ 15'. \tag{2}$$

Subtract (1) from (2), $$y = 5^\circ\ 15'.$$

Substitute the value of y in (1),
$$x + 10^\circ\ 30' = 21^\circ.$$
$$\therefore x = 10^\circ\ 30'.$$

24. *Find the value of x and of y in this set of equations:*

$$x + y = 9° \, 20' \, 15'', \qquad (1)$$
$$2x - y = 12° \, 25' \, 15''. \qquad (2)$$

Add (1) and (2), $3x = 21° \, 45' \, 30''$.

$$\therefore x = 7° \, 15' \, 10''.$$

Substitute the value of x in (1),

$$7° \, 15' \, 10'' + y = 9° \, 20' \, 15''.$$
$$\therefore y = 2° \, 5' \, 5''.$$

25. *Find the value of x and of y in this set of equations:*

$$x - y = 5' \, 5'', \qquad (1)$$
$$3x + 4y = 14° \, 50' \, 50''. \qquad (2)$$

Multiply (1) by 3, $3x - 3y = 15' \, 15''$. (3)

Subtract (3) from (2), $7y = 14° \, 35' \, 35''$.

$$\therefore y = 2° \, 5' \, 5''.$$

Multiply (1) by 4, $4x - 4y = 20' \, 20''$. (4)

Add (2) and (4), $7x = 15° \, 11' \, 10''$.

$$\therefore x = 2° \, 10' \, 10''.$$

26. *If $x < 10°$ and $y = 7° \, 30'$, what can be said as to the value of $x + y$?*

If $x < 10°$ and $y = 7° \, 30'$,

then $x + y < 10° + 7° \, 30'$,

or $x + y < 17° \, 30'$.

27. *In Ex. 26, what can be said as to the value of $x - y$?*

If $x < 10°$ and $y = 7° \, 30'$,

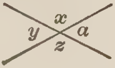
then $x - y < 10° - 7° \, 30'$,

or $x - y < 2° \, 30'$.

BOOK I

Exercise 5. Page 28

1. *In this figure, if $\angle a = 53^\circ$, how many degrees are there in $\angle y$? in $\angle x$? in $\angle z$?*



$\angle x = 180^\circ - \angle a.$ § 43

$\therefore \angle x = 180^\circ - 53^\circ = 127^\circ.$

$\angle y = \angle a.$ § 60

$\therefore \angle y = 53^\circ.$

$\angle z = \angle x.$ § 60

$\therefore \angle z = 127^\circ.$

2. *In Ex. 1, if $\angle a$ were increased to 89°, what would then be the size of $\angle$s x, y, and z?*

$\angle x = 180^\circ - \angle a.$ § 43

$\therefore \angle x = 180^\circ - 89^\circ = 91^\circ.$

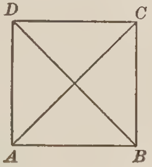
$\angle y = \angle a.$ § 60

$\therefore \angle y = 89^\circ.$

$\angle z = \angle x.$ § 60

$\therefore \angle z = 91^\circ.$

3. *In the square $ABCD$, prove that $AC = BD$.*



Given the square $ABCD$, with the lines AC and BD drawn.

To prove that $AC = BD$.

Proof. In the $\triangle$s ABC and BAD,

$AB = AB,$ Iden.

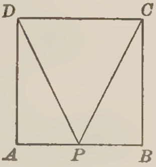
$AD = BC,$ § 65

and $\angle CBA = \angle BAD.$ § 56

$\therefore \triangle ABC$ is congruent to $\triangle BAD.$ § 68

$\therefore AC = BD$, by § 67. Q.E.D.

4. *If $ABCD$ is a square and P is the mid-point of AB, prove that $PC = PD$.*



Given the square $ABCD$, P the mid-point of AB, and the lines PC and PD drawn.

To prove that $PC = PD$.

Proof. In the $\triangle s$ PBC and PAD,

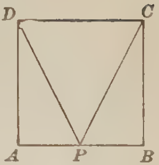
$BC = AD$, § 65

$PB = PA$, Given

and $\angle PBC = \angle PAD$. § 56

$\therefore \triangle PBC$ is congruent to $\triangle PAD$. § 68

$\therefore PC = PD$, by § 67. Q.E.D.



5. *How many degrees in an angle that equals one fourth of its complement? one tenth of its complement?*

Let x = the number of degrees in the required angle.

$$x = \tfrac{1}{4}(90 - x).$$
$$4x = 90 - x.$$
$$5x = 90.$$
$$\therefore x = 18.$$

Let x = the number of degrees in the required angle.

$$x = \tfrac{1}{10}(90 - x).$$
$$10x = 90 - x.$$
$$11x = 90.$$
$$\therefore x = 8\tfrac{2}{11}^\circ.$$

18°; $8\tfrac{2}{11}^\circ$. *Ans.*

6. *How many degrees in an angle that equals twice its supplement? one third of its supplement?*

Let x = the number of degrees in the required angle.

$$x = 2(180 - x).$$
$$x = 360 - 2x.$$
$$3x = 360.$$
$$\therefore x = 120.$$

Let x = the number of degrees in the required angle.

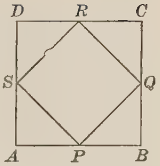
$$x = \tfrac{1}{3}(180 - x).$$
$$3x = 180 - x.$$
$$4x = 180.$$
$$\therefore x = 45.$$

120°; 45°. *Ans.*

7. *In the square $ABCD$ the points P, Q, R, S bisect the consecutive sides. Prove that $PQ = QR = RS = SP$.*

Given the square $ABCD$, P, Q, R, S the mid-points of the sides AB, BC, CD, DA, and the lines PQ, QR, RS, SP drawn.

To prove that $PQ = QR = RS = SP$.



Proof. $BC = DA$. § 65

$\therefore BQ = AS$. Ax. 4

In the $\triangle$s PBQ and PAS,

$PB = PA$, Given

$BQ = AS$,

and $\angle QBP = \angle PAS$. § 56

$\therefore \triangle PBQ$ is congruent to $\triangle PAS$. § 68

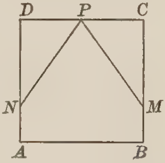
$\therefore PQ = PS$. § 67

In similar manner we may prove that $\triangle$s PBQ, RCQ, RDS, and PAS are congruent.

$\therefore PQ = QR = RS = SP$, by § 67. Q. E. D.

8. *In the square $ABCD$ the point P bisects CD, and BM is made equal to AN, as shown in this figure. Prove that $PM = PN$.*

Given the square $ABCD$, P the mid-point of CD, and BM equal to AN.



To prove that $PM = PN$.

Proof. $BC = AD$. § 65

$BM = AN$. Given

$\therefore BC - BM = AD - AN$, Ax. 2

or $CM = DN$.

In the $\triangle$s PCM and PDN,

$PC = PD$, Given

$CM = DN$,

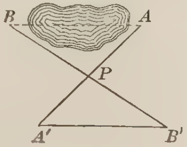
and $\angle PCM = \angle NDP$. § 56

$\therefore \triangle PCM$ is congruent to $\triangle PDN$. § 68

$\therefore PM = PN$, by § 67. Q. E. D.

9. *Prove that to determine the distance AB across a pond one may sight from A across a post P, place a stake at A' making $PA' = AP$, then sight along BP making $PB' = BP$, and finally measure $A'B'$.*

Given the lines AA' and BB' intersecting at P, with $PA' = PA$ and $PB' = PB$.



To prove that $AB = A'B'$.

Proof. In the $\triangle$s ABP and $A'B'P$,

$PA = PA'$, Given

$PB = PB'$, Given

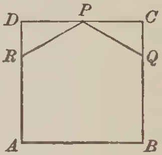
and $\angle APB = \angle A'PB'$. § 60

$\therefore \triangle ABP$ is congruent to $\triangle A'B'P$. § 68

$\therefore AB = A'B'$, by § 67. Q. E. D.

Exercise 6. Page 31

1. *In the square $ABCD$ the point P bisects CD, and PQ and PR are drawn so that $\angle QPC = 30^\circ$ and $\angle RPQ = 120^\circ$. Prove that $PQ = PR$.*



Given the square $ABCD$, P the mid-point of CD, and PQ and PR drawn so that $\angle QPC = 30^\circ$ and $\angle RPQ = 120^\circ$.

To prove that $PQ = PR$.

Proof. Since $\angle QPC = 30^\circ$, Given

and $\angle RPQ = 120^\circ$, Given

$\therefore \angle DPR = 30^\circ$. § 43

In the $\triangle s\ PCQ$ and PDR,

$\angle QPC = \angle DPR$, Ax. 8

$PC = PD$, Given

and $\angle PCQ = \angle RDP$. § 56

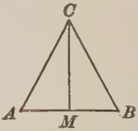
$\therefore \triangle PCQ$ is congruent to $\triangle PDR$. § 72

$\therefore PQ = PR$, by § 67. Q. E. D.

2. *In this figure prove that if CM bisects $\angle ACB$ and is also perpendicular to AB, the triangle ABC is isosceles.*

Given the $\triangle ABC$, with CM bisecting $\angle ACB$ and also $\perp$ to AB.

To prove that $CA = BC$.



Proof. In the $\triangle s\ AMC$ and BMC,

$\angle ACM = \angle MCB$, Given

$CM = CM$, Iden.

and $\angle CMA = \angle BMC$. § 56

$\therefore \triangle AMC$ is congruent to $\triangle BMC$. § 72

$\therefore CA = BC$, by § 67. Q. E. D.

3. *In the triangle ABC, $AC = BC$ and CM bisects the angle C. Prove that CM bisects the base AB.*

Given the $\triangle ABC$, with $AC = BC$, and CM bisecting $\angle C$ (see Fig. of Ex. 2).

To prove that $AM = BM$.

Proof. In the $\triangle s\ AMC$ and BMC,

$AC = BC$, Given

$\angle ACM = \angle MCB$, Given

and $CM = CM$. Iden.

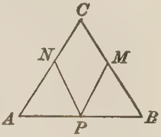
$\therefore \triangle AMC$ is congruent to $\triangle BMC$. § 68

$\therefore AM = BM$, by § 67. Q. E. D.

4. *The triangle ABC has $\angle A$ equal to $\angle B$. The point P bisects AB, and the lines PM and PN are drawn so that $\angle BPM = \angle NPA$. Prove that $BM = AN$.*

Given the $\triangle ABC$, with $\angle A = \angle B$, P the mid-point of AB, and lines PM and PN drawn so that $\angle BPM = \angle NPA$.

To prove that $BM = AN$.



Proof. In the $\triangle$s PBM and PAN,

$\angle B = \angle A$, Given

$PB = PA$, Given

and $\angle BPM = \angle NPA$. Given

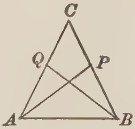
$\therefore \triangle PBM$ is congruent to $\triangle PAN$. § 72

$\therefore BM = AN$, by § 67. Q. E. D.

5. *The triangle ABC has $\angle A = \angle B$. The lines AP and BQ are so drawn that $\angle BAP = \angle QBA$. Prove that $AP = BQ$.*

Given the $\triangle ABC$, with $\angle A = \angle B$, and the lines AP and BQ drawn so that $\angle BAP = \angle QBA$.

To prove that $AP = BQ$.



Proof. In the $\triangle$s ABP and ABQ,

$\angle B = \angle A$, Given

$AB = AB$, Iden.

and $\angle BAP = \angle QBA$. Given

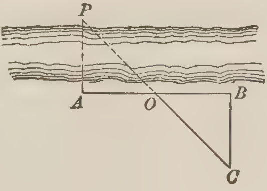
$\therefore \triangle ABP$ is congruent to $\triangle ABQ$. § 72

$\therefore AP = BQ$, by § 67. Q. E. D.

6. *Wishing to measure the distance across a river, some boys sighted from A to a point P. They then turned and measured AB at right angles to AP. They placed a stake at O, halfway from A to B, and drew a perpendicular to AB at B. They placed a stake at C, on this perpendicular, and in line with O and P. They then found the width by measuring BC. Prove that they were right.*

Given the line AB with O its mid-point, AP and $BC \perp$ to AB, and the point C in line with the points P and O.

To prove that $BC = AP$.



Proof. In the $\triangle$s OBC and OAP,

$\angle OBC = \angle OAP$, § 56

$OB = OA$, Given

and $\angle COB = \angle POA$. § 60

$\therefore \triangle OBC$ is congruent to $\triangle OAP$. § 72

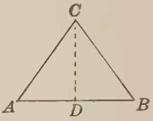
$\therefore BC = AP$, by § 67. Q. E. D.

Exercise 7. Page 33

1. *With the figure of Prop. IV, if $AC = BC$ and CD bisects $\angle C$, prove that CD is $\perp$ to AB.*

Given the isosceles $\triangle ABC$, with $AC = BC$ and CD bisecting $\angle ACB$.

To prove that CD is $\perp$ to AB.



Proof. In the $\triangle$s ADC and BDC,

$AC = BC$, Given

$\angle ACD = \angle DCB$, Given

and $CD = CD$. Iden.

$\therefore \triangle ADC$ is congruent to $\triangle BDC$. § 68

$\therefore \angle CDA = \angle BDC$. § 67

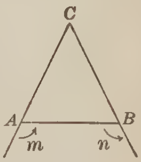
$\therefore \angle$s CDA and BDC are rt. $\angle$s. § 26

$\therefore CD$ is $\perp$ to AB, by § 27. Q. E. D.

2. *In the adjacent figure $AC = BC$. Prove that $\angle m = \angle n$.*

Given the $\triangle ABC$, with $AC = BC$, and CA and CB produced below AB.

To prove that $\angle m = \angle n$.



Proof. Now $\angle m + \angle A =$ a st. $\angle$, § 43

and $\angle n + \angle B =$ a st. $\angle$. § 43

$\therefore \angle m + \angle A = \angle n + \angle B$. Post. 6

Since $AC = BC$, Given

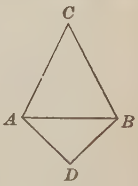
$\therefore \angle A = \angle B$. § 74

$\therefore \angle m = \angle n$, by Ax. 2. Q. E. D.

3. *In the following figure $AC = BC$ and $AD = BD$. Prove that $\angle CBD = \angle DAC$.*

Given the $\triangle$s ABC and ABD, with $AC = BC$ and $AD = BD$.

To prove that $\angle CBD = \angle DAC$.



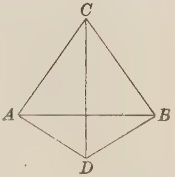
Proof. $\angle CBA = \angle BAC$, § 74

and $\angle ABD = \angle DAB$. § 74

$\therefore \angle CBA + \angle ABD = \angle BAC + \angle DAB$, Ax. 1

or $\angle CBD = \angle DAC$. Q. E. D.

4. *In the figure of Ex. 3 prove that if a line is drawn from C to D, the $\triangle DBC$ is congruent to the $\triangle DAC$.*



Given the $\triangle$s ABC and ABD, with $AC = BC$ and $AD = BD$, and the line CD drawn.

To prove that $\triangle DBC$ is congruent to $\triangle DAC$.

Proof. In the $\triangle$s DBC and DAC,

$BC = AC$, Given

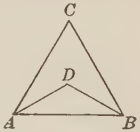
$\angle CBD = \angle DAC$, Ex. 3

and $DB = AD$. Iden.

$\therefore \triangle DBC$ is congruent to $\triangle DAC$, by § 68. Q. E. D.

5. *Two isosceles triangles, ABC and ABD, are constructed on the same side of the common base AB. Prove that $\angle CBD = \angle DAC$.*

Given the isosceles $\triangle$s ABC and ABD, with $AC = BC$ and $AD = BD$, constructed on the same side of the common base AB.



To prove that $\angle CBD = \angle DAC$.

Proof. $\angle CBA = \angle BAC$, § 74

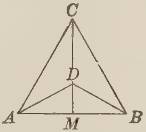
and $\angle DBA = \angle BAD$. § 74

$\therefore \angle CBA - \angle DBA = \angle BAC - \angle BAD$, Ax. 2

or $\angle CBD = \angle DAC$. Q. E. D.

6. *In the figure of Ex. 5 prove that a line drawn through C and D bisects $\angle ADB$.*

Given the isosceles $\triangle$s ABC and ABD, with $AC = BC$ and $AD = BD$, constructed on the same side of the common base AB, and the line CD drawn, which intersects AB at M.



To prove that $\angle ADM = \angle MDB$.

Proof. In the $\triangle$s ADC and BDC,

$AD = BD$, Given

$\angle DAC = \angle CBD$, Ex. 5

and $AC = BC$. Given

$\therefore \triangle ADC$ is congruent to $\triangle BDC$. § 68

$\therefore \angle CDA = \angle BDC$. § 67

Now $\angle CDA + \angle ADM =$ a st. $\angle$,

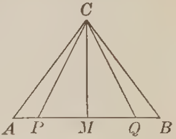
and $\angle BDC + \angle MDB =$ a st. $\angle$. § 43

$\angle CDA + \angle ADM = \angle BDC + \angle MDB$. Ax. 8

Since $\angle CDA = \angle BDC$, Proved

$\therefore \angle ADM = \angle MDB$. Ax. 2

7. *In this figure* (representing some supports to a roof) $AC = BC$ *and* $AP = BQ$. *Prove that* $PC = QC$. *Also prove that* $\angle MPC = \angle CQM$.



Given the line $AC = BC$ and $AP = BQ$.

To prove that $PC = QC$,

and that $\angle MPC = \angle CQM$.

Proof. In the $\triangle$s APC and BQC,

$AC = BC$, Given

$\angle PAC = \angle CBQ$, § 74

and $AP = BQ$. Given

$\therefore \triangle APC$ is congruent to $\triangle BQC$. § 68

$\therefore PC = QC$, by § 67. Q. E. D.

Again, since $\triangle APC$ is congruent to $\triangle BQC$, Proved

$\therefore \angle CPA = \angle BQC$. § 67

Now $\angle MPC + \angle CPA = 1$ st. $\angle$,

and $\angle BQC + \angle CQM = 1$ st. $\angle$. § 43

$\angle MPC + \angle CPA = \angle BQC + \angle CQM$, Ax. 8

$\therefore \angle MPC = \angle CQM$, by Ax. 2 Q. E. D.

8. *In this figure, if* $AC = BC$, $AP = BQ$, *and* $PM = QM$, *prove that* CM *is* $\perp$ *to* PQ. (See Fig. of Ex. 7.)

Given $AC = BC$, $AP = BQ$, and $PM = QM$.

To prove that CM is $\perp$ to PQ.

Proof. Since $AP = BQ$, Given

and $PM = QM$, Given

$\therefore AP + PM = BQ + QM$, Ax. 1

or $AM = BM$.

In the $\triangle$s AMC and BMC,

$AC = BC$, Given

$\angle MAC = \angle CBM$, § 74

and $AM = BM$.

$\therefore \triangle AMC$ is congruent to $\triangle BMC$. § 68

$\therefore \angle CMA = \angle BMC$. § 67

$\therefore \angle$s CMA and BMC are rt. $\angle$s. § 26

$\therefore CM$ is $\perp$ to PQ, by § 27. Q. E. D.

9. *In this figure* P, Q, *and* R *are mid-points of the sides of the equilateral triangle* ABC. *Prove that* PQR *is an equilateral triangle.*

Given the equilateral $\triangle ABC$, with P, Q, R the mid-points of AB, BC, CA respectively, and with the lines PQ, QR, RP drawn.

To prove that $\triangle PQR$ is equilateral.

Proof. In the $\triangle$ APR and PBQ,

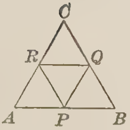
$AP = PB$, Given

$\angle A = \angle B$, § 74

and $AR = BQ$. Ax. 4

$\therefore \triangle APR$ is congruent to $\triangle PBQ$. § 68

$\therefore RP = PQ$. § 67



In similar manner

$\triangle APR$ is congruent to $\triangle RQC$,

and $RP = QR$. § 67

$\therefore RP = PQ = QR$. Ax. 8

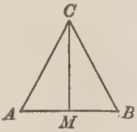
$\therefore \triangle PQR$ is equilateral, by § 62. Q.E.D.

Exercise 8. Page 37

1. *Prove that a line from the vertex to the mid-point of the base of an isosceles triangle cuts the triangle into two congruent triangles.*

Given the isosceles $\triangle ABC$, with $AC = BC$, and with CM drawn from the vertex C to M, the mid-point of AB.

To prove that $\triangle AMC$ is congruent to $\triangle BMC$.



Proof. In the $\triangle$ AMC and BMC,

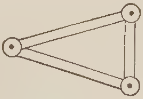
$AC = BC$, Given

$AM = BM$, Given

$CM = CM$. Iden.

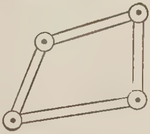
$\therefore \triangle AMC$ is congruent to $\triangle BMC$, by § 80. Q.E.D.

2. *Three iron rods are hinged at the extremities, as shown in this figure. Is the figure rigid? Why?*



The figure is rigid, since we cannot push any angle in toward, nor pull it away from, the center.

3. *Four iron rods are hinged, as shown in this figure. Is the figure rigid? If not, where would you put in the fifth rod to make it rigid? Prove that this would accomplish the result.*

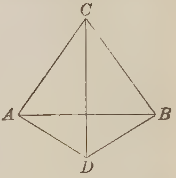


The figure is not rigid, since we may increase or decrease the size of any one of the four angles by pushing in or pulling out the angle.

To make the figure rigid a fifth rod should be put in between either pair of opposite angles. Then the question would resolve itself into Ex. 2.

4. *If two isosceles triangles are constructed on opposite sides of the same base, prove by Prop. VI and § 58 that the line through the vertices bisects the vertical angles.*

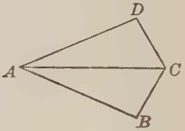
Let ABC and ABD be isosceles $\triangle$s, with $AC = BC$ and $AD = BD$, and with the line CD drawn.



To prove that $\angle ACD = \angle DCB$,
and that $\angle CDA = \angle BDC$.

Proof. In the $\triangle$s ADC and BDC,
$AC = BC$, Given
$AD = BD$, Given
and $CD = CD$. Iden.
$\therefore \triangle ADC$ is congruent to $\triangle BDC$. § 80
$\therefore \angle ACD = \angle DCB$,
and $\angle CDA = \angle BDC$, by § 67. Q.E.D.

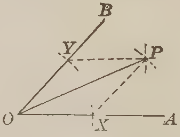
5. *In this figure $AB = AD$ and $CB = CD$. Prove that AC bisects $\angle BAD$ and $\angle DCB$.*



Given $AB = AD$ and $CB = CD$.
To prove that $\angle CAD = \angle BAC$,
and that $\angle DCA = \angle ACB$.

Proof. In the $\triangle$s ACD and ACB,
$AD = AB$, Given
$AC = AC$, Iden.
and $CD = CB$. Given
$\therefore \triangle ACD$ is congruent to $\triangle ACB$. § 80
$\therefore \angle CAD = \angle BAC$,
and $\angle DCA = \angle ACB$, by § 67. Q.E.D.

6. *In § 31, Ex. 8, it was shown how to bisect an angle, this being the figure used. Draw PX and PY, and prove by Prop. VI that PO bisects $\angle AOB$.*



Given the $\angle AOB$, with $OX = OY$, $PX = PY$, and PX and PY drawn.
To prove that $\angle AOP = \angle POB$.

Proof. In the $\triangle$s OXP and OYP,
$OX = OY$, Given
$PX = PY$, Given
and $OP = OP$. Iden.

$\therefore \triangle OXP$ is congruent to $\triangle OYP$. § 80

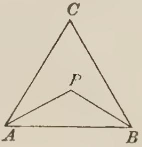
$\therefore \angle XOP = \angle POY$. § 67

That is, $\angle AOP = \angle POB$. Q. E. D.

7. *In a triangle ABC it is known that $AC = BC$. If $\angle A$ and $\angle B$ are both bisected by lines meeting at P, prove that $\triangle ABP$ is isosceles.*

Given the $\triangle ABC$, with $AC = BC$, and AP and BP bisecting the $\angle$s A and B respectively.

To prove that $\triangle ABP$ is isosceles.



Proof. Since $BC = AC$, Given

$\therefore \angle A = \angle B$. § 74

But $\angle BAP = \frac{1}{2}\angle A$,

and $\angle PBA = \frac{1}{2}\angle B$. Given

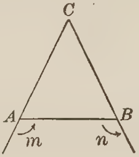
$\therefore \angle BAP = \angle PBA$. Ax. 8

$\therefore \triangle ABP$ is isosceles, by § 76. Q. E. D.

8. *In this figure it is known that $\angle m = \angle n$. Prove that $AC = BC$.*

Given AC and BC intersecting at C, and AB drawn so that $\angle m = \angle n$.

To prove that $AC = BC$.



Proof. $\angle n + \angle CBA =$ a st. $\angle$,

and $\angle m + \angle BAC =$ a st. $\angle$. § 43

$\therefore \angle n + \angle CBA = \angle m + \angle BAC$. Post. 6

But $\angle n = \angle m$. Given

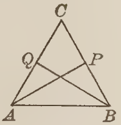
$\therefore \angle CBA = \angle BAC$. Ax. 2

$\therefore AC = BC$, by § 76. Q. E. D.

9. *From the vertices A and B of an equilateral triangle lines are drawn to the mid-points of the opposite sides. Prove that these two lines are equal.*

Given the equilateral $\triangle ABC$, with P and Q the mid-points of BC and CA respectively.

To prove that $AP = BQ$.



Proof. In the $\triangle$s ABP and ABQ,

$AB = AB$, Iden.

$BP = AQ$, Ax. 4

and $\angle PBA = \angle BAQ$. § 74

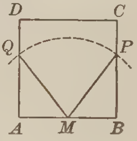
$\therefore \triangle ABP$ is congruent to $\triangle ABQ$. § 68

$\therefore AP = BQ$, by § 67. Q. E. D.

Exercise 9. Page 44

1. *$ABCD$ is a square and M is the mid-point of AB. With M as a center an arc is drawn, cutting BC at P and AD at Q. Prove that $\triangle MBP$ is congruent to $\triangle MAQ$, and write the general statement of this theorem without using letters as is done here.*

Given the square $ABCD$, with M the mid-point of AB and with $MP = MQ$.



To prove that $\triangle MBP$ is congruent to $\triangle MAQ$.

Proof. The ∠s PBM and MAQ are rt. ∠s. § 65

In the rt. $\triangle$s MBP and MAQ,

$MP = MQ$, Given

and $MB = MA$. Given

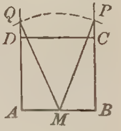
$\therefore \triangle MBP$ is congruent to $\triangle MAQ$, by § 89. Q. E. D.

A general statement of this theorem reads as follows:

If an arc is drawn, with the mid-point of one side of a square as a center, cutting the sides perpendicular to that side, then the triangles cut off by the lines drawn from the mid-point to the points of intersection of the arc with the sides are congruent.

2. *Draw a figure similar to that of Ex.* 1, *but take a radius such that the arc cuts BC produced at a point above C, and AD above D. Then prove that $\triangle MBP$ is congruent to $\triangle MAQ$.*

Given the square $ABCD$, with M the mid-point of AB, and with the arc PQ drawn with M as a center cutting BC and AD produced in the points P and Q respectively.



To prove that $\triangle MBP$ is congruent to $\triangle MAQ$.

Proof. The ∠s PBM and MAQ are rt. ∠s. § 65

In the rt. $\triangle$s MBP and MAQ,

$MP = MQ$, Given

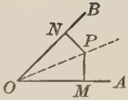
and $MB = MA$. Given

$\therefore \triangle MBP$ is congruent to $\triangle MAQ$, by § 89. Q. E. D.

3. *Prove that if from the point P the perpendiculars PM, PN to the sides of an angle AOB are equal, the point P lies on the bisector of the angle AOB. Write the general statement of this theorem without using letters as is done here.*

Given the $\angle AOB$, P a point within the angle, and PM and PN equal perpendiculars to OA and OB respectively.

To prove that P lies on the bisector of $\angle AOB$.



Proof. Draw OP.

In the rt. $\triangle$s OMP and ONP, $OP = OP$, Iden.

and $PM = PN$. Given

$\therefore \triangle OMP$ is congruent to $\triangle ONP$. § 89

$\therefore \angle MOP = \angle PON$. § 67

That is, OP is the bisector of $\angle AOB$. Q.E.D.

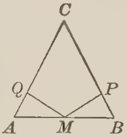
A general statement of this theorem reads as follows:

If the perpendiculars from a point within an angle upon the sides of the angle are equal, then the given point lies on the bisector of the given angle.

4. *Prove that if the perpendiculars from the mid-point M of the base AB of a triangle ABC to the sides of the triangle are equal, then $\angle A = \angle B$. What then follows as to the sides AC and BC? Write the general statement of this theorem without referring to a special figure.*

Given M the mid-point of the base AB of the $\triangle ABC$, and the $\perp$s MP and MQ upon the sides BC and CA equal.

To prove that $\angle A = \angle B$.



Proof. The $\angle$s MPB and AQM are rt. $\angle$s. Given

In the rt. $\triangle$s MBP and MAQ, $MP = MQ$, Given

and $MB = MA$. Given

$\therefore \triangle MBP$ is congruent to $\triangle MAQ$. § 89

$\therefore \angle B = \angle A$, by § 67. Q.E.D.

It follows that the sides AC and BC are equal by § 76.

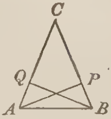
A general statement of this theorem reads as follows:

If the perpendiculars from the mid-point of the base of a triangle to the other two sides are equal, then the base angles of the triangle are equal and the triangle is isosceles.

5. *Prove that if the perpendiculars from the extremities of the base of a triangle to the other two sides are equal, the triangle is isosceles.*

Given the $\triangle ABC$ and the equal $\perp$s AP and BQ upon the sides BC and CA respectively.

To prove that $\triangle ABC$ is isosceles.



Proof. The $\angle$s APB and AQB are rt. $\angle$s. Given

In the rt. $\triangle$s ABP and BAQ, $AB = AB$, Iden.

and $AP = BQ$. Given

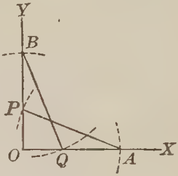
$\therefore \triangle ABP$ is congruent to $\triangle BAQ$. § 89

$\therefore \angle PBA = \angle BAQ$. § 67

$\therefore CA = BC$. § 76

$\therefore \triangle ABC$ is isosceles, by § 62. Q.E.D.

6. *Suppose $OY \perp$ to OX. With O as a center an arc is drawn cutting OX at A and OY at B. Then with A as a center an arc is drawn cutting OY at P, and with B as a center and the same radius an arc is drawn cutting OX at Q. Prove that $OP = OQ$.*



Given $OY \perp$ to OX, $OA = OB$, and $AP = BQ$.

To prove that $OP = OQ$.

Proof. In the rt. $\triangle$ OAP and OBQ,

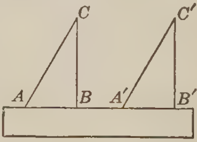
$OA = OB$, and $AP = BQ$. Given

$\therefore \triangle OAP$ is congruent to $\triangle OBQ$. § 89

$\therefore OP = OQ$, by § 67. Q.E.D.

Exercise 10. Page 52

1. *Show that if we place a draftsman's triangle against a ruler and draw AC, and move the triangle along as shown in the figure and draw $A'C'$, then AC is $\parallel$ to $A'C'$.*



Given $A'B' = AB$ and in the same straight line with AB, and $BC = B'C'$, and both $\perp$ to the line of AB and $A'B'$.

To prove that AC is $\parallel$ to $A'C'$.

Proof. In the rt. $\triangle$ ABC and $A'B'C'$,

$AB = A'B'$, Given

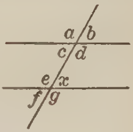
and $BC = B'C'$. Given

$\therefore \triangle ABC$ is congruent to $\triangle A'B'C'$. § 90

$\therefore \angle BAC = \angle B'A'C'$. § 67

$\therefore AC$ is $\parallel$ to $A'C'$, by § 103. Q.E.D.

2. *In the next figure $\angle x = 60°$. How many degrees in each of the other seven angles?*



Given $\angle x = 60°$.

$\angle f = \angle x = 60°$. § 60

$\angle e = 180° - \angle x = 120°$. § 43

$\angle g = 180° - \angle x = 120°$. § 43

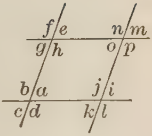
$\angle a = \angle e = 120°$. § 102

$\angle b = \angle x = 60°$. § 102

$\angle c = \angle f = 60°$. § 102

$\angle d = \angle g = 120°$. § 102

3. *In the next figure, representing two pairs of parallel lines, certain angles are equal. State these equalities in this form: $a = c = g = e = o = \cdots$, and give the reason in each case.*



$a = c.$	§ 60	$e = m.$	§ 102
$a = e.$	§ 102	$e = o.$	§ 100
$a = g.$	§ 100	$g = m.$	§ 106
$a = i.$	§ 102	$g = o.$	§ 102
$a = k.$	§ 100	$i = k.$	§ 60
$c = e.$	§ 106	$i = m.$	§ 102
$c = g.$	§ 102	$i = o.$	§ 100
$c = i.$	§ 106	$k = m.$	§ 106
$c = k.$	§ 102	$k = o.$	§ 102
$e = g.$	§ 60	$m = o.$	§ 60

$\therefore a = c = e = g = i = k = m = o$, by Ax. 8 (1)

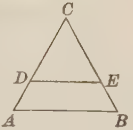
$b = d.$	§ 60	$f = n.$	§ 102
$b = f.$	§ 102	$f = p.$	§ 106
$b = h.$	§ 100	$h = n.$	§ 100
$b = j.$	§ 102	$h = p.$	§ 102
$b = l.$	§ 106	$j = l.$	§ 60
$d = f.$	§ 106	$j = n.$	§ 102
$d = h.$	§ 102	$j = p.$	§ 100
$d = j.$	§ 100	$l = n.$	§ 106
$d = l.$	§ 102	$l = p.$	§ 102
$f = h.$	§ 60	$n = p.$	§ 60

$\therefore b = d = f = h = j = l = n = p$, by Ax. 8 (2)

4. *In the figure of Ex. 3 state ten pairs of angles that are supplementary; thus:* $a + h = 180°$ *and* $d + e = 180°$.

Any angle of (1) in Ex. 3 will be supplementary to any angle of (2). By choosing those that are not adjacent, 48 pairs satisfying the given condition may be obtained.

5. *In the triangle ABC, AC = BC and DE is drawn parallel to AB. Prove that CD = CE. Write a general statement of the theorem.*



Given the $\triangle ABC$, with $AC = BC$ and $DE \parallel$ to AB.

To prove that $CD = CE$.

Proof. Since $AC = BC$, Given

$\therefore \angle CBA = \angle BAC.$ § 74

Now $\angle CED = \angle CBA,$

and $\angle EDC = \angle BAC.$ § 102

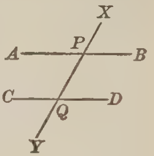
$\therefore \angle CED = \angle EDC.$ Ax. 8

$\therefore CD = CE$, by § 76. Q.E.D.

A general statement of this theorem reads as follows:

A line drawn parallel to the base of an isosceles triangle cuts off equal segments from the equal sides of the triangle.

6. *In the next figure AB is parallel to CD, and $\angle APQ$ is half of $\angle QPB$. How many degrees in the various angles?*



Given the ‖ lines AB and CD and $\angle APQ = \frac{1}{2}\angle QPB$.

Now $\angle APQ + \angle QPB = 180^\circ.$ § 43

$\therefore \angle APQ + 2\angle APQ = 180^\circ.$ Ax. 9

$3\angle APQ = 180^\circ.$

$\therefore \angle APQ = 60^\circ.$

$\angle BPX = \angle APQ = 60^\circ.$ § 60

$\angle DQP = \angle APQ = 60^\circ.$ § 100

$\angle CQY = \angle APQ = 60^\circ.$ § 102

$\angle QPB = 180^\circ - \angle APQ = 120^\circ.$ § 43

$\angle XPA = \angle QPB = 120^\circ.$ § 60

$\angle PQC = \angle XPA = 120^\circ.$ § 102

$\angle YQD = \angle PQC = 120^\circ.$ § 60

7. *If $\angle YQD = 135^\circ$, how many degrees in the various angles?*

$\angle YQD = 135^\circ.$ Given

$\angle PQC = \angle YQD = 135^\circ.$ § 60

$\angle QPB = \angle YQD = 135^\circ.$ § 102

$\angle XPA = \angle QPB = 135^\circ.$ § 60

$\angle CQY = 180^\circ - \angle YQD = 45^\circ.$ § 43

$\angle DQP = \angle CQY = 45^\circ.$ § 60

$\angle APQ = \angle CQY = 45^\circ.$ § 102

$\angle BPX = \angle APQ = 45^\circ.$ § 60

8. *Let $\angle DQP = x$ and $\angle YQD = y$. Then if $y - x = 100^\circ$, find the value of x and y.*

$\angle DQP + \angle YQD = 180^\circ.$ § 43

$\therefore x + y = 180^\circ.$ Ax. 9

But $y - x = 100^\circ.$ Given

$\therefore 2y = 280^\circ.$

$\therefore y = 140^\circ.$

$\therefore x = 180^\circ - 140^\circ = 40^\circ.$

9. *Let $\angle CQY = x$ and $\angle XPA = y$. Then if $x = \frac{1}{5}y$, find the value of x and y.*

$\angle CQY = \angle APQ.$ § 102

$\angle APQ + \angle XPA = 180^\circ.$ § 43

$\therefore x + y = 180^\circ.$ Ax. 9

$\therefore \frac{1}{5}y + y = 180^\circ.$ Ax. 9

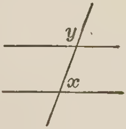
$y + 5y = 900^\circ.$

$6y = 900^\circ.$

$\therefore y = 150^\circ.$

$\therefore x = \frac{1}{5}$ of $150^\circ = 30^\circ.$

10. *In the next figure $x = 72°$ and $x = \frac{2}{3}y$. It is required to know if the lines are parallel, and why.*



If $x = 72°$ and $x = \frac{2}{3}y$,
then $y = \frac{3}{2}$ of $72° = 108°$.
The top right-hand angle $= 180° - y = 72°$. § 43
Since $x = 72°$, Given
$\therefore x =$ the top right-hand angle. Ax. 8
Therefore the lines are parallel, by § 103.

11. *In the figure of Ex. 10 suppose $x = 73°$ and $y - x = 32°$. It is required to know if the lines are parallel, and why.*

If $x = 73°$ and $y - x = 32°$,
then $y - 73° = 32°$,
and $y = 105°$.
Therefore the top right-hand angle $= 180° - y = 75°$.
Therefore that angle and x are not equal.
Hence the lines are not parallel. § 103

12. *The three angles of a triangle are x, y, and z. Find the value of z, given $x = 10°$, $y = 30°$.*

$$x + y + z = 180°. \quad \S\ 107$$
$$10° + 30° + z = 180°.$$
$$z = 180° - 10° - 30°.$$
$$\therefore z = 140°.$$

13. *The three angles of a triangle are x, y, and z. Find the value of z, given $x = 20°$, $y = 20°$.*

$$x + y + z = 180°. \quad \S\ 107$$
$$20° + 20° + z = 180°.$$
$$z = 180° - 20° - 20°.$$
$$\therefore z = 140°.$$

14. *The three angles of a triangle are x, y, and z. Find the value of z, given $x = 75°$, $y = 50°$.*

$$x + y + z = 180°. \quad \S\ 107$$
$$75° + 50° + z = 180°.$$
$$z = 180° - 75° - 50°.$$
$$\therefore z = 55°.$$

15. *The three angles of a triangle are x, y, and z. Find the value of z, given $x = 38°$, $y = 76°$.*

$$x + y + z = 180°. \quad \S\ 107$$
$$38° + 76° + z = 180°.$$
$$z = 180° - 38° - 76°.$$
$$\therefore z = 66°.$$

16. *The three angles of a triangle are x, y, and z. Find the value of z, given $x = 49°$, $y = 92°$.*

$$x + y + z = 180°. \qquad \S\ 107$$
$$49° + 92° + z = 180°.$$
$$z = 180° - 49° - 92°.$$
$$\therefore z = 39°.$$

17. *The three angles of a triangle are x, y, and z. Find the value of z, given $x = 37°$, $y = 48°$.*

$$x + y + z = 180°. \qquad \S\ 107$$
$$37° + 48° + z = 180°.$$
$$z = 180° - 37° - 48°.$$
$$\therefore z = 95°.$$

18. *The three angles of a triangle are x, y, and z. Find the value of z, given $x = 63°$, $y = 29°$.*

$$x + y + z = 180°. \qquad \S\ 107$$
$$63° + 29° + z = 180°.$$
$$z = 180° - 63° - 29°.$$
$$\therefore z = 88°.$$

19. *The three angles of a triangle are x, y, and z. Find the value of z, given $x = 75°\ 29'$, $y = 68°\ 41'$.*

$$x + y + z = 180°. \qquad \S\ 107$$
$$75°\ 29' + 68°\ 41' + z = 180°.$$
$$z = 180° - 75°\ 29' - 68°\ 41'.$$
$$\therefore z = 35°\ 50'.$$

20. *The three angles of a triangle are x, y, and z. Find the value of z, given $x = 82°\ 33'$, $y = 75°\ 48'$.*

$$x + y + z = 180°. \qquad \S\ 107$$
$$82°\ 33' + 75°\ 48' + z = 180°.$$
$$z = 180° - 82°\ 33' - 75°\ 48'.$$
$$\therefore z = 21°\ 39'.$$

21. *The three angles of a triangle are x, y, and z. Find the value of z, given $x = 69°\ 58'$, $y = 82°\ 49'$.*

$$x + y + z = 180°. \qquad \S\ 107$$
$$69°\ 58' + 82°\ 49' + z = 180°.$$
$$z = 180° - 69°\ 58' - 82°\ 49'.$$
$$\therefore z = 27°\ 13'.$$

22. *In a certain right triangle one angle is 37°. What is the size of the other acute angle?*

Since in a right triangle one angle is 90°, the sum of the other two angles must also be 90°. § 107

Therefore the required acute angle $= 90° - 37° = 53°$.

23. *In a certain right triangle one angle is* $36° 41'$. *What is the size of the other acute angle?*

The required acute angle $= 90° - 36° 41' = 53° 19'$.

24. *In a certain right triangle one angle is* $29° 48' 56''$. *What is the size of the other acute angle?*

The required acute angle $= 90° - 29° 48' 56'' = 60° 11' 4''$.

25. *In a certain right triangle one acute angle is two thirds of the other. How many degrees are there in each?*

Let $x =$ the number of degrees in the smaller acute angle.

Then $90 - x =$ the number of degrees in the larger.

$$x = \tfrac{2}{3}(90 - x).$$
$$3x = 180 - 2x.$$
$$5x = 180.$$
$$\therefore x = 36.$$
$$90 - x = 54.$$

36°; 54°. *Ans.*

26. *In a certain right triangle one acute angle is twice as large as the other. How many degrees are there in each?*

Let $x =$ the number of degrees in the smaller acute angle.

Then $2x =$ the number of degrees in the larger.

$$x + 2x = 90.$$
$$3x = 90.$$
$$x = 30.$$
$$2x = 60.$$

30°; 60°. *Ans.*

27. *In a certain right triangle the acute angles are* $2x$ *and* $5x$. *Find the value of* x *and the size of each angle.*

$$2x + 5x = 90.$$
$$7x = 90.$$
$$\therefore x = 12\tfrac{6}{7}.$$

Then $2x = 25\tfrac{5}{7}$,

and $5x = 64\tfrac{2}{7}$.

$$25\tfrac{5}{7}° = 25° 42' 51\tfrac{3}{7}'';$$
$$64\tfrac{2}{7}° = 64° 17' 8\tfrac{4}{7}''.$$

$25° 42' 51\tfrac{3}{7}''$; $64° 17' 8\tfrac{4}{7}''$. *Ans.*

28. *In a certain triangle one angle is twice as large as another and three times as large as the third. How many degrees are there in each?*

Let x = the number of degrees in the largest angle.

Then $\frac{x}{2}$ = the number of degrees in the second angle,

and $\frac{x}{3}$ = the number of degrees in the smallest angle.

$$x+\frac{x}{2}+\frac{x}{3}=180. \qquad \S\ 107$$

$$6x+3x+2x=1080.$$

$$11x=1080.$$

$$\therefore x=98\tfrac{2}{11}.$$

Then $\frac{x}{2}=49\tfrac{1}{11}$,

and $\frac{x}{3}=32\tfrac{8}{11}$. $98\tfrac{2}{11}°$; $49\tfrac{1}{11}°$; $32\tfrac{8}{11}°$. *Ans.*

29. *In a certain isosceles triangle one angle is twice another angle. How many degrees in each of the three angles?*

Let x = the number of degrees in one of the equal angles.

Then $180-2x$ = the number of degrees in the vertical angle.

If the base angles are the larger,

$$x=2(180-2x).$$

$$x=360-4x.$$

$$5x=360.$$

$$\therefore x=72.$$

$$180-2x=180-144=36.$$

If the vertical angle is the largest,

$$180-2x=2x.$$

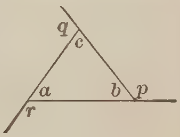
$$4x=180.$$

$$\therefore x=45.$$

$$180-2x=180-90=90.$$

72°, 72°, 36°; or 45°, 45°, 90°. *Ans.*

30. *In this figure what single angle equals $a+c$? To the sum of what angles is q equal? also r? From these relations find the number of degrees in $p+q+r$.*



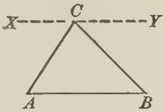
$$\angle a+\angle c=\angle p. \qquad \S\ 111$$

$$\angle q=\angle a+\angle b. \qquad \S\ 111$$

$$\angle r=\angle b+\angle c. \qquad \S\ 111$$

$$\begin{aligned}\therefore \angle p+\angle q+\angle r&=2\angle a+2\angle b+2\angle c\\&=2(\angle a+\angle b+\angle c)\\&=2\times 180°\\&=360°.\end{aligned}$$

31. *Prove Prop. XIX by first drawing a parallel to AB through C, instead of drawing BY.*



Given the triangle ABC.

To prove that $\angle A + \angle B + \angle ACB = 2$ rt. $\angle$s.

Proof. Suppose XY drawn through $C \parallel$ to AB.

Then $\angle XCA + \angle ACB + \angle BCY = 2$ rt. $\angle$s. § 34

But $\angle XCA = \angle A$,

and $\angle BCY = \angle B$. § 100

$\therefore \angle A + \angle ACB + \angle B = 2$ rt. $\angle$s. Ax. 9

That is, $\angle A + \angle B + \angle ACB = 2$ rt. $\angle$s. Q.E.D.

Exercise 11. Page 54

State in what cases it is possible to form triangles with rods of the following lengths, and give the reason:

1. 2 in., 3 in., 4 in.

It is possible to form a triangle with rods 2 in., 3 in., and 4 in. long, since the sum of any pair of rods is greater than the third rod, and the difference between any two rods is less than the third rod. § 112

2. 3 in., 4 in., 7 in.

Since 7 in. = 3 in. + 4 in., it is not possible to form a triangle with rods 3 in., 4 in., 7 in. long. § 112

3. 6 in., 7 in., 9 in.

It is possible to form a triangle with rods 6 in., 7 in., and 9 in. long, since the sum of any pair of rods is greater than the third rod, and the difference between any two rods is less than the third rod. § 112

4. 7 in., 10 in., 20 in.

Since 20 in. > 7 in. + 10 in., it is not possible to form a triangle with rods 7 in., 10 in., 20 in. long. § 112

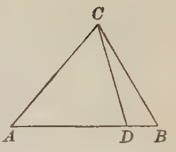
5. 8 in., $9\frac{1}{2}$ in., 18 in.

Since 18 in. > 8 in. + $9\frac{1}{2}$ in., it is not possible to form a triangle with rods 8 in., $9\frac{1}{2}$ in., 18 in. long. § 112

6. $9\frac{3}{4}$ in., $10\frac{1}{2}$ in., $12\frac{1}{4}$ in.

It is possible to form a triangle with rods $9\frac{3}{4}$ in., $10\frac{1}{2}$ in., and $12\frac{1}{4}$ in. long, since the sum of any pair of rods is greater than the third rod, and the difference between any two rods is less than the third rod. § 112

7. *In this figure prove that* $AB + BC > AD + DC$.



Given the $\triangle ABC$, with CD drawn from C to D, any point of AB.

To prove that $AB + BC > AD + DC$.

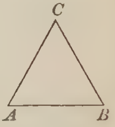
Proof. Now $DB + BC > DC$. § 112

Add AD to each member,

then $AD + DB + BC > AD + DC$, Ax. 6

or $AB + BC > AD + DC$, by Ax. 9. Q. E. D.

8. *How many degrees are there in each angle of an equiangular triangle? Prove it.*



There are 60° in each $\angle$.

Given the equiangular $\triangle ABC$.

To prove that $\angle A = \angle B = \angle C = 60°$.

Proof. $\angle A + \angle B + \angle C = 180°$. § 107

But $\angle A = \angle B = \angle C$. § 63

$\therefore \angle A + \angle A + \angle A = 180°$. Ax. 9

$3 \angle A = 180°$.

$\therefore \angle A = 60°$.

Similarly $\angle B = \angle C = 60°$. Q. E. D.

Exercise 12. Page 70

1. *What is the sum of the angles of* (a) *a pentagon?* (b) *a hexagon?* (c) *a heptagon?* (d) *an octagon?* (e) *a decagon?* (f) *a dodecagon?* (g) *a polygon of* 24 *sides?*

The sum of the $\angle$s of a polygon $= (n - 2)\ 2$ rt. $\angle$s. § 143

(a) The sum of the $\angle$s of a pentagon
$= (5 - 2)\ 2$ rt. $\angle$s $= 3 \times 2$ rt. $\angle$s $= 6$ rt. $\angle$s.

(b) The sum of the $\angle$s of a hexagon
$= (6 - 2)\ 2$ rt. $\angle$s $= 4 \times 2$ rt. $\angle$s $= 8$ rt. $\angle$s.

(c) The sum of the $\angle$s of a heptagon
$= (7 - 2)\ 2$ rt. $\angle$s $= 5 \times 2$ rt. $\angle$s $= 10$ rt. $\angle$s.

(d) The sum of the $\angle$s of an octagon
$= (8 - 2)\ 2$ rt. $\angle$s $= 6 \times 2$ rt. $\angle$s $= 12$ rt. $\angle$s.

(e) The sum of the $\angle$s of a decagon
$= (10 - 2)\ 2$ rt. $\angle$s $= 8 \times 2$ rt. $\angle$s $= 16$ rt. $\angle$s.

(f) The sum of the $\angle$s of a dodecagon
$= (12 - 2)\ 2$ rt. $\angle$s $= 10 \times 2$ rt. $\angle$s $= 20$ rt. $\angle$s.

(g) The sum of the $\angle$s of a polygon of 24 sides
$= (24 - 2)\ 2$ rt. $\angle$s $= 22 \times 2$ rt. $\angle$s $= 44$ rt. $\angle$s.

2. *What is the size of each angle of* (a) *a regular pentagon?* (b) *a regular hexagon?* (c) *a regular octagon?* (d) *a regular decagon?* (e) *a regular polygon of 32 sides?*

Each ∠ of a regular polygon of n sides $= \frac{2(n-2)}{n}$ rt. ∠s. § 145

(a) Each ∠ of a regular pentagon

$$= \frac{2(5-2)}{5} \text{ rt. } \angle s = \frac{2 \times 3}{5} \times 90^\circ = \frac{6}{5} \text{ of } 90^\circ = 108^\circ.$$

(b) Each ∠ of a regular hexagon

$$= \frac{2(6-2)}{6} \text{ rt. } \angle s = \frac{2 \times 4}{6} \times 90^\circ = 120^\circ.$$

(c) Each ∠ of a regular octagon

$$= \frac{2(8-2)}{8} \text{ rt. } \angle s = \frac{2 \times 6}{8} \times 90^\circ = 135^\circ.$$

(d) Each ∠ of a regular decagon

$$= \frac{2(10-2)}{10} \text{ rt. } \angle s = \frac{2 \times 8}{10} \times 90^\circ = 144^\circ.$$

(e) Each ∠ of a regular polygon of 32 sides

$$= \frac{2(32-2)}{32} \text{ rt. } \angle s = \frac{2 \times 30}{32} \times 90^\circ = 168\frac{3^\circ}{4} = 168^\circ\, 45'.$$

3. *How many sides has a regular polygon, each angle of which is* $1\frac{3}{4}$ *right angles?*

$$1\tfrac{3}{4} = \frac{2(n-2)}{n}. \qquad \S\ 145$$

$$7n = 8n - 16.$$

$$\therefore n = 16. \quad \textit{Ans.}$$

4. *How many sides has a regular polygon, each angle of which is* $1\frac{3}{7}$ *right angles?*

$$1\tfrac{3}{7} = \frac{2(n-2)}{n}. \qquad \S\ 145$$

$$10n = 14n - 28.$$

$$4n = 28.$$

$$\therefore n = 7. \quad \textit{Ans.}$$

5. *How many sides has a regular polygon, each angle of which is* 108°?

$$108 = \frac{2(n-2)}{n} \times 90. \qquad \S\ 145$$

$$108n = 180n - 360.$$

$$72n = 360.$$

$$\therefore n = 5. \quad \textit{Ans.}$$

6. *How many sides has a regular polygon, each angle of which is* 140°?

$$140 = \frac{2(n-2)}{n} \times 90. \qquad \S\ 145$$

$$7 = \frac{n-2}{n} \times 9.$$

$$7n = 9n - 18.$$

$$2n = 18.$$

$$\therefore n = 9. \quad \textit{Ans.}$$

7. *How many sides has a regular polygon, each angle of which is* 156°?

$$156 = \frac{2(n-2)}{n} \times 90. \qquad \S\ 145$$

$$13 = \frac{n-2}{n} \times 15.$$

$$13n = 15n - 30.$$

$$2n = 30.$$

$$\therefore n = 15. \quad \textit{Ans.}$$

8. *Four of the angles of a pentagon are* 120°, 80°, 90°, *and* 100° *respectively. Find the fifth angle.*

The sum of the ∠s of a pentagon

$= (5 - 2)\ 2$ rt. ∠s $= 3 \times 2$ rt. ∠s $= 6$ rt. ∠s $= 6 \times 90° = 540°$. § 143

∴ the fifth ∠ required

$= 540° - (120° + 80° + 90° + 100°) = 540° - 390° = 150°$.

9. *Five of the angles of a hexagon are* 100°, 120°, 130°, 150°, *and* 90° *respectively. Find the sixth angle.*

The sum of the ∠s of a hexagon

$= (6 - 2)\ 2$ rt. ∠s $= 4 \times 2$ rt. ∠s $= 8$ rt. ∠s $= 8 \times 90° = 720°$. § 143

∴ the sixth ∠ required

$= 720° - (100° + 120° + 130° + 150° + 90°) = 720° - 590° = 130°$.

10. *The angles of a quadrilateral are* x, $2x$, $2x$, *and* $3x$. *How many degrees are there in each?*

$$x + 2x + 2x + 3x = 4 \text{ rt. } \angle s. \qquad \S\ 144$$

$$\therefore 8x = 4 \times 90°.$$

$$\therefore x = 45°.$$

Then $2x = 90°$,

and $3x = 135°$. 45°, 90°, 90°, 135°. *Ans.*

11. *The angles of a quadrilateral are so related that the second is twice the first, the third three times the first, and the fourth four times the first. How many degrees in each?*

Let $x =$ the number of degrees in the first angle.

Then $2x$, $3x$, $4x =$ the number of degrees in the second, third, and fourth angles.

$$x + 2x + 3x + 4x = 4 \times 90. \qquad \S\ 144$$

$$10x = 360.$$

$$\therefore x = 36.$$

Then $2x = 72$,

$3x = 108$,

and $4x = 144$. 36°, 72°, 108°, 144°. *Ans.*

12. *The angles of a hexagon are* x, $2\frac{1}{2}x$, $3\frac{1}{2}x$, $2x$, $2x$, *and* x. *How many degrees are there in each?*

The sum of the $\measuredangle$s of a hexagon $= (6 - 2)\ 2$ rt. $\measuredangle$s $= 8$ rt. $\measuredangle$s. § 143

$$x + 2\tfrac{1}{2}x + 3\tfrac{1}{2}x + 2x + 2x + x = 8 \times 90°.$$

$$12x = 720°.$$

$$\therefore x = 60°.$$

Then $2\frac{1}{2}x = 150°$, $3\frac{1}{2}x = 210°$,

and $2x = 120°$.

60°, 150°, 210°, 120°, 120°, 60°. *Ans.*

13. *The sum of two angles of a triangle is* 100° *and their difference is* 40°. *How many degrees are there in each of the three angles of the triangle?*

Let $x =$ the number of degrees in the first angle,

and $y =$ the number of degrees in the second angle.

$$x + y = 100. \qquad (1)$$

$$x - y = 40. \qquad (2)$$

Add (1) and (2), $2x = 140$.

$$\therefore x = 70.$$

Subtract (2) from (1), $2y = 60$.

$$\therefore y = 30.$$

$$180 - (x + y) = 180 - (70 + 30) = 180 - 100 = 80.$$

70°, 30°, 80°. *Ans.*

Exercise 13. Page 72

1. *An exterior angle of a triangle is* 130° *and one of the opposite interior angles is* 52°. *Find the number of degrees in each angle of the triangle.*

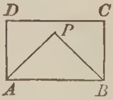
An exterior angle of a triangle is equal to the sum of the two opposite interior angles. § 111

$\therefore$ the second opposite interior $\angle = 130° - 52° = 78°$.

The third $\angle$ of the $\triangle = 180° - 130° = 50°$. 52°, 78°, 50°. *Ans.*

2. *Two consecutive angles of a rectangle are bisected by lines meeting at P. How many degrees in the angle P?*

Given the ▭ $ABCD$, with PA and PB bisecting two consecutive ∡ A and B.



Now ∡ BAD and CBA are rt. ∡. § 119

But $\angle BAP = \frac{1}{2}\angle BAD$,

and $\angle PBA = \frac{1}{2}\angle CBA$. Given

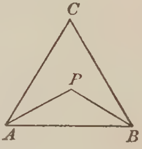
$\therefore \angle BAP = \angle PBA = 45°$.

$\therefore \angle P = 180° - (45° + 45°)$

$= 180° - 90° = 90°$.

3. *Two angles of an equilateral triangle are bisected by lines meeting at P. How many degrees in the angle P?*

Given the equilateral △ ABC, with AP and BP bisecting the ∡ A and B respectively.



$\angle A + \angle B + \angle C = 180°$. § 107

But $\angle A = \angle B = \angle C$. § 63

$\therefore \angle A = \angle B = \angle C = 60°$.

$\therefore \angle BAP = \angle PBA = 30°$. Ax. 4

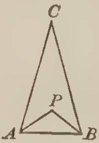
But $\angle BAP + \angle PBA + \angle P = 180°$. § 107

$\therefore 30° + 30° + \angle P = 180°$. Ax. 9

$\angle P = 120°$.

4. *The two base angles of an isosceles triangle are bisected by lines meeting at P. The vertical angle of the triangle is 30°. How many degrees in the angle P?*

Given the isosceles △ ABC, with $AC = BC$, $\angle C = 30°$, and PA and PB bisecting the ∡ A and B respectively.



Now $\angle A + \angle B + \angle C = 180°$. § 107

But $\angle A = \angle B$, § 74

and $\angle C = 30°$. Given

$\therefore \angle A + \angle A + 30° = 180°$. Ax. 9

$2\angle A = 150°$.

$\therefore \angle A = 75°$.

$\therefore \angle B = 75°$.

$\therefore \angle BAP = \angle PBA = 37\frac{1}{2}°$. Ax. 4

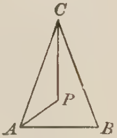
But $\angle BAP + \angle PBA + \angle P = 180°$. § 107

$\therefore 37\frac{1}{2}° + 37\frac{1}{2}° + \angle P = 180°$.

$\therefore \angle P = 105°$.

5. *The vertical angle of an isosceles triangle is 40°. This and one of the base angles are bisected by lines meeting at P. How many degrees in the angle P?*

Given the isosceles $\triangle ABC$, with $AC = BC$, $\angle C = 40^\circ$, and PA and PC bisecting the $\angle$s A and C respectively.



Now $\angle A + \angle B + \angle C = 180^\circ$. § 107

But $\angle A = \angle B$, § 74

and $\angle C = 40^\circ$. Given

$\therefore \angle A + \angle A + 40^\circ = 180^\circ$. Ax. 9

$2\angle A = 140^\circ$.

$\therefore \angle A = 70^\circ$.

$\therefore \angle PAC = 35^\circ$,

and $\angle ACP = 20^\circ$. Ax. 4

But $\angle PAC + \angle ACP + \angle P = 180^\circ$. § 107

$\therefore 35^\circ + 20^\circ + \angle P = 180^\circ$.

$\therefore \angle P = 125^\circ$.

6. *One exterior angle of a parallelogram is one eighth of the sum of the four exterior angles. How many degrees in each angle of the parallelogram?*

The sum of the exterior $\angle$s of the polygon $= 360^\circ$. § 146

$\therefore$ one exterior $\angle = \frac{1}{8}$ of $360^\circ = 45^\circ$.

$\therefore$ its interior $\angle = 180^\circ - 45^\circ = 135^\circ$.

Since any two consecutive $\angle$s of a parallelogram are supplementary, § 124

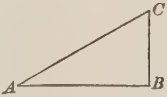
$\therefore$ the $\angle$s are 135°, 45°, 135°, and 45°.

7. *How many degrees in each exterior angle of a regular hexagon? of a regular octagon?*

The sum of the exterior $\angle$s of a polygon $= 360^\circ$. § 146

$\therefore$ each exterior $\angle$ of a regular hexagon $= \frac{1}{6}$ of $360^\circ = 60^\circ$; and each exterior $\angle$ of a regular octagon $= \frac{1}{8}$ of $360^\circ = 45^\circ$.

8. *In a right triangle one acute angle is twice the other. How many degrees in each exterior angle of the triangle?*



$\angle A + \angle C = 90^\circ$,

and $\angle C = 2\angle A$.

$\therefore \angle A + 2\angle A = 90^\circ$.

$3\angle A = 90^\circ$.

$\therefore \angle A = 30^\circ$.

$\therefore \angle C = 60^\circ$.

Since each exterior $\angle$ is the supplement of its interior $\angle$,

the exterior $\angle A = 180^\circ - 30^\circ = 150^\circ$,

the exterior $\angle B = 180^\circ - 90^\circ = 90^\circ$,

the exterior $\angle C = 180^\circ - 60^\circ = 120^\circ$.

9. *Make out a table showing the number of degrees in each interior angle and each exterior angle of regular polygons of three, four, five, · · ·, ten sides.*

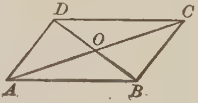
The sum of the exterior angles of a polygon is 360° and each interior angle is the supplement of its exterior angle.

Hence the following table:

NUMBER OF SIDES	EACH EXT. $\angle$	EACH INT. $\angle$
3	$360^\circ \div 3 = 120^\circ$	$180^\circ - 120^\circ = 60^\circ$
4	$360^\circ \div 4 = 90^\circ$	$180^\circ - 90^\circ = 90^\circ$
5	$360^\circ \div 5 = 72^\circ$	$180^\circ - 72^\circ = 108^\circ$
6	$360^\circ \div 6 = 60^\circ$	$180^\circ - 60^\circ = 120^\circ$
7	$360^\circ \div 7 = 51\frac{3}{7}^\circ$	$180^\circ - 51\frac{3}{7}^\circ = 128\frac{4}{7}^\circ$
8	$360^\circ \div 8 = 45^\circ$	$180^\circ - 45^\circ = 135^\circ$
9	$360^\circ \div 9 = 40^\circ$	$180^\circ - 40^\circ = 140^\circ$
10	$360^\circ \div 10 = 36^\circ$	$180^\circ - 36^\circ = 144^\circ$

10. *If the diagonals of a quadrilateral bisect each other, the figure is a parallelogram.*

Given the quadrilateral $ABCD$, with the diagonals AC and BD bisecting each other at O.



To prove that $ABCD$ is a ▱.

Proof. In the $\triangle s$ ABO and CDO,

$AO = CO$, Given

$\angle AOB = \angle COD$, § 60

and $BO = DO$. Given

$\therefore \triangle ABO$ is congruent to $\triangle CDO$. § 68

$\therefore AB = CD$. § 67

Also $\angle BAO = \angle DCO$. § 67

$\therefore AB$ is $\parallel$ to DC. § 101

$\therefore ABCD$ is a ▱, by § 130. Q.E.D.

11. *In this parallelogram $ABCD$, $AP = CR$, and $BQ = DS$. Prove that $PQRS$ is also a parallelogram.*

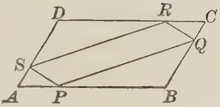
Given the ▱ $ABCD$, with the quadrilateral $PQRS$ inclosed, and AP equal to CR and BQ equal to DS.

To prove that $PQRS$ is a ▱.

Proof. Since $BC = AD$, § 125

and $BQ = DS$. Given

$\therefore QC = AS$. Ax. 2



In the $\triangle$s APS and CRQ, $QC = AS$,

$AP = CR$, Given

and $\angle A = \angle C$. § 124

$\therefore \triangle APS$ is congruent to $\triangle CRQ$. § 68

$\therefore PS = RQ$. § 67

Again, $AB = CD$, § 125

and $AP = CR$. Given

$\therefore PB = RD$. Ax. 2

In the $\triangle$s PBQ and RDS, $PB = RD$,

$\angle B = \angle D$, § 124

and $BQ = DS$. Given

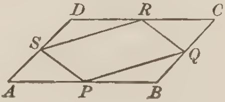
$\therefore \triangle PBQ$ is congruent to $\triangle RDS$. § 68

$\therefore PQ = RS$.

$\therefore PQRS$ is a ▱, by § 129. Q.E.D.

12. *If the mid-points of the sides of a parallelogram are connected in order, the resulting figure is also a parallelogram.*

Given the ▱ $ABCD$, with P, Q, R, S the mid-points of the sides AB, BC, CD, DA respectively.



To prove that $PQRS$ is a ▱.

Proof. Since $AB = CD$ and $BC = DA$, § 125

$\therefore AP = CR$ and $QC = SA$. Ax. 4

In the $\triangle$s APS and CRQ, $AP = CR$,

$\angle A = \angle C$, § 124

and $QC = SA$.

$\therefore \triangle APS$ is congruent to $\triangle CRQ$. § 68

$\therefore SP = QR$. § 67

Similarly $\triangle PBQ$ is congruent to $\triangle RDS$,

and $PQ = RS$.

$\therefore PQRS$ is a ▱, by § 129. Q.E.D.

Exercise 14. Page 73

1. *State without proof the locus in a plane of a point 2 in. from a fixed point O.*

The required locus is the circle described with O as a center and a radius equal to 2 in.

2. *State without proof the locus in a plane of the tip of the minute hand of a watch.*

The required locus is the circle described with the pivot of the two hands as a center and a radius equal to the length of the minute hand.

3. *State without proof the locus in a plane of the center of the hub of a carriage wheel moving straight ahead on a level road.*

The required locus is the straight line parallel to the ground at a height above the ground equal to the radius of the wheel.

4. *State without proof the locus in a plane of a point* 1 *in. from each of two parallel lines that are* 2 *in. apart.*

The required locus is the line parallel to the two given parallel lines and at a distance of 1 in. from each of them.

5. *State without proof the locus in a plane of a point on this page and* 1 *in. from the edge.*

The required locus is the straight line drawn on the page, drawn parallel to the edge at a distance of 1 in. from the edge.

6. *State without proof the locus in a plane of the point of a round lead pencil as it rolls along a desk.*

The required locus is the straight line parallel to the top of the desk at a distance above the desk equal to the radius of the pencil.

7. *State without proof the locus in a plane of the tips of a pair of shears as they open, provided the fulcrum* (*bolt or screw*) *remains always fixed in one position.*

The required locus is the circle described with the fulcrum as a center and with a radius equal to the distance from the fulcrum to the tips of the shears.

8. *State without proof the locus in a plane of the center of a circle that rolls around another circle, always just touching it.*

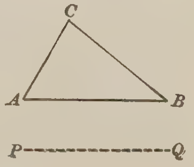
The required locus is the circle described with the center of the given fixed circle as a center and with a radius equal to the sum of the radii of the two given circles.

Exercise 15. Page 74

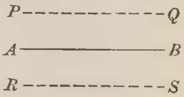
1. *Without giving the proof, draw the locus of a point* $\frac{1}{4}$ *in. below the base of a given triangle* ABC.

Given the $\triangle ABC$.

The line $PQ \parallel$ to AB at a distance $\frac{1}{4}$ in. below AB is the required locus.



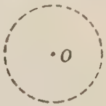
2. *Without giving the proof, draw the locus of a point $\frac{1}{2}$ in. from a given line AB.*



Given the straight line AB.

The required locus is the pair of straight lines PQ and RS, each parallel to AB and at a distance of $\frac{1}{2}$ in. from AB.

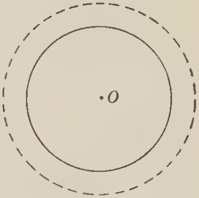
3. *Without giving the proof, draw the locus of a point 1 in. from a given point O.*



Given the point O.

The required locus is the circle described with O as a center and a radius equal to 1 in.

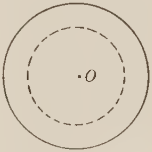
4. *Without giving the proof, draw the locus of a point $\frac{1}{2}$ in. outside the circle described about a given point O with a radius $1\frac{1}{2}$ in.*



Given the circle described about the point O with a radius $1\frac{1}{2}$ in.

The required locus is the circle described about O as a center and with a radius equal to $\frac{1}{2}$ in. + $1\frac{1}{2}$ in., or 2 in.

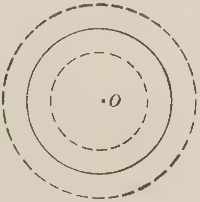
5. *Without giving the proof, draw the locus of a point $\frac{1}{2}$ in. within the circle described about a given point O with a radius $1\frac{1}{2}$ in.*



Given the circle described about the point O with a radius $1\frac{1}{2}$ in.

The required locus is the circle described about O as a center and with a radius equal to $1\frac{1}{2}$ in. — $\frac{1}{2}$ in., or 1 in.

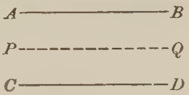
6. *Without giving the proof, draw the locus of a point $\frac{1}{2}$ in. from the circle described about a given point O with a radius $1\frac{1}{2}$ in.*



Given the circle described about the point O with a radius $1\frac{1}{2}$ in.

The required locus is the pair of circles described about O as a center, one with a radius equal to $1\frac{1}{2}$ in. + $\frac{1}{2}$ in., or 2 in., and the other with a radius equal to $1\frac{1}{2}$ in. — $\frac{1}{2}$ in., or 1 in.

7. *Without giving the proof, draw the locus of a point $\frac{1}{2}$ in. from each of two given parallel lines that are* 1 *in. apart.*



Given the two parallel lines AB and CD, 1 in. apart.

The required locus is the straight line PQ, parallel to each of the lines AB and CD, and at a distance of $\frac{1}{2}$ in. from each of them.

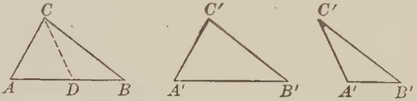
Exercise 16. Page 77

1. *If two triangles have two sides of the one equal respectively to two sides of the other, and the angles opposite two equal sides equal, the angles opposite the other two equal sides are equal or supplementary, and if equal the triangles are congruent.*

Given the $\triangle$s ABC and $A'B'C'$, with $AC = A'C'$, $BC = B'C'$, and $\angle B = \angle B'$.

To prove that $\angle$s A and $C'A'B'$ are equal or supplementary.

Proof. Place $\triangle A'B'C'$ on $\triangle ABC$ so that $B'C'$ shall coincide with BC, and $\angle$s A' and A shall be on the same side of BC.



Since $\angle B' = \angle B$, $B'A'$ will fall along BA, and A' must fall either at A or at some other point in BA, as D. § 85

If A' falls at A,

then $\triangle ABC$ is congruent to $\triangle A'B'C'$, § 66

and $\angle A = \angle B'A'C'$, by § 67. Q.E.D.

If A' falls at D,

then $\triangle DBC$ is congruent to $\triangle A'B'C'$. § 66

$\therefore \angle BDC = \angle B'A'C'$,

and $DC = A'C'$. § 67

Since $AC = A'C'$, Given

$\therefore DC = AC$. Ax. 8

$\therefore \angle A = \angle CDA$. § 74

But $\angle$s CDA and BDC are supplementary. § 43

$\therefore \angle$s A and BDC are supplementary. § 58

Or $\angle$s A and $B'A'C'$ are supplementary, by Ax. 9. Q.E.D.

Discussion. The triangles are congruent:

1. If the given angles B and B' are both right or both obtuse angles.

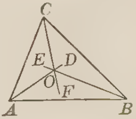
2. If the required angles A and A' are both acute, both right, or both obtuse.

3. If AC and $A'C'$ are not less than BC and $B'C'$ respectively. In case they are less, one of the above given conditions may exist, and then the triangles are congruent.

2. *The bisectors of the angles of a triangle are concurrent in a point equidistant from the sides of the triangle.*

Given AD bisecting $\angle A$, BE bisecting $\angle B$, and CF bisecting $\angle C$ of the $\triangle ABC$.

To prove that AD, BE, and CF are concurrent in a point equidistant from AB, BC, and AC.



Proof. Let AD and BE intersect at O.

Since O lies in AD,

$\therefore$ O is equidistant from AC and AB. § 152

Since O lies in BE,

$\therefore$ O is equidistant from AB and BC. § 152

$\therefore$ O is equidistant from AC and BC.

But CF is the locus of a point equidistant from AC and BC. § 152

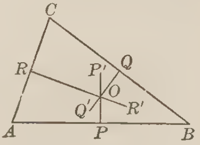
$\therefore$ O lies in the bisector CF. § 152

$\therefore$ AD, BE, and CF are concurrent in the point O equidistant from AB, BC, and AC. Q.E.D.

3. *The perpendicular bisectors of the sides of a triangle are concurrent in a point equidistant from the vertices.*

In the $\triangle ABC$ given PP' the $\perp$ bisector of AB, QQ' the $\perp$ bisector of BC, and RR' the $\perp$ bisector of CA.

To prove that PP', QQ', and RR' are concurrent in a point equidistant from A, B, and C.



Proof. Let O be the intersection of QQ' and RR'.

Since O lies in QQ',

O is equidistant from B and C. § 150

Since O lies in RR',

O is equidistant from C and A. § 150

$\therefore$ O is equidistant from A and B.

But PP' is the locus of a point equidistant from A and B. § 150

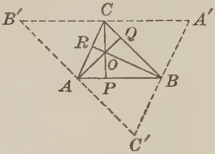
$\therefore$ O lies in the line PP'.

$\therefore$ PP', QQ', and RR' are concurrent in a point O equidistant from A, B, and C. Q.E.D.

4. *The perpendiculars from the vertices of a triangle to the opposite sides are concurrent.*

Given the $\triangle ABC$, with AQ, BR, and CP drawn $\perp$ to BC, CA, and AB respectively.

To prove that AQ, BR, and CP are concurrent.



Proof. Through A, B, C suppose $B'C'$, $C'A'$, $A'B'$ drawn $\parallel$ to CB, AC, BA respectively.

$\therefore$ $C'A = BC$, and $AB' = BC$. § 127

That is, $C'A = AB'$. Ax. 8

Then AQ is $\perp$ to $B'C'$. § 97

$\therefore$ AQ is the $\perp$ bisector of $B'C'$.

Similarly BR is the $\perp$ bisector of $C'A'$,

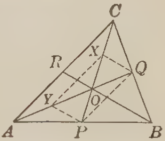
and CP is the $\perp$ bisector of $A'B'$.

$\therefore$ AQ, BR, and CP are concurrent, by Ex. 3. Q.E.D.

5. *The medians of a triangle are concurrent in a point two thirds of the distance from each vertex to the middle of the opposite side.*

Given the $\triangle ABC$, with the medians AQ, BR, and CP.

To prove that AQ, BR, and CP are concurrent in a point two thirds of the distance from each vertex to the middle of the opposite side.



Proof. Let AQ and CP meet at O.

Suppose Y is the mid-point of AO and X the mid-point of CO.

Draw XY, YP, PQ, and QX.

In the $\triangle AOC$, XY is $\parallel$ to CA and equal to $\frac{1}{2}CA$. § 136

In the $\triangle ABC$, PQ is $\parallel$ to CA and equal to $\frac{1}{2}CA$. § 136

$\therefore$ PQ is $\parallel$ to XY, § 96

and $PQ = XY$. Ax. 8

Hence $YPQX$ is a ▱. § 130

$\therefore$ $YO = OQ$, and $XO = OP$. § 131

$\therefore$ $AY = YO = OQ$, and $CX = XO = OP$. Ax. 8

Hence any median cuts off on any other median two thirds of the distance from the vertex to the middle of the opposite side.

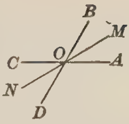
Therefore the median BR cuts off AO, two thirds of AQ.

That is, BR passes through O. Q.E.D.

6. *The bisectors of two vertical angles are in the same straight line.*

Given the vertical $\measuredangle$ AOB and COD, with OM bisecting $\angle AOB$ and ON bisecting $\angle COD$.

To prove that OM and ON are in the same straight line.



Proof. $\angle AOB = \angle COD$. § 60

$\therefore \angle AOM = \angle CON$. Ax. 4

$\therefore \angle MOB + \angle CON = \angle AOB$. Ax. 9

Now $\angle AOB + \angle BOC = 2$ rt. $\angle$s. § 43

$\therefore \angle MOB + \angle BOC + \angle CON = 2$ rt. $\angle$s. Ax. 9

$\therefore$ *OM* and *ON* are in the same straight line, by § 43. Q. E. D.

7. *The bisector of one of two vertical angles bisects the other.*

Given the vertical $\angle$s *AOB* and *COD*, with *MN* bisecting the $\angle AOB$. (See Fig. of Ex. 6.)

To prove that *MN* bisects the $\angle COD$.

Proof. $\angle CON = \angle AOM$,

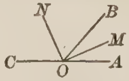
and $\angle NOD = \angle MOB$. § 60

But $\angle AOM = \angle MOB$. Given

$\therefore \angle CON = \angle NOD$, by Ax. 8. Q. E. D.

8. *The bisectors of two supplementary adjacent angles are perpendicular to each other.*

Given the supplementary adjacent $\angle$s *AOB* and *BOC*, with *OM* and *ON* their respective bisectors.



To prove that *OM* is $\perp$ to *ON*.

Proof. $\angle AOB + \angle BOC = 180°$. Given

But $\angle AOB = 2\angle MOB$, and $\angle BOC = 2\angle BON$.

$\therefore 2\angle MOB + 2\angle BON = 180°$. Ax. 9

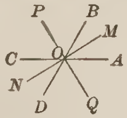
$\therefore \angle MOB + \angle BON = 90°$. Ax. 4

That is, $\angle MON = 90°$.

$\therefore$ *OM* is $\perp$ to *ON*, by § 27. Q. E. D.

9. *The bisectors of the two pairs of vertical angles formed by two intersecting lines are perpendicular to each other.*

Given the line *MN* bisecting the vertical $\angle$s *AOB* and *COD*, and the line *PQ* bisecting the vertical $\angle$s *BOC* and *DOA*.



To prove that *MN* is $\perp$ to *PQ*.

Proof. Now $\angle DOA = \angle BOC$. § 60

$\therefore \angle QOA = \angle BOP$. Ax. 4

$\angle AOM = \angle MOB$. Given

$\therefore \angle QOA + \angle AOM = \angle MOB + \angle BOP$. Ax. 1

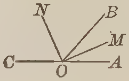
That is, $\angle QOM = \angle MOP$.

$\therefore$ $\angle$s *QOM* and *MOP* are rt. $\angle$s. § 26

$\therefore$ *MN* is $\perp$ to *PQ*, by § 27. Q. E. D.

10. *If the bisectors of two adjacent angles are perpendicular to each other, the adjacent angles are supplementary.*

Given the adjacent $\angle$s AOB and BOC, with OM and ON their bisectors respectively, and $OM \perp$ to ON.



To prove that $\angle AOB + \angle BOC = 180°$.

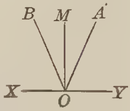
Proof. $\angle MOB + \angle BON = 90°$. Given

$\therefore 2\angle MOB + 2\angle BON = 180°$. Ax. 3

That is, $\angle AOB + \angle BOC = 180°$, by Ax. 9. Q.E.D.

11. *If an angle is bisected, and if a line is drawn through the vertex perpendicular to the bisector, this line forms equal angles with the sides of the given angle.*

Given the $\angle AOB$, with OM its bisector and XY drawn through $O \perp$ to OM.



To prove that $\angle YOA = \angle BOX$.

Proof. Since XY is $\perp$ to OM, Given

$\therefore$ $\angle$s YOM and MOX are rt. $\angle$s. § 27

$\therefore \angle YOM = \angle MOX$. § 56

But $\angle AOM = \angle MOB$. Given

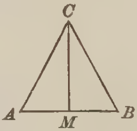
$\therefore \angle YOM - \angle AOM = \angle MOX - \angle MOB$. Ax. 2

That is, $\angle YOA = \angle BOX$. Q.E.D.

12. *The bisector of the vertical angle of an isosceles triangle bisects the base and is perpendicular to the base.*

Given the isosceles $\triangle ABC$, with $AC = BC$, and CM bisecting the $\angle C$.

To prove that $AM = MB$, and that CM is $\perp$ to AB.



Proof. In the $\triangle$s AMC and BMC,

$AC = BC$, Given

$\angle ACM = \angle MCB$, Given

and $CM = CM$. Iden.

$\therefore$ the $\triangle$s AMC and BMC are congruent. § 68

$\therefore AM = MB$, § 67

and $\angle CMA = \angle BMC$. § 67

$\therefore$ $\angle$s CMA and BMC are rt. $\angle$s. § 26

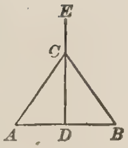
$\therefore$ CM is $\perp$ to AB, by § 27. Q.E.D.

13. *The perpendicular bisector of the base of an isosceles triangle passes through the vertex and bisects the angle at the vertex.*

Given the isosceles $\triangle ABC$, with $AC = BC$, and DE the $\perp$ bisector of AB.

To prove that DE passes through C and bisects $\angle ACB$.

Proof. $AC = BC$. Given
$\therefore$ DE passes through C. § 150
In the rt. $\triangle$s ADC and BDC,
$AC = BC$, Given
and $CD = CD$. Iden.
$\therefore$ the rt. $\triangle$s ADC and BDC are congruent. § 89
$\therefore$ $\angle ACD = \angle DCB$, by § 67. Q.E.D.



14. *If the perpendicular bisector of the base of a triangle passes through the vertex, the triangle is isosceles.*

Given the $\triangle ABC$, with DE, the $\perp$ bisector of AB, passing through the vertex C. (See Fig. in Ex. 13.)

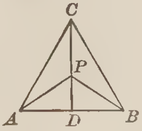
To prove that $\triangle ABC$ is isosceles.

Proof. Since C lies in the $\perp$ bisector DE, Given
$\therefore$ $AC = BC$. § 150
$\therefore$ $\triangle ABC$ is isosceles, by § 62. Q.E.D.

15. *Any point in the bisector of the vertical angle of an isosceles triangle is equidistant from the extremities of the base.*

Given the isosceles $\triangle ABC$, with $AC = BC$, and P any point in the bisector CD of the vertical $\angle C$.

To prove that $PA = PB$.



Proof. In the $\triangle$s ADC and BDC,
$AC = BC$, Given
$\angle ACD = \angle DCB$, Given
and $CD = CD$. Iden.
$\therefore$ $\triangle$s ADC and BDC are congruent. § 68
$\therefore$ $AD = DB$, § 67
and $\angle CDA = \angle BDC$.
$\therefore$ $\angle$s CDA and BDC are rt. $\angle$s. § 26
That is, CD is the $\perp$ bisector of the base AB.
$\therefore$ $PA = PB$, by § 150. Q.E.D.

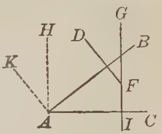
16. *If two isosceles triangles are on the same base, a line passing through their vertices is perpendicular to the base and bisects the base.*

Given the isosceles $\triangle$s ABC and ABP on the same base AB, and the line CD drawn through the vertices C and P. (See Fig. in Ex. 15.)

To prove that CD is $\perp$ to AB, and that $AD = DB$.

Proof. Since C is equidistant from A and B,
and P is equidistant from A and B. Given
$\therefore$ CD is the $\perp$ bisector of AB. § 151
That is, CD is $\perp$ to AB, and $AD = DB$. Q.E.D.

17. *Two angles whose sides are perpendicular each to each are either equal or supplementary.*



Given the $\measuredangle s$ CAB and GFD, with $DF \perp$ to AB and $GFI \perp$ to AC.

To prove that $\angle CAB$ is equal to $\angle GFD$ and is supplementary to $\angle DFI$.

Proof. Draw $AK \perp$ to AB,

and $AH \perp$ to AC.

Then AK is $\parallel$ to FD,

and AH is $\parallel$ to IG. § 95

$\therefore \angle GFD = \angle HAK$. § 123

Now $\angle BAK$ is a rt. $\angle$. Const.

$\therefore \angle BAH$ is the complement of $\angle HAK$.

Again, $\angle CAH$ is a rt. $\angle$. Const.

$\therefore \angle BAH$ is the complement of $\angle CAB$.

$\therefore \angle CAB = \angle HAK$. § 58

$\therefore \angle GFD = \angle CAB$. Ax. 8

Now $\angle DFI$ is the supplement of $\angle GFD$.

$\therefore \angle DFI$ is the supplement of $\angle CAB$, by § 58. Q. E. D.

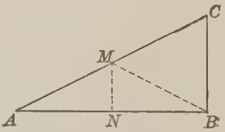
Discussion. The angles are equal if their sides extend in the same direction from their vertices, and are supplementary if their sides extend in opposite directions from their vertices.

Exercise 17. Page 80

1. *The mid-point of the hypotenuse of a right triangle is equidistant from the three vertices.*

Given M the mid-point of AC, the hypotenuse of the rt. $\triangle$ ABC.

To prove that M is equidistant from A, B, and C.



Proof. Draw $MN \perp$ to AB and draw MB.

Since MN is $\parallel$ to CB, § 95

and $AM = CM$, Given

$\therefore MN$ bisects AB. § 135

In the rt. $\triangle s$ ANM and BNM,

$AN = BN$,

and $MN = MN$. Iden.

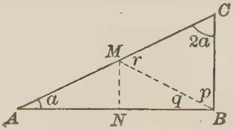
$\therefore$ the rt. $\triangle s$ ANM and BNM are congruent. § 69

$\therefore AM = BM$. § 67

But $AM = CM$. Given

$\therefore AM = BM = CM$, by Ax. 8. Q. E. D.

2. *If one acute angle of a right triangle is double the other, the hypotenuse is double the shorter side.*



In the rt. $\triangle ABC$, given $\angle A = a$, and $\angle C = 2a$.

To prove that $AC = 2\,BC$.

Proof. Draw $MN \parallel$ to CB, bisecting AC.

Draw MB. Then MN bisects AB. § 135

In the rt. $\triangle$s ANM and BNM, $AN = BN$,

and $MN = MN$. Iden.

$\therefore$ the rt. $\triangle$s ANM and BNM are congruent. § 69

$\therefore AM = BM$. § 67

Now $\angle r = \angle a + \angle q$. § 111

But $\angle a = \angle q$. § 74

$\therefore \angle r = \angle a + \angle a$. Ax. 9

$\therefore \angle r = \angle 2a$.

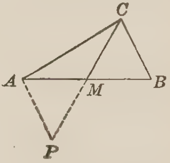
$\therefore BM = BC$. § 76

$AM = BC$. Ax. 8

$\therefore AM + BM = 2\,BC$.

That is, $AC = 2\,BC$. Q.E.D.

3. *A median of a triangle is less than half the sum of the two adjacent sides.*



Given CM a median of the $\triangle ABC$.

To prove that $CM < \frac{1}{2}(BC + CA)$.

Proof. Produce CM to P, making MP equal to CM, and draw AP.

In the $\triangle$s MBC and MAP, $MB = MA$, Given

$CM = MP$, Hyp.

and $\angle BMC = \angle AMP$. § 60

$\therefore$ the $\triangle$s MBC and MAP are congruent. § 68

$\therefore BC = AP$. § 67

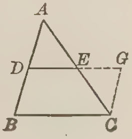
Now $CP < AP + CA$. Post. 3

$\therefore CP < BC + CA$. Ax. 9

$\therefore 2\,CM < BC + CA$. Ax. 9

$\therefore CM < \frac{1}{2}(BC + CA)$, by Ax. 4. Q.E.D.

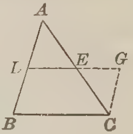
4. *The line which bisects two sides of a triangle is parallel to the third side.*



Given the line DE drawn through D and E, the mid-points of AB and AC, sides of the $\triangle ABC$.

To prove that DE is $\parallel$ to BC.

Proof. Suppose a line drawn from $C \parallel$ to BA, and suppose DE produced to meet it at G.



In the $\triangle$ CGE and ADE,

$EC = AE$, Given

$\angle CEG = \angle AED$, § 60

and $\angle GCE = \angle A$. § 100

$\therefore$ the $\triangle$ CGE and ADE are congruent. § 72

$\therefore CG = AD$. § 67

But $BD = AD$. Given

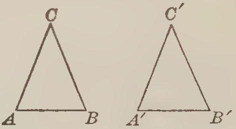
$\therefore CG = BD$. Ax. 8

$\therefore BCGD$ is a ▱. § 130

$\therefore DE$ is $\parallel$ to BC, by § 118. Q. E. D.

5. *Two isosceles triangles are congruent if a side and an angle of the one are equal respectively to the corresponding side and angle of the other.*

Given the isosceles $\triangle$ ABC and $A'B'C'$, with a side and an angle of the one equal respectively to the corresponding side and angle of the other.



To prove that the $\triangle$ ABC and $A'B'C'$ are congruent.

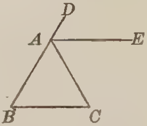
Proof. Since an angle of the one is equal to the corresponding angle of the other, and in each triangle two $\angle$ are equal, the three $\angle$ of one triangle are equal respectively to the three $\angle$ of the other.

$\therefore$ the $\triangle$ ABC and $A'B'C'$ are congruent, by § 72. Q. E. D.

6. *The bisector of an exterior angle of an isosceles triangle, formed by producing one of the equal sides through the vertex, is parallel to the base.*

Given the isosceles $\triangle ABC$, with AB equal to AC and AE bisecting the exterior $\angle CAD$, formed by producing BA through A.

To prove that AE is $\parallel$ to BC.



Proof. $\angle CAD = \angle CBA + \angle ACB$. § 111

But $\angle CAD = 2\angle EAD$, Given

and $\angle CBA = \angle ACB$. § 74

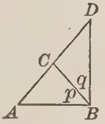
$\therefore 2\angle EAD = 2\angle CBA$. Ax. 9

$\therefore \angle EAD = \angle CBA$. Ax. 4

$\therefore AE$ is $\parallel$ to BC, by § 103. Q. E. D.

7. *If one of the equal sides of an isosceles triangle is produced through the vertex by its own length, the line joining the end of the side produced to the nearer end of the base is perpendicular to the base.*

Given the isosceles $\triangle ABC$, with AC equal to BC, AC produced through C by its own length to D, and DB drawn.



To prove that DB is $\perp$ to AB.

Proof. Now $AC = CD$, Given

and $AC = BC$. Given

$\therefore CD = BC$. Ax. 8

$\therefore \angle DBC = \angle CDB$. § 74

Also $\angle CBA = \angle BAC$. § 74

$\therefore \angle DBC + \angle CBA = \angle CDB + \angle BAC$. Ax. 1

That is, $\angle DBA = \angle CDB + \angle BAC$.

But $\angle DBA + \angle CDB + \angle BAC = 2$ rt. $\angle$s. § 107

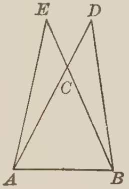
$\angle DBA + \angle DBA = 2$ rt. $\angle$s. Ax. 9

$\therefore \angle DBA = 1$ rt. $\angle$. Ax. 4

$\therefore DB$ is $\perp$ to AB, by § 27. Q.E.D.

8. *If the equal sides of an isosceles triangle are produced through the vertex so that the external segments are equal, the extremities of these segments are equidistant from the extremities of the base respectively.*

Given the isosceles $\triangle ABC$, with AC equal to BC. Given AC produced to D and BC produced to E so that $CD = CE$.



To prove that $AE = BD$.

Proof. In the $\triangle$s ACE and BCD,

$AC = BC$, Given

$\angle ECA = \angle BCD$, § 60

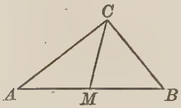
and $CE = CD$. Given

$\therefore$ the $\triangle$s ACE and BCD are congruent. § 68

$\therefore AE = BD$, by § 67. Q.E.D.

9. *If the line drawn from the vertex of a triangle to the mid-point of the base is equal to half the base, the angle at the vertex is a right angle.*

Given the $\triangle ABC$, with CM drawn to M, the mid-point of the base AB, and equal to $\frac{1}{2} AB$.



To prove that $\angle ACB = 1$ rt. $\angle$.

Proof. Since $CM = AM$, Given

$\angle A = \angle ACM$. § 74

Since $CM = BM$, Given

$\angle B = \angle MCB$. § 74

$\therefore \angle A + \angle B = \angle ACM + \angle MCB$. Ax. 1

That is, $\angle A + \angle B = \angle ACB$.

$\angle A + \angle B + \angle ACB = 2$ rt. $\angle$s. § 107

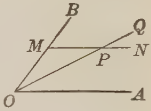
$\therefore \angle ACB + \angle ACB = 2$ rt. $\angle$s. Ax. 9

$\therefore \angle ACB = 1$ rt. $\angle$. Q.E.D.

10. *If through any point in the bisector of an angle a line is drawn parallel to either side of the angle, the triangle thus formed is isosceles.*

Given the point P in the bisector OQ of the $\angle AOB$, and MN a line drawn through $P \parallel$ to OA.

To prove that $\triangle OPM$ is isosceles.



Proof. $\angle AOP = \angle POB$. Given

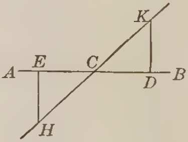
$\angle AOP = \angle MPO$. § 100

$\therefore \angle POB = \angle MPO$. Ax. 8

$\therefore MP = MO$. § 76

That is, the $\triangle OPM$ is isosceles. Q.E.D.

11. *Through any point C in the line AB an intersecting line is drawn, and from any two points in this line equidistant from C perpendiculars are drawn to AB or AB produced. Prove that these perpendiculars are equal.*



Given the line HK drawn through the point C in the line AB, and HE and KD ⊥s drawn to AB from two points in HK equidistant from C.

To prove that $HE = KD$.

Proof. In the rt. $\triangle$s HEC and KDC,

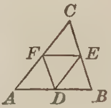
$CH = CK$, Given

and $\angle ECH = \angle DCK$. § 60

$\therefore$ the rt. $\triangle$s HEC and KDC are congruent. § 91

$\therefore HE = KD$, by § 67. Q.E.D.

12. *The lines joining the mid-points of the sides of a triangle divide the triangle into four congruent triangles.*



Given the lines DE, EF, and FD joining the mid-points of the $\triangle ABC$.

To prove that the $\triangle$s ADF, FED, DBE, and FEC are congruent.

Proof. In the $\triangle$s ADF and FED,

$DF = DF$, Iden.

$AD = FE$,

and $AF = DE$. § 136

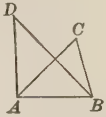
$\therefore$ the $\triangle$s ADF and FED are congruent. § 80

Similarly $\triangle FED$ is congruent to $\triangle$s DBE and FEC.

$\therefore$ $\triangle$s ADF, FED, DBE, and FEC are congruent. Q.E.D.

Exercise 18. Page 83

1. *Given ABC and ABD, two triangles on the same base AB, and on the same side of it, the vertex of each triangle being outside the other triangle. Prove that if AC equals AD, then BC cannot equal BD.*



Given the $\triangle ABC$ and ABD, the vertex of each $\triangle$ being outside the other $\triangle$, and given $AC = AD$.

To prove that BC cannot equal BD.

Proof. Assume that $BC = BD$.

Then in the $\triangle ABC$ and ABD,

$AB = AB$, Iden.

$AC = AD$, Given

and $BC = BD$. Hyp.

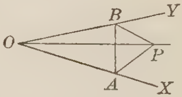
$\therefore$ the $\triangle ABC$ and ABD are congruent. § 80

That is, AC and AD coincide, and C and D coincide.

But this is contrary to the given fact that the vertex of each $\triangle$ lies outside the other $\triangle$.

$\therefore$ BC cannot equal BD. Q.E.D.

2. *On the sides of the angle XOY two equal segments, OA and OB, are taken. On AB a triangle APB is constructed, with AP greater than BP. Prove that OP cannot bisect the angle XOY.*



Given the $\angle XOY$, with $OA = OB$, and $AP > BP$.

To prove that $\angle XOP$ cannot equal $\angle POY$.

Proof. Suppose that OP is the bisector of the $\angle XOY$.

Then $\angle XOP = \angle POY$. § 24

In the $\triangle OAP$ and OBP, $OA = OB$, Given

$OP = OP$, Iden.

and $\angle XOP = \angle POY$. Hyp.

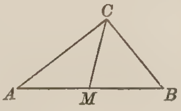
$\therefore$ the $\triangle OAP$ and OBP are congruent. § 68

$\therefore$ $AP = BP$. § 67

But this is contrary to the given fact that $AP > BP$.

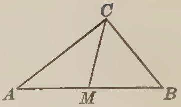
$\therefore$ $\angle XOP$ cannot equal $\angle POY$. Q.E.D.

3. *From M, the mid-point of the line AB, MC is drawn oblique to AB. Prove that CA cannot equal CB.*



Given M, the mid-point of the line AB, and MC drawn oblique to AB.

To prove that CA cannot equal CB.



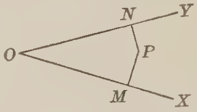
Proof. Suppose that $CA = CB$.
$MA = MB$. Given
$\therefore MC$ is $\perp$ to AB. § 151

But this is contrary to the given fact that MC is oblique to AB.

$\therefore CA$ cannot equal CB. Q.E.D.

4. *If perpendiculars are drawn to the sides of an acute angle from a point within the angle, they cannot inclose a right angle or an acute angle.*

Given the acute $\angle XOY$, and PM and PN $\perp$s drawn from P, a point within the $\angle XOY$, to the sides OX and OY respectively.



To prove that $\angle NPM$ cannot be right or acute.

Proof. Suppose that $\angle NPM = 1$ rt. $\angle$.
Then $\angle O + \angle M + \angle P + \angle N = 4$ rt. $\angle$s, § 144
But $\angle M = \angle N = 1$ rt. $\angle$, Given
and $\angle P = 1$ rt. $\angle$. Hyp.
$\therefore \angle O = 1$ rt. $\angle$.

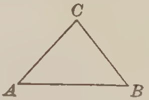
Again, suppose that $\angle NPM$ is an acute $\angle$.
Then $\angle O + \angle M + \angle P + \angle N = 4$ rt. $\angle$s. § 144
But $\angle M = \angle N = 1$ rt. $\angle$, Given
and $\angle P < 1$ rt. $\angle$. Hyp.
$\therefore \angle O$ is an obtuse angle.

But both these conclusions are contrary to the given fact that $\angle XOY$ is an acute $\angle$.

$\therefore \angle NPM$ cannot be right or acute. Q.E.D.

5. *One of the equal angles of an isosceles triangle is five ninths of a right angle. Prove that the angle at the vertex cannot be a right angle.*

Given the isosceles $\triangle ABC$, with $AC = BC$, and $\angle A = \frac{5}{9}$ rt. $\angle$.



To prove that $\angle C$ cannot be a rt. $\angle$.

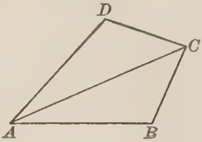
Proof. Assume $\angle C = 1$ rt. $\angle$.
Now $\angle B = \angle A$. § 74
$\therefore \angle B = \frac{5}{9}$ rt. $\angle$. Ax. 8
Now $\angle A + \angle B + \angle C = 2$ rt. $\angle$s. § 107
Then $\frac{5}{9}$ rt. $\angle + \frac{5}{9}$ rt. $\angle + 1$ rt. $\angle = 2$ rt. $\angle$s, Ax. 9
or $2\frac{1}{9}$ rt. $\angle$s $= 2$ rt. $\angle$s.

But this conclusion is contrary to the fact that the sum of the angles of a triangle is equal to 2 rt. $\angle$s. § 107

$\therefore \angle C$ cannot be a rt. $\angle$. Q.E.D.

Exercise 19. Page 85

1. *If the sides AB and AD of a quadrilateral ABCD are equal, and if the diagonal AC bisects the angle at A, then BC is equal to DC.*



Given the quadrilateral $ABCD$, with $AB = AD$, and AC bisecting the $\angle BAD$.

To prove that $BC = DC$.

Proof. In the $\triangle ABC$ and ADC,

$AB = AD$, Given

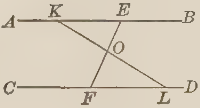
$AC = AC$, Iden.

and $\angle BAC = \angle CAD$. Given

$\therefore$ $\triangle ABC$ and ADC are congruent. § 68

$\therefore$ $BC = DC$, by § 67. Q.E.D.

2. *A line is drawn terminated by two parallel lines. Through its mid-point any line is drawn terminated by the parallels. Prove that the second line is bisected by the first.*



Given the line EF terminated by the parallel lines AB and CD, and through O, the mid-point of EF, the line KL terminated by the parallels.

To prove that $OL = OK$.

Proof. In the $\triangle FLO$ and EKO,

$OF = OE$, Given

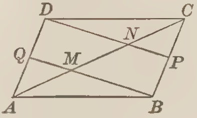
$\angle LFO = \angle KEO$, § 100

and $\angle FOL = \angle EOK$. § 60

$\therefore$ the $\triangle FLO$ and EKO are congruent. § 72

$\therefore$ $OL = OK$, by § 67. Q.E.D.

3. *In a parallelogram ABCD the line BQ bisects AD, and DP bisects BC. Prove that BQ and DP trisect AC.*



Given the quadrilateral $ABCD$, with BQ bisecting AD and DP bisecting BC.

To prove that BQ and DP trisect AC.

Proof. In the quadrilateral $QBPD$,

BP is $\parallel$ to QD. § 118

Also since $AD = BC$, § 125

$QD = BP$. Ax. 4

$\therefore$ $QBPD$ is a ▱,

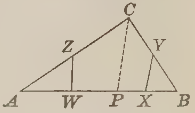
and DP is $\parallel$ to QB. § 130

In the $\triangle AND$, $AM = MN$. § 135

In the $\triangle BCM$, $MN = NC$. § 135

$\therefore$ $AM = MN = NC$, by Ax. 8. Q.E.D.

4. *On the base AB of a triangle ABC any point P is taken. The lines AP, PB, BC, and CA are bisected by W, X, Y, and Z respectively. Prove that XY is equal to WZ.*



Given the point P in the base AB of the $\triangle ABC$, and W, X, Y, and Z the mid-points of AP, PB, BC, and CA respectively.

To prove that $XY = WZ$.

Proof. Draw CP.

Then since $PX = XB$, and $CY = YB$, Given

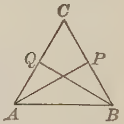
$\therefore XY = \frac{1}{2} CP$. § 136

Since $PW = WA$, and $CZ = ZA$, Given

$\therefore WZ = \frac{1}{2} CP$. § 136

$\therefore XY = WZ$, by Ax. 8. Q.E.D.

5. *In an isosceles triangle the medians drawn to the equal sides are equal.*



Given the isosceles $\triangle ABC$ having the equal sides AC and BC, with the medians AP and BQ drawn.

To prove that $AP = BQ$.

Proof. Since $AC = BC$, Given

$\therefore \angle CBA = \angle BAC$. § 74

In the $\triangle$ ABP and BAQ, $\angle PBA = \angle BAQ$,

$AB = AB$, Iden.

and $BP = AQ$. Ax. 4

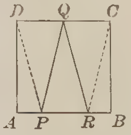
$\therefore$ the $\triangle$ ABP and BAQ are congruent. § 68

$\therefore AP = BQ$, by § 67. Q.E.D.

6. *In the square $ABCD$, CD is bisected by Q, and P and R are taken on AB so that AP equals BR. Prove that $PQ = RQ$.*

Given the square $ABCD$, with $DQ = QC$, and $AP = BR$

To prove that $PQ = RQ$.



Proof. Draw PD and RC.

In the rt. $\triangle$ PAD and RBC, $AP = BR$, Given

and $AD = BC$. § 65

$\therefore$ the rt. $\triangle$ PAD and RBC are congruent. § 90

$\therefore PD = RC$,

and $\angle ADP = \angle RCB$. § 67

$\therefore \angle PDQ = \angle QCR$. § 58

In the $\triangle$ PDQ and RCQ, $PD = RC$,

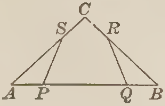
and $\angle PDQ = \angle QCR$. Proved

$DQ = QC$. Given

$\therefore$ the $\triangle$ PDQ and RCQ are congruent. § 68

$\therefore PQ = RQ$, by § 67. Q.E.D.

7. *In this figure* $AC = BC$, *and* $AP = BQ = CR = CS$. *Prove that* $QR = PS$.



Given $AC = BC$, and $AP = BQ = CR = CS$.

To prove that $QR = PS$.

Proof. Now $AC = BC$,

and $CS = CR$. Given

$\therefore AC - CS = BC - CR$. Ax. 2

That is, $AS = BR$.

In the $\triangle$s APS and BQR,

$AS = BR$, Proved

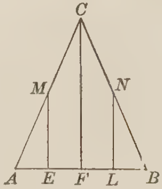
$AP = BQ$, Given

and $\angle A = \angle B$. § 74

$\therefore$ the $\triangle$s APS and BQR are congruent. § 68

$\therefore QR = PS$, by § 67. Q. E. D.

8. *From the vertex and the mid-points of the equal sides of an isosceles triangle lines are drawn perpendicular to the base. Prove that they divide the base into four equal parts.*



Given the isosceles $\triangle ABC$, with M and N the mid-points of the equal sides AC and BC, and ME, CF, and NL drawn $\perp$ to AB.

To prove that $AE = EF = FL = LB$.

Proof. In the rt. $\triangle$s AFC and BFC,

$AC = BC$, Given

and $FC = FC$. Iden.

$\therefore$ the rt. $\triangle$s AFC and BFC are congruent. § 89

$\therefore AF = BF$. § 67

Now ME is $\parallel$ to CF, § 95

and $AM = MC$. Given

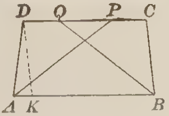
$\therefore AE = EF$. § 135

In like manner, $FL = LB$.

Since $AF = BF$,

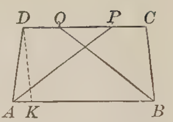
$\therefore AE = EF = FL = LB$, by Ax. 4. Q. E. D.

9. *In the quadrilateral* $ABCD$ *it is known that* AB *is parallel to* DC, *and that angle* C *equals angle* D. *On* CD *two points are taken such that* $CP = DQ$. *Prove that* $AP = BQ$.



Given the quadrilateral $ABCD$, with $AB \parallel$ to DC, $\angle C = \angle D$, and $CP = DQ$.

To prove that $AP = BQ$.



Proof. Draw $DK \parallel$ to CB.

Then $DK = CB$, § 127

and $\angle DKA = \angle B$. § 102

Since $\angle D = \angle C$, Given

and $\angle$s A and B are supplementary to $\angle$s D and C, § 104

$\therefore \angle A = \angle B$. § 58

$\therefore \angle DKA = \angle A$. Ax. 8

$\therefore DA = DK$. § 76

$\therefore DA = CB$. Ax. 8

Since $CP = DQ$, Given

$\therefore DC - CP = DC - DQ$, Ax. 2

or $DP = CQ$.

In the $\triangle$s APD and BQC, $DP = CQ$, Proved

$DA = CB$, Proved

and $\angle D = \angle C$. Given

$\therefore$ the $\triangle$s APD and BQC are congruent. § 68

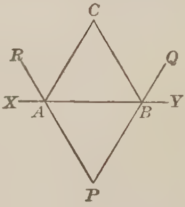
$\therefore AP = BQ$, by § 67. Q.E.D.

Exercise 20. Page 86

1. *In this figure it is given that $AC = BC$, and that BQ and AR bisect the angles YBC and CAX respectively. Prove that $\triangle APB$ is isosceles.*

Given $AC = BC$, and BQ and AR bisecting the $\angle$s YBC and CAX respectively.

To prove that $\triangle ABP$ is isosceles.



Proof. Since $AC = BC$, Given

$\therefore \angle BAC = \angle CBA$. § 74

$\therefore \angle CAX = \angle YBC$. § 58

$\therefore \frac{1}{2}\angle CAX = \frac{1}{2}\angle YBC$. Ax. 4

$\therefore \angle RAX = \angle YBQ$. Ax. 9

But $\angle PAB = \angle RAX$,

and $\angle ABP = \angle YBQ$. § 60

$\therefore \angle PAB = \angle ABP$. Ax. 8

$\therefore PA = PB$,

and the $\triangle ABP$ is isosceles, by § 76. Q.E.D.

2. *If through the vertices of an isosceles triangle lines are drawn parallel to the opposite sides, they form an isosceles triangle.*

Given the isosceles $\triangle ABC$, with $AC = BC$, and given $A'BC' \parallel$ to CA, $A'CB' \parallel$ to BA, and $B'AC' \parallel$ to CB.

To prove that $\triangle A'B'C'$ is isosceles.

Proof. The quadrilaterals $ABCB'$ and $ABA'C$ are ▱. § 118

In the ▱ $ABCB'$ and $ABA'C$,

$BC = AC$, Given

$AB = AB$, Iden.

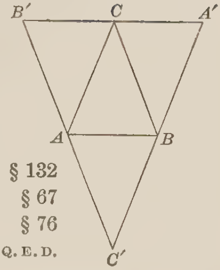
and $\angle CBA = \angle BAC$. § 74

$\therefore$ the ▱ $ABCB'$ and $ABA'C$ are congruent. § 132

$\therefore \angle A' = \angle B'$. § 67

$\therefore C'B' = C'A'$, § 76

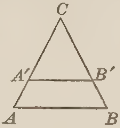
or the $\triangle A'B'C'$ is isosceles, by § 62. Q. E. D.



3. *If the vertical angles of two isosceles triangles coincide, the bases either coincide or are parallel.*

Given the isosceles $\triangle ABC$ with $AC = BC$, the isosceles $\triangle A'B'C$ with $A'C = B'C$, and the vertical $\angle$s C coinciding.

To prove that AB and $A'B'$ coincide or are parallel.



Proof. In the $\triangle$s ABC and $A'B'C$,

$AC = BC$,

and $A'C = B'C$. Given

If $A'C = AC$,

then $B'C = BC$. Ax. 8

$\therefore A'B'$ coincides with AB.

If $A'C$ is not equal to AC, in the $\triangle$s ABC and $A'B'C$,

$\angle C = \angle C$. Iden.

$\therefore \angle A' + \angle B' = \angle A + \angle B$. § 58

Since $\angle A' = \angle B'$, and $\angle A = \angle B$, § 74

$\therefore 2\angle A' = 2\angle A$. Ax. 9

$\therefore \angle A' = \angle A$ Ax. 4

$\therefore A'B'$ is $\parallel$ to AB, by § 103 Q. E. D.

4. *In which direction must the side of a triangle be produced so as to intersect the bisector of the opposite exterior angle?*

Given the $\triangle ABC$, with BN bisecting the exterior $\angle MBC$.

To find in which direction AC must be produced to intersect BN.

Solution. CASE 1. When $\angle A < \angle C$.

$\angle A + \angle A < \angle A + \angle C$, Ax. 6

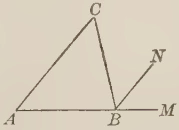
or $\angle A < \frac{1}{2}(\angle A + \angle C)$. Ax. 4

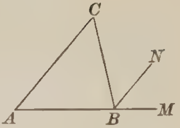
Now $\angle MBC = \angle A + \angle C$. § 111

$\therefore \angle MBN = \frac{1}{2}(\angle A + \angle C)$. Ax. 4

Then $\angle A < \angle MBN$. Ax. 9

$\therefore AC$ will intersect BN above AB. Q. E. F.





CASE 2. When $\angle A = \angle C$.

Now $\angle MBC = \angle A + \angle C$. § 111

$\therefore \angle MBC = 2\angle A$. Ax. 9

$\therefore \angle MBN = \angle A$. Ax. 4

$\therefore AC$ is $\parallel$ to BN. § 103

$\therefore AC$ will not intersect BN. Q.E.F.

CASE 3. When $\angle A > \angle C$,

$\angle A + \angle A > \angle A + \angle C$, Ax. 6

or $\angle A > \frac{1}{2}(\angle A + \angle C)$. Ax. 4

Now $\angle MBC = \angle A + \angle C$. § 111

$\therefore \angle MBN = \frac{1}{2}(\angle A + \angle C)$. Ax. 4

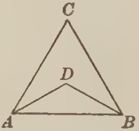
Then $\angle A > \angle MBN$. Ax. 9

$\therefore CA$ will intersect NB below AB. Q.E.F.

5. *The bisectors of the equal angles of an isosceles triangle form, together with the base, an isosceles triangle.*

Given the isosceles $\triangle ABC$, with $AC = BC$, and AD and BD bisecting the $\angle$s BAC and CBA respectively.

To prove that $\triangle ABD$ is isosceles.



Proof. $\angle BAC = \angle CBA$. § 74

Also $\angle BAD = \frac{1}{2}\angle BAC$,

and $\angle DBA = \frac{1}{2}\angle CBA$. Given

$\therefore \angle BAD = \angle DBA$. Ax. 8

$\therefore BD = AD$,

and $\triangle ABD$ is isosceles, by § 76. Q.E.D.

6. *The bisectors of the base angles of an equilateral triangle form an angle equal to the exterior angle at the vertex of the triangle.*

Given the equilateral $\triangle ABC$, with AD and BD bisecting the base $\angle$s BAC and CBA.

To prove that $\angle ADB = \angle BCE$.

Proof. $\angle ADB + \angle BAD + \angle DBA = 2$ rt. $\angle$s. § 107

But $\angle DBA = \frac{1}{2}\angle B$, Given

and $\angle BAD = \frac{1}{2}\angle A$. Given

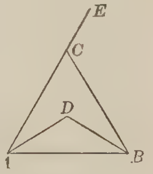
$\therefore \angle DBA + \angle BAD = \frac{1}{2}(\angle B + \angle A)$ Ax. 1

$= \frac{1}{2}(\frac{2}{3}$ rt. $\angle + \frac{2}{3}$ rt. $\angle)$ Ax. 9

$= \frac{2}{3}$ rt. $\angle$.

$\therefore \angle ADB + \frac{2}{3}$ rt. $\angle = 2$ rt. $\angle$s. Ax. 9

$\therefore \angle ADB = \frac{4}{3}$ rt. $\angle$s.



Also $\angle BCE = \angle A + \angle B$

$= \frac{2}{3}$ rt. $\angle + \frac{2}{3}$ rt. $\angle$

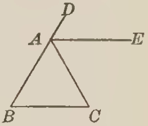
$= \frac{4}{3}$ rt. $\angle$s.

$\therefore \angle ADB = \angle BCE$, by Ax. 8 Q.E.D

7. *If the bisector of an exterior angle of a triangle is parallel to the opposite side, the triangle is isosceles.*

Given the $\triangle ABC$, with AE bisecting the exterior $\angle CAD$ and $\parallel$ to BC.

To prove that $\triangle ABC$ is isosceles.



Proof. $\angle EAD = \angle B$, § 102

and $\angle CAE = \angle C$. § 100

But $\angle EAD = \angle CAE$. Given

$\therefore \angle B = \angle C$. Ax. 8

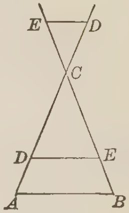
$\therefore AC = AB$,

and $\triangle ABC$ is isosceles, by § 76. Q. E. D.

8. *A line drawn parallel to the base of an isosceles triangle makes equal angles with the sides or the sides produced.*

Given the isosceles $\triangle ABC$, with $AC = BC$, and DE drawn $\parallel$ to the base.

To prove that $\angle EDC = \angle CED$.



Proof. Case 1. When DE cuts the sides.

Now $\angle EDC = \angle A$,

and $\angle CED = \angle B$. § 102

But $\angle A = \angle B$. § 74

$\therefore \angle EDC = \angle CED$, by Ax. 8. Q. E. D.

Case 2. When DE cuts the sides produced.

Now $\angle EDC = \angle A$,

and $\angle CED = \angle B$. § 100

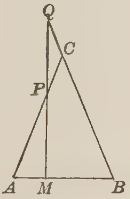
But $\angle A = \angle B$. § 74

$\therefore \angle EDC = \angle CED$, by Ax. 8. Q. E. D.

9. *A line drawn at right angles to AB, the base of an isosceles triangle ABC, cuts AC at P and BC produced at Q. Prove that PCQ is an isosceles triangle.*

Given the isosceles $\triangle ABC$, with $AC = BC$, and the line $QM \perp$ to AB, cutting AC at P, and BC produced at Q.

To prove that $\triangle PCQ$ is isosceles.



Proof. In the rt. $\triangle$s AMP and BMQ,

$\angle A = \angle B$. § 74

$\therefore \angle APM = \angle MQB$. § 108

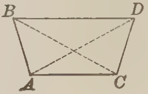
But $\angle APM = \angle CPQ$. § 60

$\therefore \angle CPQ = \angle MQB$. Ax. 8

$\therefore CQ = CP$,

and $\triangle PCQ$ is isosceles, by § 76. Q. E. D.

10. *In this figure, if $AB = CD$ and $\angle A = \angle C$, then BD is parallel to AC.*



Given $AB = CD$, and $\angle A = \angle C$.

To prove that BD is $\parallel$ to AC.

Proof. Draw AD and BC.

In the $\triangle$s ACB and CAD,

$AB = CD$, Given

$AC = AC$, Iden.

and $\angle A = \angle C$. Given

$\therefore$ the $\triangle$s ACB and CAD are congruent. § 68

$\therefore AD = BC$. § 67

In the $\triangle$s ADB and CBD,

$AD = BC$, Proved

$AB = CD$, Given

and $BD = BD$. Iden.

$\therefore$ the $\triangle$s ADB and CBD are congruent. § 80

$\therefore \angle B = \angle D$. § 67

But $\angle A + \angle B + \angle C + \angle D = 4$ rt. $\angle$s. § 144

$\therefore \angle A + \angle B + \angle A + \angle B = 4$ rt. $\angle$s. Ax. 9

$\therefore 2(\angle A + \angle B) = 4$ rt. $\angle$s.

$\therefore \angle A + \angle B = 2$ rt. $\angle$s. Ax. 4

$\therefore BD$ is $\parallel$ to AC, by § 105. Q. E. D.

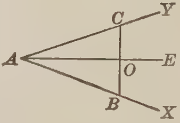
Exercise 21. Page 87

1. *A perpendicular to the bisector of an angle forms with the sides an isosceles triangle.*

Given the line AE, bisecting the $\angle XAY$, and COB drawn $\perp$ to AE.

To prove that $\triangle ABC$ is isosceles.

Proof. In the rt. $\triangle$s AOB and AOC,



$AO = AO$, Iden.

$\angle AOB = \angle COA$, § 56

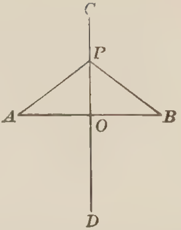
and $\angle BAO = \angle OAC$. Given

$\therefore$ the $\triangle$s AOB and AOC are congruent. § 72

$\therefore AB = AC$, § 67

and $\triangle ABC$ is isosceles. Q. E. D.

2. *If two lines bisect each other at right angles, any point in either is equidistant from the extremities of the other.*



Given $CD \perp$ to AB at the point O, $OA = OB$, and $OC = OD$, and P any point in either line.

To prove that $PA = PB$.

Proof. In the rt. $\triangle$ AOP and BOP,

$OA = OB$, Given

and $OP = OP$. Iden.

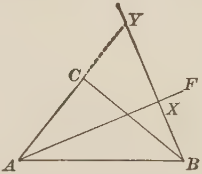
$\therefore$ the rt. $\triangle$ AOP and BOP are congruent. § 90

$\therefore$ $PA = PB$, by § 67. Q. E. D.

3. *From B a perpendicular is drawn to the bisector of the angle A of the triangle ABC, meeting it at X, and meeting AC or AC produced at Y. Prove that $BX = XY$.*

Given the $\triangle ABC$, with AF bisecting the $\angle A$, and BY drawn $\perp$ to AF and meeting AF at X, and AC produced at Y.

To prove that $BX = XY$.



Proof. In the $\triangle$ BAX and YAX,

$AX = AX$, Iden.

$\angle BAX = \angle XAY$, Given

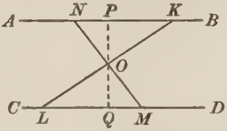
and $\angle AXB = \angle YXA$. § 56

$\therefore$ the $\triangle$ BAX and YAX are congruent. § 72

$\therefore$ $BX = XY$, by § 67. Q. E. D.

4. *If through any point equally distant from two parallel lines two lines are drawn cutting the parallels, they intercept equal segments on these parallels.*

Given the point O equally distant from the $\parallel$ lines AB and CD, and KL and MN two lines drawn through O and cutting the segments LM and KN on the parallels.



To prove that $LM = KN$.

Proof. Through O draw $PQ \perp$ to CD.

Then PQ is $\perp$ to AB. § 97

In the $\triangle$ OQL and OPK,

$OQ = OP$, Given

$\angle LOQ = \angle KOP$, § 60

and $\angle OQL = \angle OPK$. § 56

$\therefore$ the $\triangle$ OQL and OPK are congruent. § 72

$\therefore$ $LQ = PK$. § 67

Similarly the $\triangle$ OQM and OPN are congruent,

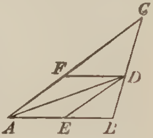
and $QM = PN$.

$\therefore$ $LQ + QM = PK + PN$ Ax. 1

That is, $LM = KN$. Q. E. D.

5. *If from the point where the bisector of an angle of a triangle meets the opposite side, parallels are drawn to the other two sides, and terminated by the sides, these parallels are equal.*

Given the $\triangle ABC$, with the bisector of $\angle A$ meeting BC at D, and DE and DF drawn $\parallel$ to CA and BA respectively.



To prove that $DE = DF$.

Proof. The quadrilateral $AEDF$ is a ▱. § 118

$\therefore \angle EAD = \angle FDA$, § 100

and $\angle DAF = \angle ADE$. § 100

$\angle EAD = \angle DAF$. Given

$\therefore \angle ADE = \angle FDA$. Ax. 8

In the $\triangle$s AED and ADF,

$\angle EAD = \angle DAF$, Given

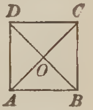
$\angle ADE = \angle FDA$, Proved

and $AD = AD$, Iden.

$\therefore$ $\triangle$s AED and ADF are congruent. § 72

$\therefore DE = DF$, by § 67. Q. E. D

6. *The diagonals of a square are perpendicular to each other and bisect the angles of the square.*



Given the square $ABCD$, with its diagonals AC and BD intersecting at O.

To prove that AC is $\perp$ to BD, and that the diagonals bisect the angles of the square.

Proof. Since $BA = BC$, and $DA = DC$, § 65

$\therefore BD$ is the $\perp$ bisector of AC. § 151

That is, BD and AC are $\perp$ to each other.

In the rt. $\triangle$s ABO and CBO,

$OB = OB$, Iden.

and $AB = BC$. § 65

$\therefore$ the rt. $\triangle$s ABO and CBO are congruent. § 89

$\therefore \angle OBA = \angle CBO$. § 67

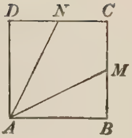
Similarly

$\angle OCB = \angle DCO$, $\angle ODC = \angle ADO$, and $\angle OAD = \angle BAO$. Q. E. D.

7. *If from the vertex of a square there are drawn line-segments to the mid-points of the two sides not adjacent to the vertex, these line-segments are equal.*

Given the square $ABCD$, and AM and AN drawn to the mid-points of the sides BC and CD respectively.

To prove that $AM = AN$.



Proof. Since $BC = CD$, § 65

$\therefore BM = DN$. Ax. 4

In the rt. △s ABM and ADN,

$AB = AD$, § 65

and $BM = DN$.

$\therefore$ the rt. △s ABM and ADN are congruent. § 90

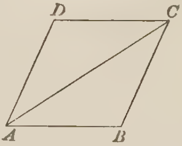
$\therefore AM = AN$, by § 67. Q.E.D.

8. *If either diagonal of a parallelogram bisects one of the angles, the sides of the parallelogram are all equal.*

Given the ▱ $ABCD$, and the diagonal AC bisecting the $\angle A$.

To prove that $AB = BC = CD = DA$.

Proof. The △s ABC and ADC are congruent. § 126



$\therefore \angle ACB = \angle CAD$. § 67

But $\angle BAC = \angle CAD$. Given

$\therefore \angle ACB = \angle BAC$. Ax. 8

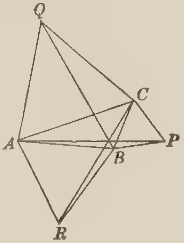
$\therefore AB = BC$. § 76

Similarly $CD = DA$.

But $AB = CD$. § 125

$\therefore AB = BC = CD = DA$, by Ax. 8. Q.E.D

9. *On the sides of any triangle ABC equilateral triangles BPC, CQA, ARB are constructed. Prove that $AP = BQ = CR$.*



Given the equilateral △s BPC, CQA, and ARB constructed on the sides of the $\triangle ABC$.

To prove that $AP = BQ = CR$.

Proof. Each ∠ of an equilateral △ is equal to $\frac{2}{3}$ rt. ∠. § 145

$\therefore \angle PBA = \angle CBA + \frac{2}{3}$ rt. ∠, Ax. 11

and $\angle CBR = \angle CBA + \frac{2}{3}$ rt. ∠.

$\therefore \angle PBA = \angle CBR$. Ax. 8

In the △s ABP and RBC,

$AB = BR$, Given

$BP = BC$, Given

and $\angle PBA = \angle CBR$. Proved

$\therefore$ the △s ABP and RBC are congruent. § 68

$\therefore AP = CR$. § 67

In like manner the △s ACP and BCQ are congruent,

and $AP = BQ$.

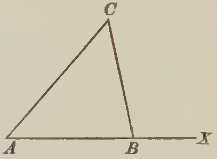
$\therefore AP = BQ = CR$, by Ax. 8. Q.E.D.

Exercise 22. Page 88

1. *An exterior angle of an acute triangle or of a right triangle cannot be acute.*

Given the $\triangle ABC$, with no angle greater than a rt. $\angle$, and the exterior $\angle XBC$.

To prove that $\angle XBC$ cannot be less than a rt. $\angle$.



Proof. $\angle A + \angle C + \angle CBA = 2$ rt. $\angle$s. § 107

$\therefore \angle A + \angle C = 2$ rt. $\angle$s $- \angle CBA$. Ax. 2

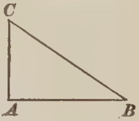
Also $\angle A + \angle C = \angle XBC$. § 111

$\therefore \angle XBC = 2$ rt. $\angle$s $- CBA$. Ax. 8

But $\angle CBA$ is not greater than a rt. $\angle$. Given

$\therefore \angle XBC$ is not less than a rt. $\angle$. Q.E.D.

2. *If the sum of two angles of a triangle equals the third angle, the triangle is a right triangle.*



Given the $\triangle ABC$, with $\angle B + \angle C = \angle A$.

To prove that $\angle A = 1$ rt. $\angle$.

Proof. Now $\angle A + \angle B + \angle C = 2$ rt. $\angle$s, § 107

and $\angle B + \angle C = \angle A$. Given

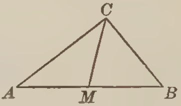
$\therefore 2 \angle A = 2$ rt. $\angle$s. Ax. 9

$\therefore \angle A = 1$ rt. $\angle$, by Ax. 4. Q.E.D.

3. *If the line joining any vertex of a triangle to the mid-point of the opposite side divides the triangle into two isosceles triangles, the original triangle is a right triangle.*

Given the $\triangle ABC$, with the median CM dividing the $\triangle ABC$ into two isosceles $\triangle$s AMC and MBC that have $CM = AM$ and $CM = BM$.

To prove that $\angle ACB = 1$ rt. $\angle$.



Proof. In the $\triangle AMC$, $\angle A = \angle ACM$, § 74

and in the $\triangle MBC$, $\angle B = \angle MCB$. § 74

$\therefore \angle A + \angle B = \angle ACM + \angle MCB$. Ax. 1

That is, $\angle A + \angle B = \angle ACB$.

But $\angle A + \angle B + \angle ACB = 2$ rt. $\angle$s. § 107

$\therefore 2 \angle ACB = 2$ rt. $\angle$s. Ax. 9

$\therefore \angle ACB = 1$ rt. $\angle$, by Ax. 4. Q.E.D.

4. *If the vertical angles of two isosceles triangles are supplements one of the other, the base angles of the one are complements of those of the other.*

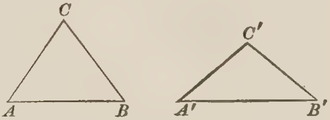
Given the isosceles $\triangle$s ABC and $A'B'C'$, and $\angle$s C and C' supplementary.

To prove that $\angle A + \angle A' = 1$ rt. $\angle$.

Proof. Now $\angle A + \angle B + \angle C = 2$ rt. $\angle$s,

and $\angle A' + \angle B' + \angle C' = 2$ rt. $\angle$s. § 107

$\therefore \angle A + \angle A' + \angle B + \angle B' + \angle C + \angle C' = 4$ rt. $\angle$s. Ax. 1



But $\angle A = \angle B$,

and $\angle A' = \angle B'$, § 74

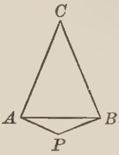
and $\angle C + \angle C' = 2$ rt. $\angle$s. Given

$\therefore 2\angle A + 2\angle A' + 2$ rt. $\angle$s $= 4$ rt. $\angle$s. Ax. 9

$\therefore 2\angle A + 2\angle A' = 2$ rt. $\angle$s. Ax. 2

$\therefore \angle A + \angle A' = 1$ rt. $\angle$, by Ax. 4. Q.E.D.

5. *From the extremities of the base AB of a triangle ABC perpendiculars to the other two sides are drawn, meeting at P. Prove that the angle P is the supplement of the angle C.*



Given the $\triangle ABC$, and $AP \perp$ to AC and $BP \perp$ to BC.

To prove that $\angle P + \angle C = 2$ rt. $\angle$s.

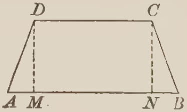
Proof. $\angle P + \angle C + \angle PAC + \angle CBP = 4$ rt. $\angle$s. § 143

But $\angle PAC = \angle CBP = 1$ rt. $\angle$. Given

$\therefore \angle P + \angle C + 1$ rt. $\angle + 1$ rt. $\angle = 4$ rt. $\angle$s. Ax. 9

$\therefore \angle P + \angle C = 2$ rt. $\angle$s, by Ax. 2. Q.E.D.

6. *If two sides of a quadrilateral are parallel, and the other two sides are equal but not parallel, the sums of the two pairs of opposite angles are equal.*



Given the quadrilateral $ABCD$, with $DC \parallel$ to AB, $AD = BC$, and AD not $\parallel$ to BC.

To prove that $\angle A + \angle C = \angle B + \angle D$.

Proof. Draw DM and $CN \perp$ to AB.

Now DM is $\parallel$ to CN. § 95

In the rt. $\triangle$s AMD and BNC, $AD = BC$, Given

and $DM = CN$. § 127

$\therefore$ the rt. $\triangle$s AMD and BNC are congruent. § 89

$\therefore \angle ADM = \angle NCB$. § 67

But $\angle MDC$ and $\angle DCN$ are rt. $\angle$s. § 97

$\therefore \angle ADC = \angle ADM + 1$ rt. $\angle$,

and $\angle DCB = \angle NCB + 1$ rt. $\angle$. Ax. 11

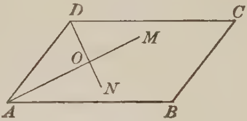
$\therefore \angle ADC = \angle DCB$. Ax. 8

But $\angle A = \angle B$. § 67

$\therefore \angle A + \angle DCB = \angle B + \angle ADC$. Ax. 1

That is, $\angle A + \angle C = \angle B + \angle D$. Q.E.D.

7. *The bisectors of two consecutive angles of a parallelogram are perpendicular to each other.*



Given the ▱ $ABCD$, with MA and ND bisecting the $\measuredangle A$ and D respectively and intersecting at O.

To prove that $\angle MOD = 1$ rt. $\angle$.

Proof. $\angle BAD + \angle ADC = 2$ rt. $\measuredangle$. § 104

$\therefore \frac{1}{2}\angle BAD + \frac{1}{2}\angle ADC = 1$ rt. $\angle$, Ax. 4

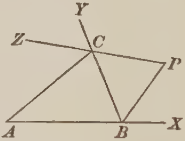
or $\angle OAD + \angle ADO = 1$ rt. $\angle$. Ax. 9

But $\angle MOD = \angle OAD + \angle ADO$. § 111

$\therefore \angle MOD = 1$ rt. $\angle$, by Ax. 8. Q. E. D.

8. *The exterior angles at B and C of any triangle ABC are bisected by lines meeting at P. Prove that the angle at P together with half the angle A equals a right angle.*

Given BP and CZ bisecting the exterior $\measuredangle XBC$ and YCA of the $\triangle ABC$ and meeting at P.



To prove that $\angle P + \frac{1}{2}\angle A = 1$ rt. $\angle$.

Proof. $\angle ACP = \angle ACB + \angle BCP$. Ax. 11

But $\angle BCP = \angle YCZ$, § 60

and $\angle YCA = \angle A + \angle CBA$. § 111

Since $\frac{1}{2}\angle YCA = \angle YCZ$, Given

$\therefore \angle BCP = \frac{1}{2}(\angle A + \angle CBA)$. Ax. 8

$\therefore \angle ACP = \angle ACB + \frac{1}{2}(\angle A + \angle CBA)$. Ax. 9

In like manner, $\angle PBA = \angle CBA + \frac{1}{2}(\angle A + \angle ACB)$.

$\therefore \angle ACP + \angle PBA = \angle A + \frac{3}{2}(\angle ACB + \angle CBA)$.

Now $\angle A + \angle CBA + \angle ACB = 2$ rt. $\measuredangle$. § 107

$\therefore \angle CBA + \angle ACB = 2$ rt. $\measuredangle - \angle A$.

$\therefore \angle ACP + \angle PBA = \angle A + 3$ rt. $\measuredangle - \frac{3}{2}\angle A$. Ax. 9

That is, $\angle ACP + \angle PBA = 3$ rt. $\measuredangle - \frac{1}{2}\angle A$.

$\angle P + \angle A + \angle ACP + \angle PBA = 4$ rt. $\measuredangle$. § 144

$\therefore \angle P + \angle A + 3$ rt. $\measuredangle - \frac{1}{2}\angle A = 4$ rt. $\measuredangle$. Ax. 9

$\angle P + \frac{1}{2}\angle A = 1$ rt. $\angle$. Q. E. D.

9. *The opposite angles of the quadrilateral formed by the bisectors of the interior angles of any quadrilateral are supplemental.*

Given the quadrilateral $ABCD$ and the quadrilateral $NMLP$ formed by the bisectors of the $\measuredangle$ of $ABCD$.

To prove that $\angle MNP + \angle PLM = 2$ rt. $\measuredangle$,

and $\angle LMN + \angle NPL = 2$ rt. $\measuredangle$.

Proof. $\angle MNP = \angle AND$. § 60

$\angle DNA = 2$ rt. $\angle$s $- \frac{1}{2}(\angle A + \angle D)$. § 107

$\therefore \angle MNP = 2$ rt. $\angle$s $- \frac{1}{2}(\angle A + \angle D)$. Ax. 8

In like manner, $\angle PLM = 2$ rt. $\angle$s $- \frac{1}{2}(\angle B + \angle C)$.

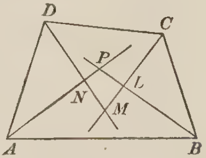
$\therefore \angle MNP + \angle PLM = 4$ rt. $\angle$s $- \frac{1}{2}(\angle A + \angle B + \angle C + \angle D)$. Ax. 1

But $\angle A + \angle B + \angle C + \angle D = 4$ rt. $\angle$s. § 144

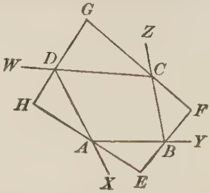
$\therefore \angle MNP + \angle PLM = 2$ rt. $\angle$s.

In like manner,

$\angle LMN + \angle NPL = 2$ rt. $\angle$s. Q. E. D.



10. *Show that Ex. 9 is true if the bisectors of the exterior angles are taken.*



Given the quadrilateral $ABCD$ and the quadrilateral $EFGH$ formed by the bisectors of the exterior $\angle$s of $ABCD$.

To prove that $\angle E + \angle G = 2$ rt. $\angle$s,

and $\angle F + \angle H = 2$ rt. $\angle$s.

Proof. Now $\angle ABE = \frac{1}{2}\angle YBC$,

and $\angle CDG = \frac{1}{2}\angle WDA$. § 60

In the $\triangle EAB$, $\angle E = 2$ rt. $\angle$s $- \frac{1}{2}(\angle XAB + \angle YBC)$,

and in the $\triangle GDC$, $\angle G = 2$ rt. $\angle$s $- \frac{1}{2}(\angle ZCD + \angle WDA)$. § 107

$\therefore \angle E + \angle G = 4$ rt. $\angle$s $- \frac{1}{2}(\angle XAB + \angle YBC + \angle ZCD + \angle WDA)$. Ax. 1

But $\angle XAB + \angle YBC + \angle ZCD + \angle WDA = 4$ rt. $\angle$s. § 146

$\therefore \angle E + \angle G = 4$ rt. $\angle$s $- 2$ rt. $\angle$s $= 2$ rt. $\angle$s. Ax. 9

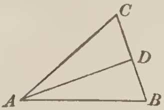
In like manner, $\angle F + \angle H = 2$ rt. $\angle$s. Q. E. D.

Exercise 23. Page 89

1. *In the triangle ABC the angle A is bisected by a line meeting BC at D. Prove that BA is greater than BD, and CA greater than CD.*

Given the $\triangle ABC$, with AD bisecting the $\angle A$ and meeting BC at D.

To prove that $BA > BD$, and that $CA > CD$.



Proof. $\angle ADB = \angle DAC + \angle C$. § 111

That is, $\angle ADB = \frac{1}{2}\angle A + \angle C$.

$\therefore \angle ADB > \angle BAD$. Ax. 11

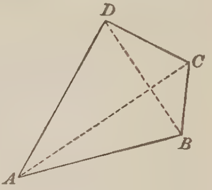
$\therefore AB > BD$. § 114

Similarly $CA > CD$. Q. E. D.

2. *In the quadrilateral ABCD it is known that AD is the longest side and BC the shortest side. Prove that the angle B is greater than the angle D, and the angle C greater than the angle A.*

Given the quadrilateral $ABCD$, with AD the longest side and BC the shortest.

To prove that $\angle B > \angle D$, and $\angle C > \angle A$.



Proof. Draw BD.

Since $AD > AB$, Given

$\therefore \angle DBA > \angle ADB$. § 113

Also, since $DC > BC$, Given

$\therefore \angle CBD > \angle BDC$. § 113

$\therefore \angle DBA + \angle CBD > \angle ADB + \angle BDC$. Ax. 7

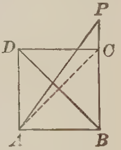
$\therefore \angle B > \angle D$. Ax. 11

Similarly, by drawing AC, $\angle C > \angle A$. Q.E.D.

3. *A line is drawn from the vertex A of a square $ABCD$ so as to cut CD and to meet BC produced in P. Prove that AP is greater than DB.*

Given the square $ABCD$, the diagonal BD, and the line AP drawn to meet BC produced in P.

To prove that $AP > DB$.



Proof. Draw AC.

In the rt. $\triangle$ ABC and BAD,

$BC = AD$, § 65

and $AB = AB$. Iden.

$\therefore$ the rt. $\triangle$ ABC and BAD are congruent. § 90

$\therefore AC = BD$. § 67

Now $\angle PCA = \angle CBA + \angle BAC$ § 111

$= 1 \text{ rt.} \angle + \angle BAC$.

$\therefore \angle PCA > \angle APC$. § 109

$\therefore AP > AC$. § 114

But $AC = BD$.

$\therefore AP > BD$, by Ax. 9. Q.E.D.

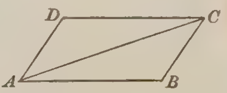
4. *If the angle between two adjacent sides of a parallelogram is increased, the length of the sides remaining unchanged, the diagonal from the vertex of this angle is diminished.*

Given the ▱ $ABCD$, the diagonal, and the variable $\angle$s A, B, C, D.

To prove that if $\angle A$ is increased, AC is diminished.

Proof. Now $\angle A + \angle B = 2$ rt. $\angle$s. § 104

$\therefore \angle B = 2$ rt. $\angle$s $- \angle A$. Ax. 2



If $\angle A$ is increased, 2 rt. $\angle$s $- \angle A$ will be diminished; that is, $\angle B$ will become smaller, but the sides BA and BC of the $\triangle ABC$ will remain unchanged.

$\therefore AC$ will be diminished, by § 115. Q.E.D.

5. *Within a triangle ABC a point P is taken such that $CP = CB$. Prove that AB is always greater than AP.*

Given the point P within the $\triangle ABC$, and $CP = CB$.

To prove that $AB > AP$.

Proof. Draw CM bisecting the $\angle PCB$, and draw PM.

In the $\triangle$ PMC and BMC, $CP = CB$, Given

$CM = CM$, Iden.

and $\angle PCM = \angle MCB$. Const.

$\therefore$ the $\triangle$ PMC and BMC are congruent. § 68

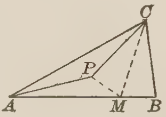
$\therefore MP = MB$. § 67

But $AP < AM + MP$. § 112

$\therefore AP < AM + MB$. Ax. 9

That is, $AP < AB$,

or $AB > AP$. Q. E. D.



6. *In a quadrilateral $ABCD$ it is known that AD equals BC and that the angle C is less than the angle D. Prove that the diagonal AC is greater than the diagonal BD.*

Given the quadrilateral $ABCD$, with $AD = BC$ and $\angle C$ less than $\angle D$, and the diagonals AC and BD.

To prove that $AC > BD$.

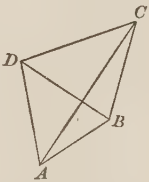
Proof. In the $\triangle$ ADC and BCD,

$AD = BC$, Given

$DC = DC$, Iden.

and $\angle D > \angle C$. Given

$\therefore AC > BD$, by § 115. Q. E. D.



7. *In the quadrilateral $ABCD$ it is known that AD equals BC and that the angle D is greater than the angle C. Prove that the angle B is greater than the angle A.*

Given the quadrilateral $ABCD$, with $AD = BC$ and $\angle D > \angle C$.

To prove that $\angle B > \angle A$.

Proof. Draw the diagonals AC and BD.

In the $\triangle$ ACD and BCD, $AD = BC$, Given

$DC = DC$, Iden.

and $\angle D > \angle C$. Given

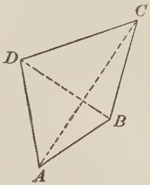
$\therefore AC > BD$. § 115

In the $\triangle$ ABC and ABD, $BC = AD$, Given

$AB = AB$, Iden.

and $AC > BD$. Proved

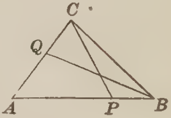
$\therefore \angle B > \angle A$, by § 116. Q. E. D.



8. *In the triangle ABC the side AB is greater than AC. On AB and AC respectively BP is taken equal to CQ. Prove that BQ is greater than CP.*

Given the $\triangle ABC$, with $AB > AC$, and $BP = CQ$.

To prove that $BQ > CP$.



Proof. Since $AB > AC$, Given

$\therefore \angle C > \angle B$. § 113

In the $\triangle$s BCQ and PBC, $CQ = BP$, Given

$CB = CB$, Iden.

and $\angle C > \angle B$.

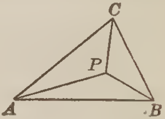
$\therefore BQ > CP$, by § 115. Q.E.D.

9. *The sum of the distances of any point from the three vertices of a triangle is greater than half the sum of the sides.*

Given the $\triangle ABC$, and PA, PB, and PC lines drawn from a point P to the three vertices.

To prove that

$$PA + PB + PC > \tfrac{1}{2}(AB + BC + CA).$$



Proof. Now $PA + PB > AB$,

$PB + PC > BC$,

and $PC + PA > CA$. § 112

$\therefore 2PA + 2PB + 2PC > AB + BC + CA$. Ax. 7

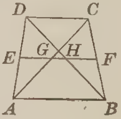
$\therefore PA + PB + PC > \tfrac{1}{2}(AB + BC + CA)$, by Ax. 6. Q.E.D.

Exercise 24. Page 90

1. *The line joining the mid-points of the nonparallel sides of a trapezoid passes through the mid-points of the two diagonals.*

Given the trapezoid $ABCD$, AB and DC the $\parallel$ sides, AC and BD the diagonals, and EF joining the mid-points of AD and BC.

To prove that EF bisects AC and BD.



Proof. Let G be the mid-point of AC and H of BD.

Then EF is $\parallel$ to AB and DC. § 137

But E is the mid-point of AD. Given

$\therefore EF$ passes through H. § 135

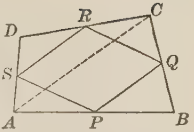
Similarly EF passes through G.

$\therefore EF$ bisects AC and BD. Q.E.D.

2. *The lines joining the mid-points of the consecutive sides of any quadrilateral form a parallelogram.*

Given the quadrilateral $ABCD$, and the quadrilateral $PQRS$ formed by joining the mid-points of the consecutive sides.

To prove that $PQRS$ is a $\square$.



Proof. Draw AC.

PQ and SR are $\parallel$ to AC, § 136

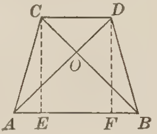
and $PQ = SR = \frac{1}{2} AC$. § 136

$\therefore PQ$ is $\parallel$ to SR. § 96

$\therefore PQRS$ is a ▱, by § 130. Q.E.D.

3. *If the diagonals of a trapezoid are equal, the trapezoid is isosceles.*

Given the trapezoid $ABCD$, with AB and CD the $\parallel$ sides, and with equal diagonals AD and BC.



To prove that $AC = BD$.

Proof. Draw CE and $DF \perp$ to AB.

Then CE is $\parallel$ to DF. § 95

$\therefore CE = DF$. § 127

In the rt. $\triangle$s AFD and BEC, $DF = CE$,

and $AD = BC$. Given

$\therefore$ the rt. $\triangle$s AFD and BEC are congruent. § 89

$\therefore \angle FAD = \angle CBA$. § 67

In the $\triangle$s ABC and BAD, $AB = AB$, Iden.

$BC = AD$, Given

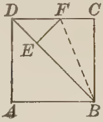
and $\angle CBA = \angle FAD$. Proved

$\therefore$ the $\triangle$s ABC and BAD are congruent. § 68

$\therefore AC = BD$, by § 67. Q.E.D.

4. *If from the diagonal DB of a square ABCD, BE is cut off equal to BC, and EF is drawn perpendicular to BD, meeting DC at F, then DE is equal to EF and also to FC.*

Given the square $ABCD$, the diagonal DB, $BE = BC$, and $EF \perp$ to BD.



To prove that $DE = EF = FC$.

Proof. $\angle BDC = \angle CBD$. § 74

Since $\angle BDC + \angle CBD = 1$ rt. $\angle$,

$\therefore \angle BDC = \frac{1}{2}$ rt. $\angle$.

$\therefore \angle BDC + \angle FED = \frac{1}{2}$ rt. $\angle + 1$ rt. $\angle$

$= \frac{3}{2}$ rt. $\angle$. Ax. 1

$\therefore \angle DFE = 2$ rt. $\angle$s $- \frac{3}{2}$ rt. $\angle = \frac{1}{2}$ rt. $\angle$. § 107

$\therefore \angle DFE = \angle EDF$. Ax. 8

$\therefore \triangle DEF$ is isosceles and $DE = EF$. § 76

Draw BF.

In the rt. $\triangle$s BEF and BCF, $BF = BF$, Iden.

and $BE = BC$. Given

$\therefore$ the rt. $\triangle$s BEF and BCF are congruent. § 89

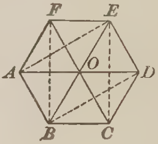
$\therefore EF = FC$. § 67

$\therefore DE = EF = FC$, by Ax. 8. Q.E.D

5. *If the opposite sides of a hexagon are equal and parallel, the diagonals that join opposite vertices meet in a point.*

Given the hexagon $ABCDEF$, with AB equal and parallel to ED, BC equal and parallel to FE, and CD equal and parallel to AF.

To prove that AD, BE, and CF meet in a point.



Proof. Draw AE, BF, BD, and CE.

Since FE is equal and parallel to BC, Given

∴ the quadrilateral $BCEF$ is a ▱. § 130

Let O be the point of bisection of the diagonals CF and BE of the ▱ $BCEF$.

Then $OE = OB$. § 131

Similarly $ABDE$ is a ▱, and its diagonals AD and BE bisect each other.

It has been proved that O is the mid-point of BE.

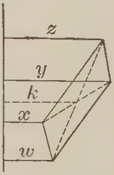
∴ AD passes through the point O.

That is, AD, BE, and CF meet in a point. Q.E.D.

6. *If perpendiculars are drawn from the four vertices of a parallelogram to any line outside the parallelogram, the sum of the perpendiculars from one pair of opposite vertices equals the sum of those from the other pair.*

Given z, y, x, w, all $\perp$s to the line, each drawn from the vertex of the ▱.

To prove that $x + y = w + z$.



Proof. Draw the diagonals of the ▱, and from their point of intersection draw $k \perp$ to the given line.

These diagonals bisect each other. § 131

Now x is ∥ to y, and y is ∥ to k. § 95

The nonparallel sides of the trapezoid of which x and y are the bases are bisected by k. § 134

$$\therefore k = \tfrac{1}{2}(x + y). \qquad \S\ 137$$

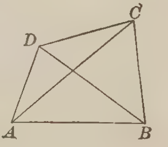
Similarly $k = \tfrac{1}{2}(w + z)$.

$$\therefore \tfrac{1}{2}(x + y) = \tfrac{1}{2}(w + z). \qquad \text{Ax. } 8$$

∴ $x + y = w + z$, by Ax. 3. Q.E.D.

Exercise 25. Page 91

1. *The sum of the four sides of any quadrilateral is greater than the sum of the diagonals.*

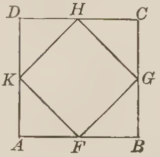


Given the quadrilateral $ABCD$ and the diagonals AC and BD.

To prove that $AB + BC + CD + DA > AC + BD$.

Proof. $BC + CD > BD,\ DA + AB > BD,$
$DA + CD > AC$, and $AB + BC > AC$. § 112
$\therefore AB + BC + CD + DA > AC + BD$, by Ax. 7. Q.E.D.

2. *The lines joining the mid-points of the sides of a square, taken in order, form a square.*



Given the square $ABCD$, with FG, GH, HK, KF lines joining the mid-points of the sides.

To prove that $FGHK$ is a square.

Proof. Now $AB = BC = CD = DA$. § 65

$\therefore AF = FB = BG = GC = CH = HD$
$= DK = KA$. Ax. 4

Again, $\angle A = \angle B = \angle C = \angle D$. § 65

In the rt. $\triangle$s FBG, GCH, HDK, and KAF,
$FB = GC = HD = KA$,
and $BG = CH = DK = AF$.

$\therefore$ the rt. $\triangle$s FBG, GCH, HDK, and KAF are congruent. § 90
$\therefore FG = GH = HK = KF$.

In the $\triangle AFK$, since $\angle FAK = 1$ rt. $\angle$,
$\therefore \angle KFA + \angle AKF = 1$ rt. $\angle$. § 107

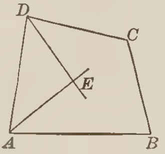
Since $KA = AF$,
$\therefore \angle KFA = \angle AKF$. § 74
$\therefore \angle KFA = \angle AKF = \frac{1}{2}$ rt. $\angle$.

Similarly $\angle BFG = \angle FGB = \frac{1}{2}$ rt. $\angle$.

Now $\angle KFA + \angle GFK + \angle BFG = 2$ rt. $\angle$s. § 43
$\therefore \frac{1}{2}$ rt. $\angle + \angle GFK + \frac{1}{2}$ rt. $\angle = 2$ rt. $\angle$s. Ax. 9
$\therefore \angle GFK = 1$ rt. $\angle$.

Similarly $\angle HGF = \angle KHG = \angle FKH = 1$ rt. $\angle$.
$\therefore FGHK$ is a square, by § 65. Q.E.D.

3. *In a quadrilateral the angle between the bisectors of two consecutive angles is one half the sum of the other two angles.*



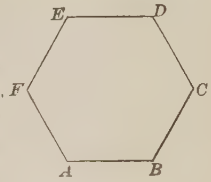
Given the quadrilateral $ABCD$, with AE bisecting the $\angle BAD$ and DE bisecting the $\angle ADC$.

To prove that $\angle DEA = \frac{1}{2}(\angle B + \angle C)$.

Proof. $\angle DEA + \frac{1}{2}(\angle A + \angle D) = 2$ rt. $\angle$s. § 107
$\therefore \angle DEA = 2$ rt. $\angle$s $- \frac{1}{2}(\angle A + \angle D)$.

$\angle A + \angle D + \angle B + \angle C = 4$ rt. $\angle$s. § 144
$\therefore \angle A + \angle D = 4$ rt. $\angle$s $- (\angle B + \angle C)$. Ax. 2
$\therefore \frac{1}{2}(\angle A + \angle D) = 2$ rt. $\angle$s $- \frac{1}{2}(\angle B + \angle C)$. Ax. 4
$\therefore \angle DEA = 2$ rt. $\angle$s $- 2$ rt. $\angle$s $+ \frac{1}{2}(\angle B + \angle C)$. Ax. 9
$\therefore \angle DEA = \frac{1}{2}(\angle B + \angle C)$. Q.E.D.

4. *If the opposite sides of a hexagon are equal, does it follow that they are parallel? Give reasons for your answer.*

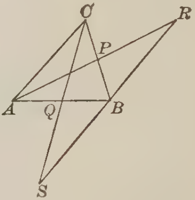


Given the hexagon $ABCDEF$, with $AB = ED$, $BC = FE$, $CD = AF$.

It does not follow that the opposite sides are parallel.

If either two, three, or four consecutive, or two opposite vertices are fixed in position, the remaining ones may be shifted so that no two opposite sides need be parallel.

5. *In a triangle ABC the side BC is bisected at P and AB is bisected at Q. AP is produced to R so that $AP = PR$, and CQ is produced to S so that $CQ = QS$. Prove that S, B, and R are in a straight line.*



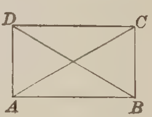
Given the $\triangle ABC$, with BC bisected at P and AB at Q, with AP produced to R so that $AP = PR$, and with CQ produced to S so that $CQ = QS$.

To prove that RBS is a straight line.

Proof. In the $\triangle$ APC and RPB,

$PA = PR$, Given
$PC = PB$, Given
and $\angle CPA = \angle BPR$. § 60
$\therefore$ $\triangle$ APC and RPB are congruent. § 68
$\therefore \angle PAC = \angle PRB$. § 67
$\therefore$ BR is $\parallel$ to AC. § 100
Similarly the $\triangle$ AQC and SQB are congruent,
and SB is $\parallel$ to AC.
$\therefore$ RBS is a straight line, by § 94. Q. E. D.

6. *If the diagonals of a parallelogram are equal, all of the angles of the parallelogram are equal.*



Given the $\square$ $ABCD$, with $AC = BD$.

To prove that $\angle A = \angle B = \angle C = \angle D$.

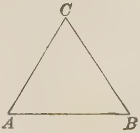
Proof. In the $\triangle$ ABD and ABC,

$AD = BC$, § 125
$BD = AC$, Given
and $AB = AB$. Iden.
$\therefore$ the $\triangle$ ABD and ABC are congruent. § 80
$\therefore \angle A = \angle B$. § 67
Similarly $\angle A = \angle D$,
and $\angle B = \angle C$.
$\therefore \angle A = \angle B = \angle C = \angle D$, by Ax. 8. Q. E. D

7. *In the triangle ABC, $\angle A = 60°$ and $\angle B > \angle C$. Which is the longest and which is the shortest side of the triangle? Prove it.*

Given the $\triangle ABC$, with $\angle A = 60°$ and $\angle B > \angle C$.

To prove that $AC > BC > AB$.



Proof. $\angle A + \angle B + \angle C = 2 \text{ rt. } \angle s$. § 107

But $\angle A = 60° = \frac{2}{3} \text{ rt. } \angle$. Given

$\therefore \frac{2}{3} \text{ rt. } \angle + \angle B + \angle C = 2 \text{ rt. } \angle s$. Ax. 9

$\therefore \angle B + \angle C = \frac{4}{3} \text{ rt. } \angle$. Ax. 2

But $\angle B > \angle C$. Given

$\therefore \angle C$ must be less than $\frac{2}{3}$ rt. $\angle$.

$\therefore \angle C < \angle A$.

$\therefore AC > BC > AB$, by § 114. Q.E.D.

8. *How many sides has a polygon each of whose interior angles is equal to 175°?*

Solution. $175° = \frac{175}{90} \text{ rt. } \angle = \frac{35}{18} \text{ rt. } \angle$.

$$\therefore \frac{2(n-2)}{n} \text{ rt. } \angle = \frac{35}{18} \text{ rt. } \angle. \qquad \S\ 145$$

$$\therefore \frac{2(n-2)}{n} = \frac{35}{18}.$$

$$\therefore 36n - 72 = 35n.$$

$$\therefore n = 72.$$

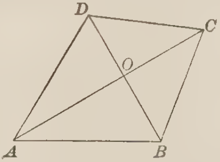
Hence the polygon has 72 sides.

9. *Given the quadrilateral $ABCD$, with AB equal to AD, and BC equal to CD. Prove that the diagonal AC bisects the angle DCB and is perpendicular to the diagonal BD.*

Given the quadrilateral $ABCD$, with $AB = AD$ and $BC = CD$, and with the diagonals AC and BD intersecting at O.

To prove that $\angle ACB = \angle DCA$,

and that AC is $\perp$ to BD.



Proof. In the $\triangle s$ ABC and ADC,

$AB = AD$,

$BC = CD$, Given

and $AC = AC$. Iden.

$\therefore \triangle s$ ABC and ADC are congruent. § 80

$\therefore \angle ACB = \angle DCA$. § 67

In the $\triangle s$ DOC and BOC, $CD = BC$, Given

$CO = CO$, Iden.

and $\angle DCA = \angle ACB$.

$\therefore$ the $\triangle s$ DOC and BOC are congruent. § 80

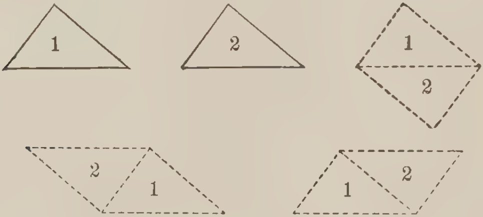
$\therefore \angle COD = \angle BOC$. § 67

$\therefore AC$ is $\perp$ to BD, by §§ 26, 27. Q.E.D.

10. *In how many ways can two congruent triangles be put together to form a parallelogram? Draw the diagrams.*

There are three ways in which two congruent triangles can be put together to form a parallelogram.

The diagrams to illustrate are shown below.



11. *The sides of a polygon of an odd number of sides are produced to meet, thus forming a star-shaped figure. What is the sum of the angles at the points of the star?*

Given the polygon $ABC \cdots$, having an odd number, n, of sides, and $\triangle$s XAB, YBC, $\cdots$, formed by producing the sides.

Required to find the sum of the angles at the points of the star.

Solution. Since the polygon has n sides, there are n triangles formed by producing the sides of the polygon.

The sum of the angles of these n $\triangle$s $= n \cdot 2$ rt. $\angle$s. § 107

But the sum of the base angles of these n $\triangle$s $= 8$ rt. $\angle$s. § 146

$\therefore$ the sum of the angles at the points of the star

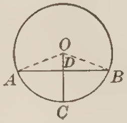
$= n \cdot 2$ rt. $\angle$s $- 8$ rt. $\angle$s Ax. 2

$= (n - 4) 2$ rt. $\angle$s.

BOOK II

Exercise 27. Page 102

1. *A radius that bisects an arc bisects its subtending chord and is perpendicular to it.*



Given the radius OC, bisecting the arc AB at C and intersecting the chord AB at D.

To prove that $AD = DB$ and that OC is $\perp$ to AB.

Proof. Draw the radii OA and OB.

In the $\triangle$ ADO and BDO, $OA = OB$, § 162

$OD = OD$, Iden.

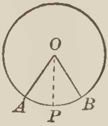
and $\angle AOD = \angle DOB$. § 167

$\therefore$ the $\triangle$ ADO and BDO are congruent. § 68

$\therefore AD = DB$, § 67

and OC is $\perp$ to AB, by § 176. Q.E.D.

2. *On a circle the point P is equidistant from two radii OA and OB. Prove that P bisects the arc AB.*



Given the point P on the ⊙ with center O equidistant from the radii OA and OB.

To prove that arc AP = arc PB.

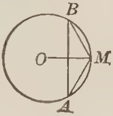
Proof. Draw OP.

Since P is equidistant from OA and OB, Given

$\therefore OP$ bisects the $\angle AOB$. § 152

$\therefore$ arc AP = arc PB, by § 166. Q.E.D.

3. *In this circle the chords AM and MB are equal. Prove that M bisects the arc AB and that the radius OM bisects the chord AB.*



Given the equal chords AM and MB, the chord AB, and the radius OM.

To prove that arc AM = arc MB,

and that OM bisects AB.

Proof. Since chord AM = chord MB,

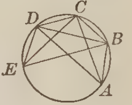
$\therefore$ arc AM = arc MB. § 172

The point O is equidistant from A and B, § 159

and the point M is equidistant from A and B. Given

$\therefore OM$ is the $\perp$ bisector of AB, by § 151. Q.E.D.

4. *On a circle are five points A, B, C, D, E so placed that AB, BC, CD, DE are equal chords. Prove that AC, BD, CE are equal chords, and that AD and BE are also equal chords.*



Given the equal chords AB, BC, CD, and DE, the chords AC, BD, and CE, and the chords AD and BE.

To prove that the chords AC, BD, and CE are equal, and that the chords AD and BE are equal.

Proof. The chords AB, BC, CD, and DE are equal. Given

$\therefore$ the arcs AB, BC, CD, and DE are equal. § 172

$\therefore$ arc AB + arc BC = arc BC + arc CD = arc CD + arc DE. Ax. 1

That is, arc AC = arc BD = arc CE.

$\therefore$ chord AC = chord BD = chord CE. § 170

Also arc AB + arc BC + arc CD = arc BC + arc CD + arc DE. Ax. 1

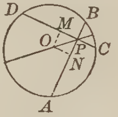
That is, arc AD = arc BE.

$\therefore$ chord AD = chord BE, by § 170. Q.E.D.

5. *If two chords intersect and make equal angles with the diameter through their point of intersection, these chords are equal.*

Given the chords AB and CD intersecting at P and making equal $\angle$s OPA and DPO with the diameter through P.

To prove that $AB = CD$.



Proof. Draw $ON \perp$ to AB and $OM \perp$ to CD.

In the rt. $\triangle$s ONP and OMP, $OP = OP$, Iden.

and $\angle OPN = \angle MPO$. Given

$\therefore$ rt. $\triangle$s ONP and OMP are congruent. § 91

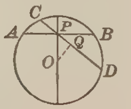
$\therefore ON = OM$. § 67

$\therefore AB = CD$, by § 178. Q.E.D.

Exercise 28. Page 104

1. *The shortest chord that can be drawn through a given point within a circle is that which is perpendicular to the diameter through the point.*

Given the point P within the ⊙, with center O, the chord AB through $P \perp$ to the diameter through P, and CD any other chord through P.



To prove that $AB < CD$.

Proof. Draw $OQ \perp$ to CD.

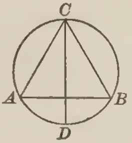
Now $OP > OQ$. § 86

$\therefore AB < CD$, by § 180. Q.E.D.

2. *The diameter CD bisects the arc AB. Prove that $\angle CBA = \angle BAC$.*

Given the diameter CD, bisecting the arc AB.

To prove that $\angle CBA = \angle BAC$.



Proof. Now arc CAD = arc CBD, § 175

and arc AD = arc BD. Given

$\therefore$ arc CAD − arc AD = arc CBD − arc BD. Ax. 2

That is, arc CA = arc CB.

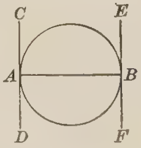
$\therefore CA = CB$. § 172

$\therefore \angle CBA = \angle BAC$, by § 74. Q.E.D.

3. *Tangents at the extremities of a diameter are parallel.*

Given CD and EF tangent to the ⊙ at A and B respectively, the extremities of the diameter AB.

To prove that CD is $\parallel$ to EF.



Proof. CD is $\perp$ to AB,

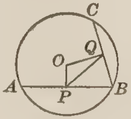
and EF is $\perp$ to AB. § 185

$\therefore CD$ is $\parallel$ to EF, by § 95. Q.E.D.

4. *The arc AB is greater than the arc BC. OP and OQ are perpendiculars from the center to AB and BC respectively. Prove that $\angle QPO$ is greater than $\angle OQP$.*

Given the arc AB greater than the arc BC, and OP and OQ perpendiculars from the center to the chords AB and BC respectively.

To prove that $\angle QPO > \angle OQP$.



Proof. Since arc $AB >$ arc BC, Given

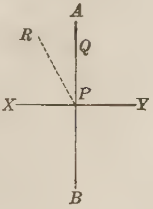
$\therefore$ chord $AB >$ chord BC. § 170

$\therefore OQ > OP$. § 179

Then in the $\triangle OPQ$,

$\angle QPO > \angle OQP$, by § 113. Q.E.D.

5. *What is the locus of the center of a circle tangent to the line XY at the point P? Prove it.*



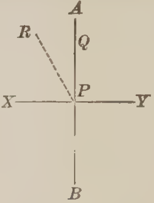
The locus of the center of a circle tangent to the line XY at the point P is the line perpendicular to XY at P.

Given the point P in the line XY, and the line $AB \perp$ to XY at P.

To prove that AB is the locus of the center of a ⊙ tangent to XY at P.

Proof. Let Q be any point on AB and R be any point not on AB.

Suppose a ⊙ drawn with Q as center and QP as radius.



Then XY is tangent to this ⊙. § 184

Suppose a ⊙ drawn with R as center and RP as radius.

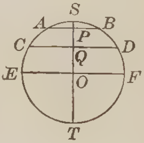
Since R is not on the line AB, Hyp.

RP is not ⊥ to XY. § 57

Then XY is not tangent to this ⊙. § 185

Therefore AB is the required locus, by § 148. Q. E. D.

6. *What is the locus of the mid-points of a number of parallel chords of a circle? Prove it.*



The locus of the mid-points of a number of parallel chords of a circle is the diameter perpendicular to the parallel chords.

Given the ⊙ with center O, and the parallel chords AB, CD, EF whose mid-points are P, Q, O respectively, and given the diameter $ST \perp$ to AB.

To prove that ST is the locus of the mid-points of the parallel chords.

Proof. Since ST is ⊥ to AB, Given

and CD and EF are ∥ to AB, Given

ST is also ⊥ to CD and EF. § 97

Now ST bisects the chords AB, CD, and EF. § 174

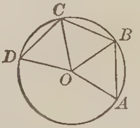
Similarly the mid-point of every other chord ∥ to AB must lie in ST, the ⊥ bisector of each one of these ∥ chords.

Therefore ST is the required locus. Q. E. D.

7. *Three equal chords, AB, BC, CD, are placed end to end, and the radii OA, OB, OC, OD are drawn. Prove that $\angle AOC = \angle BOD$.*

Given the equal chords AB, BC, and CD, and the radii OA, OB, OC, and OD.

To prove that $\angle AOC = \angle BOD$.



Proof. Since $AB = BC = CD$, Given

$\therefore$ arc $AB =$ arc $BC =$ arc CD. § 172

$\therefore$ arc $AB +$ arc $BC =$ arc $BC +$ arc CD Ax. 1

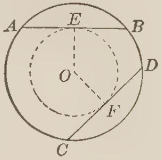
That is, arc $AC =$ arc BD.

$\therefore \angle AOC = \angle BOD$, by § 167. Q. E. D.

8. *All equal chords of a circle are tangent to a concentric circle.*

Given AB and CD equal chords of the ⊙ with center O.

To prove that AB and CD are tangent to a concentric ⊙.



Proof. Draw $OE \perp$ to AB, and $OF \perp$ to CD.

Then $OE = OF$. § 178

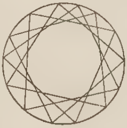
Also OE is the shortest line from O to AB,

and OF is the shortest line from O to CD. § 86

∴ the ⊙ described with center O and radius OE is tangent to the chords AB and CD. § 160

That is, AB and CD are tangent to this ⊙. Q.E.D.

9. *If a number of equal chords are drawn in this circle, the figure gives the impression of a second circle inside the first and concentric with it. Explain the reason.*



Since the chords are all equal, they are equally distant from the center of the circle. The chords form a polygon in which the distance from the center to each side is constant and the large number of chords makes each side of the polygon short. Hence each side of the polygon gives the impression of being nearly an arc of the circle. If the number of chords is greatly increased, the number of sides of the polygon is also greatly increased, and the polygon becomes nearer and nearer to a circle, and may be made so near a circle that the eye cannot detect the difference.

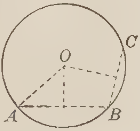
Exercise 29. Page 108

1. *Show that the reasoning of* § 190 *will not hold for four points, and hence that a circle cannot always be drawn through four points.*

The reasoning of § 190 is given thus:

A, B, and C are the three given points not in a straight line.

AB and BC are drawn.

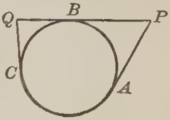


The $\perp$ bisectors of AB and BC are drawn and shown to intersect at a point O, which is the center of the circle required.

If now the circle is to pass through a fourth given point D, that point D must lie at the distance OA from O, or the circle will not pass through D.

Hence, in general, D will lie outside or inside the circle, and a circle cannot be drawn through four given points.

2. *Tangents to a circle at A, B, C, points on the circle, meet in P and Q, as here shown. Prove that $AP + QC = PQ$.*



Given AP, PQ, and QC, tangents to the $\odot ABC$ at the points A, B, and C respectively.

To prove that $AP + QC = PQ$.

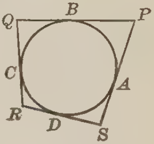
Proof, $AP = BP$,

and $QC = QB$. § 192

$\therefore AP + QC = BP + QB$. Ax. 1

That is, $AP + QC = PQ$. Q.E.D.

3. *If a quadrilateral has each side tangent to a circle, the sum of one pair of opposite sides equals the sum of the other pair.*



Given the quadrilateral $RSPQ$, with its sides tangent to the $\odot$ in the points D, A, B, C respectively.

To prove that $SP + QR = PQ + RS$.

Proof. Now $PA = PB$,

$SA = SD$,

$RC = RD$,

and $QC = QB$. § 192

$\therefore PA + SA + RC + QC = PB + SD + RD + QB$. Ax. 1

But $PA + SA = SP$,

$RC + QC = QR$,

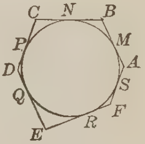
$PB + QB = PQ$,

and $SD + RD = RS$. Ax. 11

$\therefore SP + QR = PQ + RS$, by Ax. 9 Q.E.D.

4. *The hexagon here shown has each side tangent to the circle. Prove that $AB + CD + EF = BC + DE + FA$.*

Given the hexagon $ABCDEF$, with its sides tangent to the $\odot$ in the points M, N, P, Q, R, S respectively.



To prove that $AB + CD + EF = BC + DE + FA$.

Proof. Now $AM = AS$,

$BM = BN$,

$CP = CN$,

$DP = DQ$,

$ER = EQ$,

and $FR = FS$. § 192

$\therefore AM + BM + CP + DP + ER + FR = AS + BN + CN + DQ + EQ + FS$. Ax. 1

But $AM + BM = AB$,

$CP + DP = CD$,

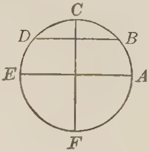
$ER + FR = EF$,

$BN + CN = BC,$
$DQ + EQ = DE,$
and $AS + FS = FA.$ Ax. 11
$\therefore AB + CD + EF = BC + DE + FA$, by Ax. 9. Q.E.D.

5. *In this figure CF is a diameter perpendicular to the parallel chords DB and EA, and arc $AB = 40°$ and arc $BC = 50°$. How many degrees are there in arcs CD, DE, EF, and FA?*

Given the diameter $CF \perp$ to the $\parallel$ chords DB and EA, arc $AB = 40°$, and arc $BC = 50°$.

Required to find the number of degrees in arcs CD, DE, EF, and FA.

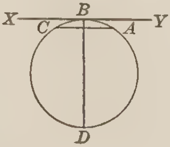


Solution. Now arc CD = arc BC. § 174
But arc $BC = 50°$. Given
$\therefore$ arc $CD = 50°$. Ax. 8
Also arc DE = arc AB. § 189
But arc $AB = 40°$. Given
$\therefore$ arc $DE = 40°$. Ax. 8
Also arc EF = arc FA. § 174
Now arc AB + arc BC + arc CD + arc DE + arc EF + arc $FA = 360°$. § 44
$40° + 50° + 50° + 40°$ + arc EF + arc $EF = 360°$. Ax. 9
2 arc $EF = 180°$.
$\therefore$ arc $EF = 90°$. Ax. 3
$\therefore$ arc $FA = 90°$. Ax. 8
50°, 40°, 90°, 90°. *Ans.*

6. *In this figure XY is tangent to the circle at B, the chord CA is perpendicular to the diameter BD, and the arc $CD = 150°$. How many degrees are there in arc AB?*

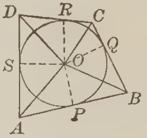
Given XY tangent to the $\odot$ at B, the chord $CA \perp$ to the diameter BD, and arc $CD = 150°$.

Required to find the number of degrees in arc AB.



Solution. BD is $\perp$ to XY at B. § 185
$\therefore XY$ is $\parallel$ to CA. § 95
$\therefore$ arc AB = arc BC. § 189
Now arc BCD = arc BC + arc CD. Ax. 11
arc BCD = arc AB + arc CD. Ax. 9
arc BCD = arc $DAB = 180°$. § 175
arc $CD = 150°$. Given
$\therefore 180°$ = arc $AB + 150°$. Ax. 9
That is, arc $AB = 30°$. 30°. *Ans.*

7. *If a quadrilateral has each side tangent to a circle, the sum of the angles at the center subtended by any two opposite sides is equal to a straight angle.*



Given the quadrilateral $ABCD$, with its sides tangent to the ⊙ with center O at the points P, Q, R, and S respectively, and the $\angle$s at the center AOB, BOC, COD, and DOA subtended by the sides AB, BC, CD, and DA respectively.

To prove that $\angle BOC + \angle DOA = 1$ st. $\angle$,
and that $\angle AOB + \angle COD = 1$ st. $\angle$.

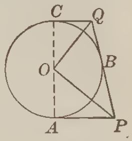
Proof. Draw the radii OP, OQ, OR, and OS.

Then in the $\triangle$s OQC and ORC, $OC = OC$, Iden.
$OQ = OR$, § 162
and $CQ = CR$. § 192
$\therefore$ the $\triangle$s OQC and ORC are congruent. § 80
$\therefore \angle QOC = \angle COR$. § 67
Similarly $\angle BOQ = \angle POB$,
$\angle SOA = \angle AOP$,
and $\angle DOS = \angle ROD$.
$\therefore \angle QOC + \angle BOQ + \angle SOA + \angle DOS = \angle COR + \angle POB + \angle AOP + \angle ROD$. Ax. 1
$\therefore \angle BOC + \angle DOA = \angle AOB + \angle COD$. Ax. 9
But $\angle BOC + \angle DOA + \angle AOB + \angle COD = 2$ st. $\angle$s. § 41
$\therefore \angle BOC + \angle DOA = 1$ st. $\angle$,
and $\angle AOB + \angle COD = 1$ st. $\angle$. Q.E.D.

8. *AP and CQ are parallel tangents meeting a third tangent QP, as shown in the figure. O being the center, prove that the angle POQ is a right angle.*

Given the ∥ tangents AP and CQ meeting the third tangent in P and Q, and the lines OP and OQ drawn from the center of the ⊙.

To prove that $\angle POQ = 1$ rt. $\angle$.



Proof. Draw the radii OA and OC.
Since OA is $\perp$ to AP, § 185
$\therefore$ OA produced is $\perp$ to CQ. § 97
But OC is $\perp$ to CQ. § 185
$\therefore$ OC and OA produced coincide. § 82
That is, AO and OC form a straight line.
Then in the quadrilateral $APQC$,
$\angle PAC + \angle QPA + \angle CQP + \angle ACQ = 4$ rt. $\angle$s. § 144

But $\angle$s PAC and ACQ are rt. $\angle$s. § 185

$\therefore \angle QPA + \angle CQP = 2$ rt. $\angle$s. Ax. 2

Now $\angle QPO = \frac{1}{2} \angle QPA$,

and $\angle OQP = \frac{1}{2} \angle CQP$. § 192

$\therefore \angle QPO + \angle OQP = 1$ rt. $\angle$. Ax. 4

But $\angle POQ + \angle QPO + \angle OQP = 2$ rt. $\angle$s. § 107

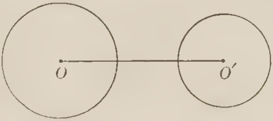
$\therefore \angle POQ + 1$ rt. $\angle = 2$ rt. $\angle$s. Ax. 9

$\therefore \angle POQ = 1$ rt. $\angle$, by Ax. 2. Q. E. D.

Exercise 30. Page 109

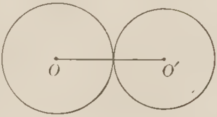
1. *Describe the relative position of two circles if the line-segment joining the centers is greater than the sum of the radii, and illustrate by a figure.*

If the line-segment joining the centers O and O' of two circles is greater than the sum of the radii, the circles must lie each wholly outside the other, as shown in the figure.



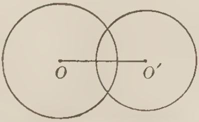
2. *Describe the relative position of two circles if the line-segment joining the centers is equal to the sum of the radii, and illustrate by a figure.*

If the line-segment joining the centers O and O' of two circles is equal to the sum of the radii, the circles must be tangent externally to each other, as shown in the figure.



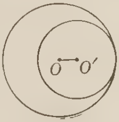
3. *Describe the relative position of two circles if the line-segment joining the centers is less than the sum but greater than the difference of the radii, and illustrate by a figure.*

If the line-segment joining the centers O and O' of two circles is less than the sum but greater than the difference of the radii, the circles must intersect each other, as shown in the figure.



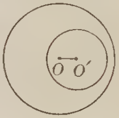
4. *Describe the relative position of two circles if the line-segment joining the centers is equal to the difference of the radii, and illustrate by a figure.*

If the line-segment joining the centers O and O' of two circles is equal to the difference of the radii, the circles must be tangent internally to each other, as shown in the figure.



5. *Describe the relative position of two circles if the line-segment joining the centers is less than the difference of the radii, and illustrate by a figure.*

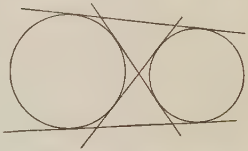
If the line-segment joining the centers O and O' of two circles is less than the difference of the radii, the smaller circle must lie wholly within the larger, as shown in the figure.



Exercise 31. Page 110

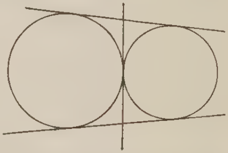
1. *Describe the relative position of two circles having two common external and two common internal tangents, and illustrate by a figure.*

If two circles have two common external and two common internal tangents, the circles must be each wholly outside the other, as shown in the figure.



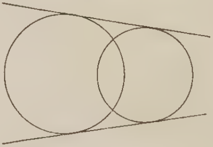
2. *Describe the relative position of two circles having two common external tangents and one common internal tangent, and illustrate by a figure.*

If two circles have two common external tangents and one common internal tangent, the circles must be tangent to each other externally, as shown in the figure.



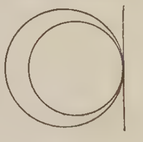
3. *Describe the relative position of two circles having two common external tangents and no common internal tangent, and illustrate by a figure.*

If two circles have two common external tangents and no common internal tangent, the circles must intersect each other, as shown in the figure.



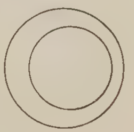
4. *Describe the relative position of two circles having one common external and no common internal tangent, and illustrate by a figure.*

If two circles have one common external and no common internal tangent, the two circles must be tangent to each other internally, as shown in the figure.

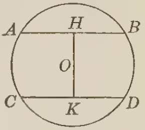


5. *Describe the relative position of two circles having no common tangent, and illustrate by a figure.*

If two circles have no common tangent, the two circles must lie one wholly within the other, as shown in the figure.



6. *The line which passes through the mid-points of two parallel chords passes through the center of the circle.*



Given the ∥ chords AB and CD, O the center of the ⊙, and HK, the line passing through H and K, the mid-points of AB and CD respectively.

To prove that HK passes through O.

Proof. Draw OH and OK.

OH is $\perp$ to AB, and OK is $\perp$ to CD. § 176

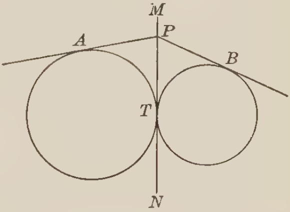
But OH is $\perp$ to CD. § 97

$\therefore$ OH and OK coincide and form one straight line. § 82

$\therefore$ the line HK coincides with this line. Post. 1

That is, HK passes through O. Q.E.D.

7. *If two circles are tangent externally, the tangents to them from any point of the common internal tangent are equal.*



Given MN, the common internal tangent to the two ⓢ tangent externally at the point T, and PA and PB tangents to the ⓢ from P, any point in MN.

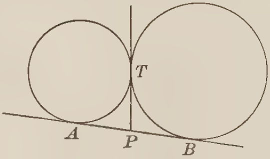
To prove that $PA = PB$.

Proof. Now $PA = PT$,

and $PB = PT$. § 192

$\therefore PA = PB$, by Ax. 8. Q.E.D.

8. *If two circles tangent externally are tangent to a line AB at A and B, their common internal tangent bisects AB.*



Given the two ⓢ, tangent externally at T, and tangent to the line AB at A and B respectively, and TP, the common internal tangent intersecting AB at P.

To prove that $PA = PB$.

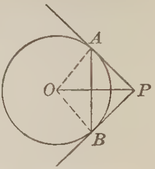
Proof. Now $PA = PT$,

and $PB = PT$. § 192

$\therefore PA = PB$, by Ax. 8. Q.E.D.

9. *The line drawn from the center of a circle to the point of intersection of two tangents is the perpendicular bisector of the chord joining the points of contact.*

Given PA and PB, tangents from a point P to the ⊙ with center O, AB the chord joining the points of contact, and OP drawn.



To prove that OP is the ⊥ bisector of AB.

Proof. Draw the radii OA and OB.

Then $OA = OB$, § 162
and $PA = PB$. § 192

∴ OP is the ⊥ bisector of AB, by § 151. Q.E.D.

10. *The diameters of two circles are respectively* 2.74 *in. and* 3.48 *in. Find the distance between the centers of the circles if they are tangent externally. Find the distance between the centers of the circles if they are tangent internally.*

Solution. The radius of the first ⊙ = $\frac{1}{2}$ of 2.74 in. = 1.37 in.

The radius of the second ⊙ = $\frac{1}{2}$ of 3.48 in. = 1.74 in.

If the Ⓢ are tangent externally, the distance between the centers = the sum of the radii = 1.37 in. + 1.74 in. = 3.11 in.

If the Ⓢ are tangent internally, the distance between the centers = the difference between the radii = 1.74 in. − 1.37 in. = 0.37 in.

11. *Three circles of diameters* 4.8 *in.,* 3.6 *in., and* 4.2 *in. are externally tangent, each to the other two. Find the perimeter of the triangle formed by joining the centers.*

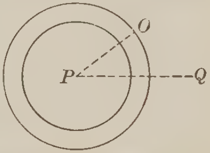
Solution. The radii of the Ⓢ are $\frac{1}{2}$ of 4.8 in., or 2.4 in.; $\frac{1}{2}$ of 3.6 in., or 1.8 in.; and $\frac{1}{2}$ of 4.2 in., or 2.1 in.

The sides of the △ are 2.4 in. + 1.8 in., or 4.2 in.; 1.8 in. + 2.1 in., or 3.9 in.; and 2.1 in. + 2.4 in., or 4.5 in.

∴ perimeter of △ = 4.2 in. + 3.9 in. + 4.5 in. = 12.6 in.

12. *A circle of center O and radius r' rolls around a fixed circle of radius r. What is the locus of O? Prove it.*

The required locus is the circle concentric with the given circle and with radius equal to $r + r'$.



Given the concentric Ⓢ with centers at P, and with radii r and $r + r'$.

To prove that the ⊙ with radius $r + r'$ is the locus of the center of a ⊙ with radius r' that rolls around the ⊙ with radius r.

Proof. Let O be any point on the ⊙ with radius $r + r'$, and let Q be any point not on that ⊙.

Draw PO and PQ.

The ⊙ with O as center and r' as radius is tangent to the ⊙ with radius r.

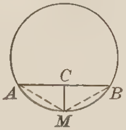
The ⊙ with Q as center and r' as radius either does not cut or touch the ⊙ with radius r, or does cut it, according as $PQ > r + r'$ or $PQ < r + r'$.

∴ the ⊙ with center P and radius $r + r'$ is the required locus, by § 148. Q.E.D.

Discussion. If the rolling circle is supposed to roll around the fixed circle within that circle, the locus is evidently the circle with center P and radius $r - r'$.

13. *The line drawn from the mid-point of a chord to the mid-point of its subtended arc is perpendicular to the chord.*

Given the line CM drawn from C, the mid-point of the chord AB, to M, the mid-point of the arc AB.



To prove that CM is ⊥ to AB.

Proof. Draw AM and BM.

Since arc AM = arc BM, Given

$\therefore AM = BM$. § 170

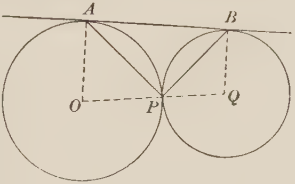
Also $AC = BC$. Given

$\therefore CM$ is the ⊥ bisector of AB. § 151

That is, CM is ⊥ to AB. Q.E.D.

14. *If two circles tangent externally at P are tangent to a line AB at A and B, the angle BPA is a right angle.*

Given the two Ⓢ, with centers O and Q, tangent externally at the point P, and tangent to the line AB at the points A and B, and the chords PA and PB.



To prove that $\angle BPA = 1$ rt. ∠.

Proof. Draw the radii OA and QB, and the line of centers OQ.

Then OQ passes through P. § 197

Since $OA = OP$, § 162

$\therefore \angle APO = \angle OAP$. § 74

Since $QP = QB$, § 162

$\therefore \angle PBQ = \angle QPB$. § 74

Now $\angle APO + \angle BPA + \angle QPB = 180°$, § 33

and $\angle BPA + \angle ABP + \angle PAB = 180°$. § 107

$\therefore \angle APO + \angle PAB + \angle QPB + \angle ABP + 2\angle BPA = 360°$. Ax. 1

$\therefore \angle OAP + \angle PAB + \angle PBQ + \angle ABP + 2\angle BPA = 360°$. Ax. 9

But $\angle OAP + \angle PAB = 90°$,

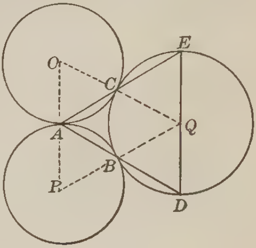
and $\angle PBQ + \angle ABP = 90°$. § 185

$\therefore 90° + 90° + 2\angle PBA = 360°$. Ax. 9

$\therefore 2\angle PBA = 180°$. Ax. 2

$\therefore \angle PBA = 90°$, by Ax. 4. Q.E.D.

15. *Three circles are tangent externally at the points A, B, and C, and the chords AB and AC are produced to cut the circle BC at D and E. Prove that DE is a diameter.*



Given the three Ⓢ with centers O, P, and Q, tangent externally at the points A, B, and C.

Given the chords AB and AC produced to cut the ⊙ with center Q in the points D and E, and the chord DE drawn.

To prove that the chord DE is a diameter.

Proof. Draw the lines of centers OP, PQ, and QO.

Then OP passes through A,

PQ passes through B,

and QO passes through C. § 197

Draw QE and QD.

Since $QE = QC$, § 162

$\therefore \angle QCE = \angle CEQ$. § 74

$OC = OA$. § 162

$\therefore \angle CAO = \angle OCA$. § 74

But $\angle QCE = \angle OCA$. § 60

$\therefore \angle CEQ = \angle CAO$. Ax. 8

$\therefore EQ$ is ∥ to OP. § 101

Similarly, $\angle BDQ = \angle BAP$,

and QD is ∥ to OP.

But EQ and QD pass through Q. Const.

$\therefore EQ$ and QD form a straight line. § 94

Since Q is the center of the ⊙ BC, Given

$\therefore ED$ is a diameter, by § 163. Q.E.D.

16. *If two radii of a circle, at right angles to each other, when produced are cut by a tangent to the circle at A and B, the other tangents from A and B are parallel to each other.*

Given the ⊥ radii OM and ON produced to meet the tangent AB at A and B respectively, and AD and BC the other tangents to the ⊙ from A and B.

To prove that AD and BC are ∥.

Proof. Draw the radii OD and OC.

Then OD is $\perp$ to AD, and OC is $\perp$ to BC. § 185

Now $\angle OAD = \angle BAO$,

and $\angle CBO = \angle OBA$. § 192

$\therefore \angle OAD + \angle CBO = \angle BAO + \angle OBA$. Ax. 1

Since $\angle BAO + \angle OBA = 1$ rt. $\angle$, § 107

$\therefore \angle OAD + \angle CBO = 1$ rt. $\angle$. Ax. 8

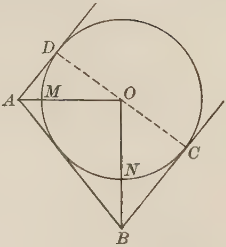
Now $\angle OAD = 1$ rt. $\angle - \angle DOA$,

and $\angle CBO = 1$ rt. $\angle - \angle BOC$. § 107

$\therefore 1$ rt. $\angle - \angle DOA + 1$ rt. $\angle - \angle BOC$

$= 1$ rt. $\angle$. Ax. 9

$\therefore \angle DOA + \angle BOC = 1$ rt. $\angle$. Ax. 2



But $\angle AOB = 1$ rt. $\angle$. Given

$\therefore \angle DOA + \angle AOB + \angle BOC = 2$ rt. $\measuredangle$. Ax. 1

$\therefore$ DOC is a straight line. § 33

$\therefore$ AD and BC are $\parallel$, by § 95. Q. E. D.

17. *If two common external tangents or two common internal tangents are drawn to two circles, the line-segments intercepted between the points of contact are equal.*

Given AB and CD, two common external tangents to the two ⓢ, and EF and HG, two common internal tangents.

To prove that $AB = CD$, and $EF = HG$.

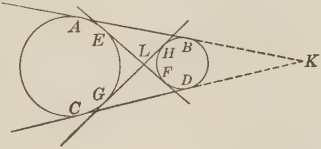
Proof. Produce AB and CD to meet at K.

Then $KA = KC$,

and $KB = KD$. § 192

$\therefore KA - KB = KC - KD$. Ax. 2

That is, $AB = CD$.



Let L be the point of meeting of EF and HG.

Then $LE = LG$,

and $LF = LH$. § 192

$\therefore LE + LF = LG + LH$. Ax. 1

That is, $EF = HG$. Q. E. D.

Exercise 32. Page 113

1. *Find a common measure of* 32 *in. and* 24 *in.*

Since 32 in. and 24 in. each contains 8 in. an integral number of times, 8 in. is a common measure of 32 in. and 24 in.

2. *Find a common measure of* 48 *ft. and* 18 *ft.*

Since 48 ft. and 18 ft. each contains 3 ft. an integral number of times, 3 ft. is a common measure of 48 ft. and 18 ft.

3. *Find a common measure of* $5\frac{1}{2}$ *in. and* $3\frac{1}{2}$ *in.*

Since $5\frac{1}{2}$ in. and $3\frac{1}{2}$ in. each contains $\frac{1}{2}$ in. an integral number of times,

$\frac{1}{2}$ in. is a common measure of $5\frac{1}{2}$ in. and $3\frac{1}{2}$ in.

4. *Find a common measure of* $2\frac{3}{4}$ *lb. and* $1\frac{1}{2}$ *lb.*

Since $2\frac{3}{4}$ lb. and $1\frac{1}{2}$ lb. each contains $\frac{1}{4}$ lb. an integral number of times,

$\frac{1}{4}$ lb. is a common measure of $2\frac{3}{4}$ lb. and $1\frac{1}{2}$ lb.

5. *Find a common measure of* $6\frac{1}{3}$ *da. and* $2\frac{2}{3}$ *da.*

Since $6\frac{1}{3}$ da. and $2\frac{2}{3}$ da. each contains $\frac{1}{3}$ da. an integral number of times,

$\frac{1}{3}$ da. is a common measure of $6\frac{1}{3}$ da. and $2\frac{2}{3}$ da.

6. *Find a common measure of* 14.4 *in. and* 1.2 *in.*

Since 14.4 in. and 1.2 in. each contains 1.2 in. an integral number of times,

1.2 in. is a common measure of 14.4 in. and 1.2 in.

7. *Find the greatest common measure of* 64 *yd. and* 24 *yd.*

$$64 \text{ yd.} = 2 \times 2 \times 2 \times 2 \times 2 \times 2 \text{ yd.}$$
$$24 \text{ yd.} = 2 \times 2 \times 2 \times 3 \text{ yd.}$$

Therefore the greatest common measure is $2 \times 2 \times 2$ yd., or 8 yd.

8. *Find the greatest common measure of* 51 *ft. and* 17 *ft.*

$$51 \text{ ft.} = 3 \times 17 \text{ ft.}$$
$$17 \text{ ft.} = 17 \text{ ft.}$$

Therefore the greatest common measure is 17 ft.

9. *Find the greatest common measure of* 7.5 *in. and* 1.25 *in.*

$$7.5 \text{ in.} = 7.50 \text{ in.} = 2 \times 3 \times 5 \times 5 \times 5 \times 0.01 \text{ in.}$$
$$1.25 \text{ in.} = 5 \times 5 \times 5 \times 0.01 \text{ in.}$$

Therefore the greatest common measure is $5 \times 5 \times 5 \times 0.01$ in., or 1.25 in.

10. *Find the greatest common measure of* $3\frac{1}{3}$ *in. and* $0.33\frac{1}{3}$ *in.*

$$3\tfrac{1}{3} \text{ in.} = \tfrac{10}{3} \text{ in.} = 2 \times 5 \times \tfrac{1}{3} \text{ in.}$$
$$0.33\tfrac{1}{3} \text{ in.} = \tfrac{1}{3} \text{ in.}$$

Therefore the greatest common measure is $\frac{1}{3}$ in.

11. *Find the greatest common measure of* $2\frac{3}{4}$ *ft. and* 0.25 *ft.*

$$2\tfrac{3}{4}\text{ ft.} = \tfrac{11}{4}\text{ ft.} = 11 \times \tfrac{1}{4}\text{ ft.}$$
$$0.25\text{ ft.} = \tfrac{1}{4}\text{ ft.}$$

Therefore the greatest common measure is $\frac{1}{4}$ ft.

12. *Find the greatest common measure of* $75°$ *and* $7° 30'$.

$$75° = 3 \times 5 \times 5°.$$
$$7° 30' = 7.5° = 3 \times 5 \times 5 \times 0.1°.$$

Therefore the greatest common measure is $3 \times 5 \times 5 \times 0.1°$, or $7.5°$, or $7° 30'$.

13. *If* $a : b = \sqrt{3}$, *find an approximate value of this ratio that shall differ from the true value by less than* 0.001.

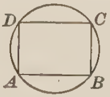
Since the $\sqrt{3} = 1.73205 \cdots$,
1.732 is an approximate value of $a : b$ that differs from the true value by less than 0.001.

Exercise 33. Page 120

1. *A parallelogram inscribed in a circle is a rectangle.*

Given the ▱ $ABCD$ inscribed in a ⊙.

To prove that $ABCD$ is a ▭.

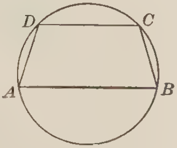


Proof. Now $AB = DC$,
and $BC = AD$. § 125
$\therefore$ arc AB = arc DC,
and arc BC = arc AD. § 172
$\therefore$ arc AB + arc BC = arc DC + arc AD. Ax. 1
$\therefore$ arc ABC = arc CDA.
$\therefore \angle D = \angle B = 90°$. § 215
Similarly $\angle A = \angle C = 90°$.
$\therefore$ $ABCD$ is a ▭ by § 119. Q.E.D.

2. *A trapezoid inscribed in a circle is isosceles.*

Given the trapezoid $ABCD$ inscribed in the ⊙, and $AB \parallel$ to DC.

To prove that $ABCD$ is isosceles.



Proof. Since AB and DC are $\parallel$, Given
$\therefore$ arc AD = arc BC. § 189
$\therefore AD = BC$. § 170
$\therefore$ the trapezoid $ABCD$ is isosceles, by § 118. Q.E.D.

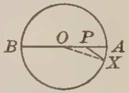
3. *The shorter segment of the diameter through a given point within a circle is the shortest line that can be drawn from that point to the circle.*

Given PA, the shorter segment of the diameter through the point P.

To prove that PA is the shortest line from P to the ⊙.

Proof. Draw any other line PX from P to the ⊙.

Draw the radius OX.



Now $OX < OP + PX$. § 112

But $OX = OA$. § 162

$\therefore OA < OP + PX$. Ax. 9

That is, $OP + PA < OP + PX$.

$\therefore PA < PX$, by Ax. 6. Q.E.D.

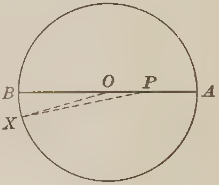
4. *The longer segment of the diameter through a given point within a circle is the longest line that can be drawn from that point to the circle.*

Given the point P within the ⊙, and AB, the diameter through P.

To prove that PB is the longest line that can be drawn from P to the ⊙.

Proof. Draw any other line PX from P to the ⊙.

Draw the radius OX.



Now $PO + OX > PX$. § 112

But $OB = OX$. § 162

$\therefore PO + OB > PX$. Ax. 9

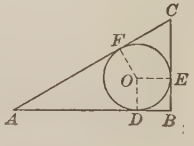
That is, $PB > PX$. Q.E.D.

5. *The diameter of the circle inscribed in a right triangle is equal to the difference between the hypotenuse and the sum of the other two sides.*

Given the ⊙ with center O inscribed in the rt. △ ABC, with rt. ∠ at B, and D, E, and F the points of contact.

To prove that the diameter of the ⊙ $= AB + BC - AC$.

Proof. Draw the radii OD, OE, and OF.



Then OD is ⊥ to AB, and OE is ⊥ to BC. § 185

$\therefore OD$ is ∥ to CB, and OE is ∥ to AB. § 95

$\therefore OD = BE$, and $OE = BD$. § 127

Now $EC = FC$, and $AD = AF$. § 192

$\therefore OD + OE = BE + BD$ Ax. 1

$$= (BC - EC) + (AB - AD)$$

$$= BC - FC + AB - AF$$

$$= AB + BC - (AF + FC)$$

$$= AB + BC - AC.$$

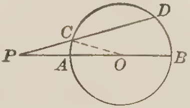
That is, the diameter of the ⊙ $= AB + BC - AC$. Q.E.D.

6. *A line from a given point outside a circle passing through the center contains the shortest line-segment that can be drawn from that point to the circle.*

Given the ⊙ with center O, P the point outside the ⊙, PAB a secant passing through O, and PCD any other secant from P.

To prove that $PA < PC$.

Proof. Draw the radius CO.



Now $PA + AO < PC + CO$. § 112

But $AO = CO$. § 162

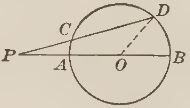
$\therefore$ $PA < PC$, by Ax. 6. Q.E.D.

7. *A line from a given point outside a circle passing through the center contains the longest line-segment (to the concave arc) that can be drawn from that point to the circle.*

Given the ⊙ with center O, P the point outside the ⊙, PAB a secant passing through O, and PCD any other secant from P.

To prove that $PB > PD$.

Proof. Draw the radius OD.



Now $PO + OD > PD$. § 112

But $OB = OD$. § 162

$\therefore$ $PO + OB > PD$. Ax. 9

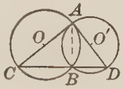
That is, $PB > PD$. Q.E.D.

8. *Through one of the points of intersection of two circles a diameter of each circle is drawn. Prove that the line joining the ends of the diameters passes through the other point of intersection.*

Given the ⓢ with centers O and O' intersecting in the points A and B, AC and AD the diameters, and the line CD.

To prove that CD passes through B.

Proof. Draw AB, BC, and BD.

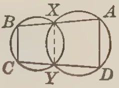


Now $\angle$s ABC and DBA are rt. $\angle$s. § 215

$\therefore$ CBD is a straight line. § 43

That is, CD passes through B. Q.E.D.

9. *If two circles intersect and a line is drawn through each point of intersection terminated by the circles, the chords joining the corresponding ends of these lines are parallel.*



Given the two ⓢ intersecting at X and Y, the lines BXA and CYD drawn through X and Y, and the chords BC and AD.

To prove that BC is ‖ to AD.

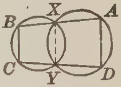
Proof. Draw the common chord XY.

Now $\angle XAD$ is measured by $\frac{1}{2}$ arc XYD,

and $\angle DYX$ is measured by $\frac{1}{2}$ arc DAX. § 214

Since arc XYD + arc $DAX = 360°$,

$\therefore \angle XAD = 180° - \angle DYX$. Ax. 9



Similarly $\angle CBX = 180° - \angle XYC$.

But $\angle DYX = 180° - \angle XYC$. § 43

$\therefore \angle DYX = \angle CBX$, Ax. 8

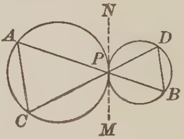
and $\angle XAD = 180° - \angle CBX$. Ax. 9

$\therefore \angle XAD + \angle CBX = 180°$. Ax. 1

$\therefore CB$ is $\parallel$ to DA, by § 105. Q.E.D.

Exercise 34. Page 124

1. *If two circles touch each other and two lines are drawn through the point of contact terminated by the circles, the chords joining the ends of these lines are parallel.*



Given the two ⓢ tangent at P, AB and CD lines drawn through P terminated by the ⓢ, and the chords AC and BD.

To prove that AC is $\parallel$ to BD.

Proof. Through P draw the common tangent MN.

Now $\angle A$ is measured by $\frac{1}{2}$ arc CP, § 214

and $\angle CPM$ is measured by $\frac{1}{2}$ arc CP. § 220

$\therefore \angle A = \angle CPM$.

Again, $\angle B$ is measured by $\frac{1}{2}$ arc DP, § 214

and $\angle DPN$ is measured by $\frac{1}{2}$ arc DP. § 220

$\therefore \angle B = \angle DPN$.

But $\angle CPM = \angle DPN$. § 60

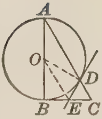
$\therefore \angle A = \angle B$. Ax. 8

$\therefore AC$ is $\parallel$ to DB, by § 101. Q.E.D.

2. *If one side of a right triangle is the diameter of a circle, the tangent at the point where the circle cuts the hypotenuse bisects the other side.*

Given the ⊙ with center O described on AB as diameter, AB being one side of the rt. △ with rt. ∠ at B, and DE, cutting BC at E, tangent to the ⊙ at D, the point where the hypotenuse cuts the ⊙.

To prove that $BE = EC$.



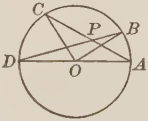
Proof. Draw OE and OD.
Then EB is tangent at B. § 184
$\therefore \angle BOE = \angle EOD$. § 192
But $\angle BOD = \angle OAD + \angle ADO$. § 111
Since $OD = OA$, § 162
$\therefore \angle OAD = \angle ADO$. § 74
$\therefore \angle BOE = \angle OAD$. Ax. 4
$\therefore$ OE is ∥ to AC. § 103
Since $OA = OB$, § 162
$\therefore BE = EC$, by § 135. Q. E. D.

3. *If from the extremities of a diameter AD two chords, AC and DB, are drawn, intersecting at P so as to make $\angle APB = 45°$, then $\angle BOC$ is a right angle.*

Given the ⊙ with center O, AD a diameter, AC and DB chords so drawn that $\angle APB = 45°$, and the radii OB and OC drawn.

To prove that $\angle BOC = 1$ rt. $\angle$.

Proof. Now $\angle APB$ is measured by $\frac{1}{2}$(arc AB + arc CD),
and $\angle CPD$ is measured by $\frac{1}{2}$(arc AB + arc CD). § 219
$\therefore \angle APB + \angle CPD$ is measured by arc AB + arc CD. Ax. 1
Now $\angle CPD = \angle APB$. § 60
But $\angle APB = 45°$. Given
$\therefore \angle CPD = 45°$. Ax. 8
$\therefore \angle APB + \angle CPD = 90°$. Ax. 1
$\therefore$ arc AB + arc $CD = 90°$.
Now arc AB + arc BC + arc $CD = 180°$. Given
$\therefore 90°$ + arc $BC = 180°$ Ax. 9
$\therefore$ arc $BC = 90°$. Ax. 2
Now $\angle BOC$ is measured by arc BC. § 213
$\therefore \angle BOC = 1$ rt. $\angle$. Q. E. D.

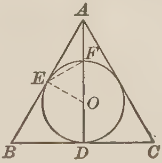


4. *The radius of the circle inscribed in an equilateral triangle is equal to one third the altitude of the triangle.*

Given the ⊙ with center O inscribed in the equilateral $\triangle ABC$, D and E the points of contact of the ⊙ with BC and AB respectively, and AD the altitude of the $\triangle ABC$ intersecting the ⊙ in point F.

To prove that $OD = \frac{1}{3} AD$.

Proof. Draw OE and EF.
Since $CA = AB$, Given
$\therefore CD = BD$. § 85



In the quadrilateral $BDOE$,

$\angle$s ODB and BEO are rt. $\angle$s, § 185

and $\angle DBE = 60°$. §§ 75, 107

$\therefore \angle EOD = 120°$. § 144

$\therefore \angle FOE = 60°$. § 43

Now $\angle OEF + \angle FOE + \angle EFO = 180°$. § 107

$\therefore \angle OEF + \angle EFO = 120°$. Ax. 2

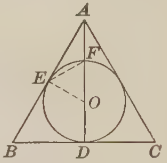
Since $OF = OE$, § 162

$\therefore \angle OEF = \angle EFO$. § 74

$\therefore \angle OEF = 60°$. Ax. 4

$\therefore \angle OEF = \angle EFO = \angle FOE = 60°$. Ax. 8

$\therefore OF = OE = EF$. § 77



Now in the $\triangle EFA$,

$\angle AFE = 1$ st. $\angle - \angle EFO$

$= 120°$, § 43

and $\angle FEA = \angle OEA - \angle OEF = 30°$.

$\therefore \angle EAF = 30°$. § 107

$\therefore \angle EAF = \angle FEA$. Ax. 8

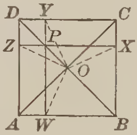
$\therefore EF = AF$. § 76

$\therefore AF = FO = OD$. Ax. 9

That is, $OD = \frac{1}{3} AD$. Q. E. D.

5. *If two lines are drawn through any point in a diagonal of a square parallel to the sides, the points where these lines meet the sides lie on the circle whose center is the point of intersection of the diagonals.*

Given the square $ABCD$, with diagonals AC and BD intersecting at O, P any point in either diagonal, and ZX and YW lines drawn through P parallel to the sides of the square and meeting the sides in the points Z, X, Y, and W.



To prove that Z, X, Y, and W are on the ⊙ whose center is O.

Proof. Draw OW, OX, OY, and OZ.

In the $\triangle$s ODA and ODC, $OD = OD$, Iden.

$DA = DC$, § 65

and $OA = OC$. § 131

$\therefore$ the $\triangle$s ODA and ODC are congruent. § 80

$\therefore \angle ADO = \angle ODC$. § 67

Now PZ is $\perp$ to DA, and PY is $\perp$ to CD. § 97

Since P is in the bisector of $\angle ADC$,

$\therefore PZ = PY$. § 152

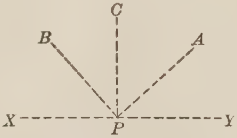
But $YD = PZ$, and $DZ = PY$. § 127
$\therefore YD = DZ$. Ax. 8
Now in the $\triangle$ DOY and DOZ,
$YD = DZ$, Proved
$OD = OD$, Iden.
and $\angle ODY = \angle ZDO$. Proved
$\therefore$ the $\triangle$ DOY and DOZ are congruent. § 68
$\therefore OY = OZ$. § 67
In a similar manner,
the $\triangle$ DOY and COX are congruent.
$\therefore OY = OX$. § 67
And in a similar manner,
the $\triangle$ BOX and BOW are congruent.
$\therefore OX = OW$. § 67
$\therefore OW = OX = OY = OZ$. Ax. 8
$\therefore$ W, X, Y, and Z are on the ⊙ whose center is O and radius OW, by § 160. Q.E.D.

Exercise 35. Page 130

1. *To construct an angle of* 45°; *of* 135°.

1. Required to construct an angle of 45°.

Construction. Draw a straight line XY.
At any point P in XY erect the $\perp$ PC. § 228
Draw PA, bisecting the $\angle YPC$. § 231
Then $\angle YPA$ is the angle of 45° required. Q.E.F.



Proof. The $\angle YPC = 90°$, Const.
and $\angle YPA = \frac{1}{2} \angle YPC$. Const.
$\therefore \angle YPA = 45°$. Q.E.D.

2. Required to construct an angle of 135°.

Construction. Draw a straight line XY.
At any point P in XY erect the $\perp$ PC. § 228
Draw PB, bisecting the $\angle CPX$. § 231
Then $\angle YPB$ is the angle of 135° required. Q.E.F.

Proof. $\angle YPC = \angle CPX = 90°$. Const.
$\angle CPB = \frac{1}{2} \angle CPX$. Const.
$\therefore \angle CPB = 45°$.
$\therefore \angle YPC + \angle CPB = 90° + 45°$. Ax. 1
That is, $\angle YPB = 135°$. Q.E.D.

2. *To construct an angle of* 22° 30′; *of* 157° 30′.

1. Required to construct an angle of 22° 30′.

Construction. Draw a straight line XY.

At any point P in XY erect the $\perp PC$. § 228

Draw PD, bisecting the $\angle YPC$. § 231

Draw PA, bisecting the $\angle YPD$. § 231

Then $\angle YPA$ is the angle of 22° 30′ required. Q.E.F.

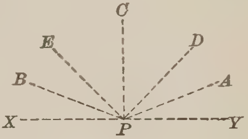
Proof. $\angle YPC = 90°$, Const.

and $\angle YPD = \frac{1}{2}\angle YPC$, Const.

and $\angle YPA = \frac{1}{2}\angle YPD$. Const.

$\therefore \angle YPA = \frac{1}{4}\angle YPC$. Ax. 4

$\therefore \angle YPA = 22° 30'$. Q.E.D.



2. Required to construct an angle of 157° 30′.

Construction. Draw a straight line XY.

At any point P in XY erect the $\perp PC$. § 228

Draw PE, bisecting the $\angle CPX$. § 231

Draw PB, bisecting the $\angle EPX$. § 231

Then $\angle YPB$ is the angle of 157° 30′ required. Q.E.F.

Proof. $\angle CPX = 90°$. Const.

$\angle CPE = \frac{1}{2}\angle CPX$, Const.

$\therefore \angle CPE = 45°$.

$\angle EPX = \frac{1}{2}\angle CPX$, Const.

and $\angle EPB = \frac{1}{2}\angle EPX$. Const.

$\therefore \angle EPB = 22° 30'$.

Now $\angle YPB = \angle YPC + \angle CPE + \angle EPB$. Ax. 11

$\therefore \angle YPB = 90° + 45° + 22° 30'$ Ax. 9

$= 157° 30'$. Q.E.D.

3. *To construct an equilateral triangle, having given one side, and thus to construct an angle of* 60°.

Given the line AB.

Required to construct an equilateral $\triangle$ with each side equal to AB.

Construction. With A as center and radius AB construct an arc, and with B as center and radius AB construct an arc intersecting the first arc at C.

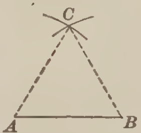
Draw AC and BC.

Then $\triangle ABC$ is the equilateral $\triangle$ required. Q.E.F.

Proof. $AC = AB$,

and $BC = AB$. Const.

$\therefore AC = BC = AB$. Ax. 8



Hence $\triangle ABC$ is an equilateral $\triangle$ with sides equal to AB. Q.E.D.

Discussion. $\angle A = \angle B = \angle C.$ § 75

But $\angle A + \angle B + \angle C = 180°.$ § 107

$\therefore \angle A = 60°$, by Ax. 4

4. *To construct an angle of* 30°, *and thus to trisect a right angle.*

Required to construct an angle of 30°.

Construction. Draw a straight line AP.

At the point P erect the $\perp PB$. § 228

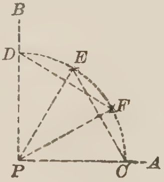
With P as center and any convenient radius describe an arc cutting AP in C and BP in D.

With C as center and the same radius cut the arc CD at E.

With D as center and the same radius cut the arc CD at F.

Draw PE and PF.

Then $\angle CPF$ is the angle of 30° required. Q.E.F.



Proof. Draw CE and DF.

The $\triangle s\ PCE$ and PFD are equilateral. Const.

$\therefore$ the $\triangle s\ PCE$ and PFD are equiangular. § 75

$\therefore \angle CPE = \angle FPD = 60°.$ § 107

Now $\angle CPF = \angle CPD - \angle FPD.$

$\therefore \angle CPF = 90° - 60° = 30°.$ Q.E.D.

Discussion. It will be noted that

$$\angle CPF = \angle FPE = \angle EPD = 30°.$$

That is, the rt. $\angle APB$ has been trisected.

5. *To construct an angle of* 15°; *of* 7° 30′; *of* 195°; *of* 345°.

1. Required to construct an angle of 15°.

Construction. Draw a straight line AB of convenient length.

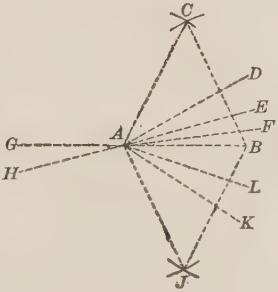
With A and B as centers and radius AB describe arcs intersecting at C.

Draw AC and BC.

Draw AD, bisecting the $\angle BAC$. § 231

Draw AE, bisecting the $\angle BAD$. § 231

Then $\angle BAE$ is the angle of 15° required. Q.E.F.



Proof. $AB = BC = AC.$ Const.

$\therefore \angle ACB = \angle BAC = \angle CBA.$ § 75

$\therefore \angle BAC = 60°.$ § 107

Now $\angle BAD = \frac{1}{2} \angle BAC.$ Const.

$\therefore \angle BAD = 30°.$

Again, $\angle BAE = \frac{1}{2} \angle BAD.$ Const.

$\therefore \angle BAE = 15°.$ Q.E.D.

2. Required to construct an angle of 7° 30′.

Construction. Construct the $\angle BAE$ equal to 15°. Const. 1

Draw AF bisecting the $\angle BAE$. § 231

Then $\angle BAF$ is the angle of 7° 30′ required. Q.E.F.

Proof. $\angle BAE = 15°$. Const.

Now $\angle BAF = \frac{1}{2}\angle BAE$. Const.

$\therefore \angle BAF = 7° 30'$. Q.E.D.

3. Required to construct an angle of 195°.

Construction. Construct the $\angle BAE$ equal to 15°. Const. 1

Produce BA through A to a point G, and produce EA through A to a point H.

Then the reflex $\angle BAH$ is the angle of 195° required. Q.E.F.

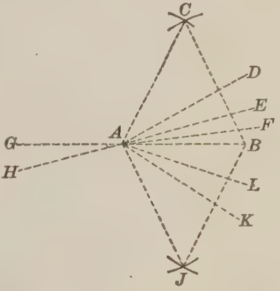
Proof. $\angle GAH = \angle BAE$. § 60

But $\angle BAE = 15°$. Const.

$\therefore \angle GAH = 15°$. Ax. 9

Now $\angle BAG = 180°$. § 43

$\therefore \angle BAG + \angle GAH = 195°$. Ax. 1



That is, the reflex $\angle BAH = 195°$. Q.E.D.

4. Required to construct an angle of 345°.

Construction. Draw the straight line AB of convenient length.

With A and B as centers and AB as radii describe arcs intersecting at J below the line AB.

Draw AK bisecting the $\angle JAB$. § 231

Draw AL bisecting the $\angle KAB$. § 231

Then the reflex $\angle BAL$ is the angle of 345° required. Q.E.F.

Proof. $AB = AJ = BJ$. Const.

$\therefore \angle BJA = \angle ABJ = \angle JAB$. § 75

$\therefore \angle JAB = 60°$. § 107

Now $\angle KAB = \frac{1}{2}\angle JAB$. Const.

$\therefore \angle KAB = 30°$.

Again, $\angle LAB = \frac{1}{2}\angle KAB$. Const.

$\therefore \angle LAB = 15°$.

Now the perigon about $A = 360°$. § 44

$\therefore$ perigon $- \angle LAB = 345°$. Ax. 2

That is, the reflex $\angle LAB = 345°$. Q.E.D.

6. *To construct a triangle having two of its angles equal to 75°. Is the triangle definitely determined?*

Required to construct a △ having two of its angles equal to 75°.

Construction. Draw a straight line AP of convenient length.

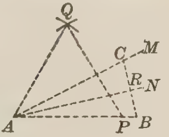
With P and A as centers and AP as radius describe arcs intersecting at Q.

Draw AQ and PQ.

Draw AM, bisecting the $\angle PAQ$. § 231

Draw AN, bisecting the $\angle PAM$. § 231

At some convenient point R in AN erect a $\perp$ cutting AM in C and AP or AP produced in B.



Then $\triangle ABC$ is the $\triangle$ required. Q. E. F.

Proof. $AP = AQ = PQ$. Const.

$\therefore \angle AQP = \angle PAQ = \angle QPA$. § 75

$\therefore \angle PAQ = 60°$. § 107

Now $\angle PAM = \frac{1}{2} \angle PAQ$. Const.

$\therefore \angle PAM = 30°$.

Again, $\angle PAN = \frac{1}{2} \angle PAM$. Const.

$\therefore \angle PAN = 15°$.

Now $\angle ARB = 90°$. Const.

$\therefore \angle RBA = 75°$. § 107

Similarly $\angle ACR = 75°$.

$\therefore$ the $\triangle ABC$ has two of its $\angle$s equal to 75°. Q. E. D.

Discussion. The $\triangle ABC$ is not definitely determined, for the point R may be taken at any point in AN, and therefore there are an infinite number of constructions possible, each of which will fulfill the conditions of the problem.

In order that a triangle may be definitely determined, some side or other line in the triangle must be known.

Exercise 36. Page 132

1. *To divide a given line into four equal parts.*

Given the line AB.

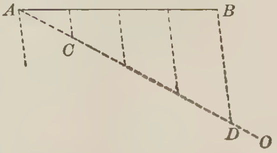
Required to divide the line AB into four equal parts.

Construction. From A draw the line AO, making any convenient angle with AB.

Take any convenient length AC, and by describing arcs apply it to AO four times.

From D, the last point thus found on AO, draw DB.

From the division points on AO draw parallels to DB. § 233



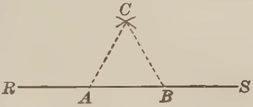
These lines divide AB into four equal parts. Q. E. F.

Proof. These lines divide AB into four equal parts, by § 134. Q. E. D.

2. *To construct an equilateral triangle, given the perimeter.*

Given the straight line RS.

Required to construct an equilateral $\triangle$ with perimeter equal to RS.



Construction. Trisect RS in the points A and B. § 234

With A and B as centers and the radius AB describe arcs intersecting at C.

Draw AC and BC.

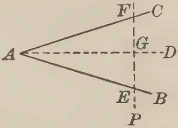
Then $\triangle ABC$ is the equilateral $\triangle$ required. Q.E.F.

Proof. $AB = AC = BC$. Const.

But $AB = \frac{1}{3} RS$. Const.

$\therefore AB + AC + BC = RS$. Q.E.D.

3. *Through a given point, to draw a line which shall make equal angles with the two sides of a given angle.*



Given the point P and the $\angle BAC$.

Required to draw through P a line which shall make equal $\angle$s with AB and AC.

Construction. Draw the line AD, bisecting the $\angle BAC$. § 231

Through P draw a line $\perp$ to AD, cutting AB at E, AD at G, and AC at F. § 227

Then PEF is the line required. Q.E.F.

Proof. In the rt. $\triangle$s AGF and AGE,

$AG = AG$, Iden.

and $\angle GAF = \angle EAG$. Const.

$\therefore$ the rt. $\triangle$s AGF and AGE are congruent. § 72

$\therefore \angle AFG = \angle GEA$, by § 67. Q.E.D.

4. *Through a given point, to draw two lines so that they shall form with two intersecting lines two isosceles triangles.*

Given the point P and lines AB and CD intersecting at O.

Required to draw through P two lines which shall form with AB and CD two isosceles $\triangle$s.

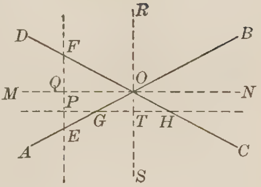
Construction. Draw ROS, bisecting the $\angle AOC$. § 231

Draw MON, bisecting the $\angle DOA$. § 231

Through P draw a line $EF \perp$ to OM, cutting AB at E, CD at F, and OM at Q. § 227

Through P draw a line $GH \perp$ to OS, cutting AB at G, CD at H, and OS at T. § 227

Then EPF and PGH are the lines required. Q.E.F.



Proof. In the rt. $\triangle$s FQO and EQO,

$OQ = OQ$, Iden.

and $\angle FOQ = \angle QOE$. Const.

$\therefore$ the rt. $\triangle$s FQO and EQO are congruent. § 72

$\therefore OF = OE$. § 67

Hence $\triangle EOF$ is isosceles. § 62

Similarly the rt. $\triangle$s GTO and HTO are congruent. § 72

$\therefore OG = OH$. § 67

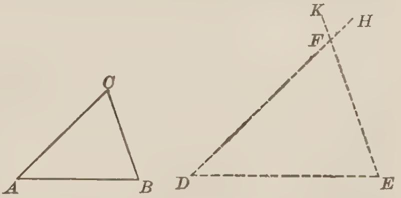
Hence $\triangle GHO$ is isosceles, by § 62. Q.E.D.

5. *To construct a triangle having its three angles respectively equal to the three angles of a given triangle.*

Given the $\triangle ABC$.

Required to construct a $\triangle$ having its three $\angle$s equal to A. B, and C.

Construction. Draw a convenient line DE.



From D draw DH, making $\angle EDH$ equal to $\angle A$. § 232

From E draw EK, making $\angle KED$ equal to $\angle B$. § 232

Let F be the intersection of DH and EK.

Then $\triangle DEF$ is the $\triangle$ required. Q.E.F.

Proof. $\angle D = \angle A$, and $\angle E = \angle B$. Const.

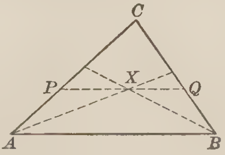
$\therefore \angle DFE = \angle C$, by § 107. Q.E.D.

Exercise 37. Page 141

1. *In a triangle ABC, to draw PQ parallel to the base AB, cutting the sides in P and Q, so that PQ shall equal $AP + BQ$.*

Given the $\triangle ABC$.

Required to draw, in the $\triangle ABC$, $PQ \parallel$ to AB, cutting AC in P and BC in Q, so that $PQ = AP + BQ$.



Analysis. Assume the problem solved, and PQ the required line.

Then AP must equal some part of PQ, as PX, and BQ must equal QX.

Now if $AP = PX$,

then $\angle PXA = \angle XAP$. § 74

Since PQ is $\parallel$ to AB,

$\therefore \angle PXA = \angle BAX$. § 100

$\therefore \angle XAP = \angle BAX$. Ax. 8

Similarly $\angle QBX = \angle XBA$.

Construction. Draw lines bisecting the $\angle$s A and B. § 231

Through X, the point of intersection of these bisectors, draw

$PQ \parallel$ to AB. § 233

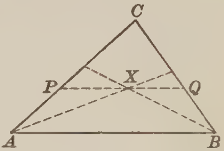
Then PQ is the line required. Q.E.F.

Proof. $\angle BAX = \angle XAP$. Const.

But $\angle BAX = \angle PXA$. § 100

$\therefore \angle PXA = \angle XAP$. Ax. 8

$\therefore AP = PX$. § 76



Similarly $BQ = QX$.

$\therefore PX + QX = AP + BQ$. Ax. 1

That is, $PQ = AP + BQ$, by Ax. 11. Q.E.D.

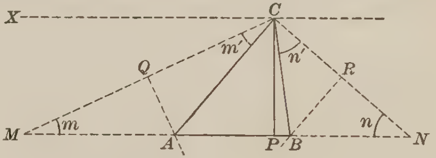
2. *To construct a triangle, having given the perimeter, one angle, and the altitude from the vertex of the given angle.*

Given MN, the perimeter of a $\triangle$, $\angle C$ the $\angle$ at the vertex, and CP the altitude from C.

Required to construct the $\triangle$.

Analysis. Assume the problem solved and ABC the required triangle.

Produce AB to N, making BN equal to BC, and produce BA to M, making AM equal to AC.



$\therefore \angle m = \angle m'$,

and $\angle n = \angle n'$. § 74

Then $\angle m + \angle n + \angle MCN = 180°$. § 107

But $\angle MCN = \angle m' + \angle ACB + \angle n'$. Ax. 11

$\therefore \angle m + \angle n + \angle m' + \angle ACB + \angle n' = 180°$. Ax. 9

$\therefore 2\angle m + 2\angle n + \angle ACB = 180°$. Ax. 9

$\therefore \angle m + \angle n + \frac{1}{2}\angle ACB = 90°$. Ax. 4

$\therefore \angle m + \angle n = 90° - \frac{1}{2}\angle ACB$. Ax. 2

$\therefore \angle MCN = 90° - \frac{1}{2}\angle ACB + \angle ACB$. Ax. 1

That is, $\angle MCN = 90° + \frac{1}{2}\angle ACB$.

$\therefore \angle MCN$ is known.

Construction. On MN construct a segment in which $\angle MCN = 90° + \frac{1}{2}\angle C$ may be inscribed. § 247

At the distance CP, the given altitude, draw $XC \parallel$ to MN, cutting the arc at C. § 233

Draw MC and NC.

Bisect MC at Q, and NC at R. § 229

Draw $QA \perp$ to MC, cutting MN at A,

and draw $RB \perp$ to NC, cutting MN at B. § 228

Draw AC and BC.

Then $\triangle ABC$ is the $\triangle$ required. Q. E. F.

Proof. Since A is on the $\perp$ bisector of MC, Const.

$\therefore AM = AC$. § 150

Since B is on the $\perp$ bisector of NC, Const.

$\therefore BN = BC$. § 150

Now $AB = AB$. Iden.

$\therefore AC + AB + BC = AM + AB + BN$. Ax. 1

But $AM + AB + BN = MN$. Ax. 11

$\therefore AC + AB + BC = MN$. Ax. 8

Again, $\angle AMC = \angle MCA$, and $\angle CNB = \angle BCN$. § 74

$\angle NMC + \angle CNM + \angle MCN = 180°$. § 107

$\angle MCA + \angle BCN + \angle MCN = 180°$. Ax. 9

$\angle MCA + \angle ACB + \angle BCN + \angle MCN = 180° + \angle ACB$. Ax. 1

$\angle MCN + \angle MCN = 180° + \angle ACB$.

Now $\angle MCN = 90° + \frac{1}{2}\angle C$. Const.

$\therefore 180° + \angle C = 180° + \angle ACB$. Ax. 9

That is, $\angle ACB = \angle C$, the given angle.

Again, the altitude CP equals the required altitude, by § 128. Q. E. D.

3. *To draw through two sides of a triangle a line parallel to the third side, so that the part intercepted between the sides shall have a given length.*

Given the $\triangle ABC$ and the line d.

Required to draw through AC and BC a line $\parallel$ to AB and equal to d.

Analysis. Assume the problem solved and PQ the required line.

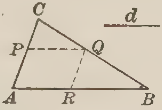
From Q draw $QR \parallel$ to CA. § 233

Then $ARQP$ is a ▱. § 118

$\therefore AR = PQ$. § 125

But $PQ = d$.

$\therefore AR = d$. Ax. 8

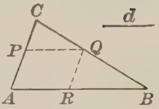


Construction. On AB take AR equal to d.

From R draw $RQ \parallel$ to AC. § 233

Draw $QP \parallel$ to BA. § 233

Then PQ is the line required. Q. E. F.



Proof. Since RQ is $\parallel$ to AC,

and QP is $\parallel$ to BA, Const.

$\therefore$ the quadrilateral $ARQP$ is a ▱. § 118

$\therefore PQ = AR$. § 125

But $AR = d$. Const.

$\therefore PQ = d$, by Ax. 8. Q.E.D.

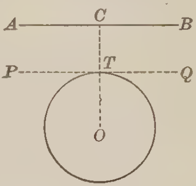
4. *To draw a tangent to a given circle so that it shall be parallel to a given line.*

Given the circle with center O and the line AB.

Required to draw a tangent to the circle $\parallel$ to AB.

Analysis. Assume the problem solved and PQ the required tangent.

Draw the radius OT and produce it to meet AB at C.



Since PQ is $\parallel$ to AB, Given

and OC is $\perp$ to PQ, § 185

$\therefore OC$ is $\perp$ to AB. § 97

Construction. From the center of the circle draw $OC \perp$ to AB, and cutting the circle at T. § 227

Through T draw $PQ \perp$ to OC. § 228

Then PQ is the tangent required. Q.E.F.

Proof. Since PQ is $\perp$ to OT at its extremity T, Const.

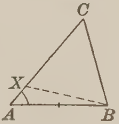
$\therefore PQ$ is tangent to the circle at T. § 184

Again, PQ is $\parallel$ to AB, by § 95. Q.E.D.

5. *To construct a triangle, having given a side, an adjacent angle, and the difference of the other sides.*

Given AB a side, the $\angle A$, and AX a line.

Required to construct a $\triangle$, with a side equal to AB, an adjacent $\angle$ equal to $\angle A$, and the difference of the other sides equal to AX.



Analysis. Assume the problem solved and ABC the required $\triangle$.

Let the side AC be greater than the side BC.

On AC take AX equal to the given difference, and draw XB.

Then $XC = BC$, and the $\triangle XBC$ is isosceles.

Hence C lies on the $\perp$ bisector of XB, § 150

and also on the line AX.

Construction. Construct the $\triangle ABX$, with two sides equal to AB and AX, and the included $\angle$ equal to $\angle A$. § 235

Find the mid-point of XB, § 229

and at that point draw a line $\perp$ to XB. § 228

Produce AX to meet this $\perp$ at C, and draw CB.

Then the $\triangle ABC$ is the $\triangle$ required. Q. E. F.

Proof. Since C is in the $\perp$ bisector of XB, Const.

$\therefore XC = BC$. § 150

But $AX = AC - XC$. Ax. 11

$\therefore AX = AC - BC$. Ax. 9

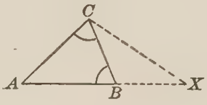
Also $\angle A =$ the given $\angle$,

and $AB =$ the given side, by Const. Q. E. D.

6. *To construct a triangle, having given two angles and the sum of two sides.*

Given the $\angle$s B and C and the line AX.

Required to construct a $\triangle$ with two angles equal to $\angle$s B and C, and the sum of two sides equal to AX.



Analysis. Assume the problem solved and ABC the required $\triangle$.

Produce AB to X, making BX equal to BC, and draw CX.

Then $\angle CXB = \angle BCX$. § 74

But $\angle CBA = \angle CXB + \angle BCX$. § 111

$\therefore \angle CBA = \angle CXB + \angle CXB$. Ax. 9

$\therefore \angle CXB = \frac{1}{2} \angle CBA$. Ax. 4

Again, $\angle BAC + \angle CBA + \angle ACB = 180°$. § 107

$\therefore \angle BAC = 180° - (\angle CBA + \angle ACB)$. Ax. 2

Therefore the $\triangle AXC$ is determined.

Construction. Construct the $\triangle AXC$, with AX given, the $\angle XAC$ equal to $180° - (\angle B + \angle C)$, and the $\angle CXB$ equal to $\frac{1}{2} \angle B$. § 236

Bisect XC, § 229

and draw its $\perp$ bisector, cutting AX at B. § 228

Draw BC.

Then the $\triangle ABC$ is the $\triangle$ required. Q. E. F.

Proof. Since B is in the $\perp$ bisector of XC, Const.

$\therefore BC = BX$. § 150

$\therefore AB + BC = AB + BX$. Ax. 1

That is, $AB + BC = AX$. Ax. 11

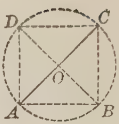
Again, $\angle CXB = \angle BCX$. § 74

But $\angle CBA = \angle CXB + \angle BCX$. § 111

$\therefore \angle CBA = 2 \angle CXB = \angle B$.

Again, $\angle A = 180° - (\angle B + \angle C)$, by Const. Q. E. D.

7. *To construct a square, having given the diagonal.*



Given the line AC.

Required to construct a square having the given line AC as diagonal.

Analysis. Assume the problem solved and $ABCD$ the required square.

Draw the diagonal BD, cutting AC at O.

Then $AO = OC$. § 131

With O as center and radius OC describe a ⊙.

Since $\angle CBA$ and $\angle ADC$ are rt. ∠s, § 65

∴ B and D are points on the ⊙. § 215

Also, since $BA = BC = DA = DC$, § 65

∴ B and D are points in the ⊥ bisector of AC. § 150

Construction. Bisect AC at O. § 229

With O as center and radius OA describe a ⊙.

Through O draw a line ⊥ to AC, cutting the ⊙ at B and D. § 228

Draw AB, BC, CD, and DA.

Then $ABCD$ is the square required. Q.E.F.

Proof. ∠s A, B, C, and D are rt. ∠s. § 215

Now $AB = BC$, and $CD = DA$. § 150

Also $DA = AB$, and $BC = CD$. § 150

∴ $AB = BC = CD = DA$. Ax. 8

∴ $ABCD$ is a square, by § 65. Q.E.D.

8. *To draw through a given point P between the sides of an angle AOB a line terminated by the sides of the angle and bisected at P.*

Given the $\angle AOB$ and the point P between the sides of the $\angle AOB$.

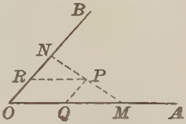
Required to draw through P a line terminated by the sides of the $\angle AOB$ and bisected at P.

Analysis. Assume the problem solved and MN the required line.

Draw $PR \parallel$ to AO, and $PQ \parallel$ to BO. § 233

Then $OR = RN$, and $OQ = QM$. § 135

Hence the points N and M are determined.



Construction. Draw $PR \parallel$ to AO. § 233

On OB take RN equal to OR.

Draw NP and produce it to meet AO at M.

Then MN is the line required. Q.E.F.

Proof. Since $OR = RN$, Const.

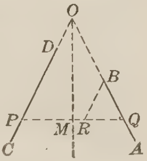
and PR is $\parallel$ to MO, Const.

∴ $NP = PM$, by § 135. Q.E.D.

9. *To draw a line that would bisect the angle formed by two lines if those lines were produced to meet.*

Given the lines AB and CD.

Required to draw a line that would bisect the angle formed by AB and CD if those lines were produced to meet.



Analysis. Assume the problem solved and OM the required line.

Produce AB and CD to meet at O.

Then the bisector of the $\angle O$ would be the $\perp$ bisector of PQ, a line drawn so as to make equal angles with AB and CD.

Construction. From any point B of AB draw a line $\parallel$ to DC. § 233
On this line and on BA take BR equal to BQ.
Draw QR and produce it to meet CD at P.
Bisect PQ at M. § 229
Draw OM, the $\perp$ bisector of PQ. § 228
Then OM is the line required. Q. E. F.

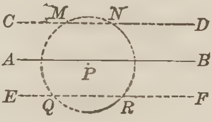
Proof. Since $BQ = BR$, Const.
$\therefore \angle QRB = \angle BQR$. § 74
But BR is $\parallel$ to DP. Const.
$\therefore \angle QRB = \angle QPD$. § 102
$\therefore \angle BQR = \angle QPD$. Ax. 8

$\therefore$ the $\triangle$ formed by the two given lines QB and PD, and PQ is isosceles. § 76

Since OM is the $\perp$ bisector of PQ, Const.
$\therefore$ OM goes through the vertex of the $\triangle$. § 150
$\therefore$ OM bisects the $\angle$ formed by QB and PD, by § 83. Q. E. D.

Exercise 38. Page 143

1. *To find a point that is $\frac{1}{4}$ in. from a given point and $\frac{3}{16}$ in. from a given line.*



Given the line AB and the point P.

Required to find a point that is $\frac{1}{4}$ in. from P and $\frac{3}{16}$ in. from AB.

Construction. With P as center and radius $\frac{1}{4}$ in. describe a $\odot$.

Draw CD and EF $\parallel$ to AB at a distance of $\frac{3}{16}$ in. from AB. § 233
Let CD and EF intersect the $\odot$ in the points M, N, Q, and R.
Then M, N, Q, or R is the required point. Q. E. F.

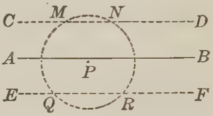
Proof. The ⊙ MNR is the locus of a point $\frac{1}{4}$ in. from P. § 160

The ∥ lines CD and EF are the locus of a point $\frac{3}{16}$ in. from AB. § 147

∴ M, N, Q, or R is the required point, by § 254. Q.E.D.

Discussion. In general, there are four points that satisfy the two given conditions.

There are three points that satisfy the given conditions when P lies between CD and EF and is distant $\frac{1}{4}$ in. from the further one of these parallel lines, for then the circle is tangent to that line, and either M and N coincide, or Q and R coincide.



There are two points that satisfy the given conditions when P is more than $\frac{1}{4}$ in. distant from one of the parallel lines CD and EF, but less than $\frac{1}{4}$ in. from the other, for then the circle evidently cuts one of the parallel lines in two points, but does not cut or touch the other.

There is one point that satisfies the given condition when P is $\frac{1}{4}$ in. distant from one of the parallel lines CD and EF and more than $\frac{1}{4}$ in. distant from the other, for then the circle evidently is tangent to one of the parallel lines, but does not cut or touch the other.

There is no point that satisfies the given conditions when P is more than $\frac{1}{4}$ in. distant from each of the parallel lines CD and EF, for then the circle evidently does not cut or touch either line.

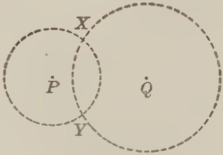
2. *To find a point that is $\frac{1}{2}$ in. from one given point and $\frac{3}{4}$ in. from another given point.*

Given the points P and Q.

Required to find a point $\frac{1}{2}$ in. from P and $\frac{3}{4}$ in. from Q.

Construction. With P as center and $\frac{1}{2}$ in. as radius describe a ⊙, and with Q as center and $\frac{3}{4}$ in. as radius describe a ⊙, intersecting the first ⊙ in the points X and Y.

Then X or Y is the point required. Q.E.F.



Proof. The ⊙ with center P is the locus of a point $\frac{1}{2}$ in. from P, and the ⊙ with center Q is the locus of a point $\frac{3}{4}$ in. from Q. § 160

∴ X or Y is the point required, by § 254. Q.E.D.

Discussion. In general, there are two points that satisfy the given conditions.

If $PQ < \frac{1}{2}$ in. $+ \frac{3}{4}$ in., but $> \frac{3}{4}$ in. $- \frac{1}{2}$ in., there are two solutions.

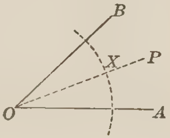
If $PQ = \frac{1}{2}$ in. $+ \frac{3}{4}$ in., there is one solution, for then the circles are tangent to each other.

If $PQ > \frac{1}{2}$ in. $+ \frac{3}{4}$ in., there is no solution, for then the circles do not intersect.

3. *To find a point that is $\frac{1}{2}$ in. from the vertex of an angle and equidistant from the sides of the angle.*

Given the $\angle AOB$.

Required to find a point $\frac{1}{2}$ in. from O and also equidistant from OA and OB.



Construction. Draw OP, bisecting the $\angle AOB$. § 231

With O as center and radius $\frac{1}{2}$ in. describe a ⊙ cutting the line OP at X.

Then X is the point required. Q.E.F.

Proof. The ⊙ is the locus of a point $\frac{1}{2}$ in. from O. § 160

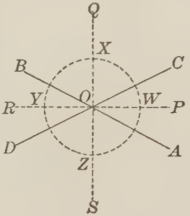
OP is the locus of a point equidistant from OA and OB. § 152

$\therefore$ X is the point required, by § 254. Q.E.D.

4. *To find a point that is equidistant from two intersecting lines and $\frac{1}{4}$ in. from their point of intersection.*

Given the lines AB and CD, intersecting at O.

Required to find a point $\frac{1}{4}$ in. from O and equidistant from AB and CD.



Construction. Through O draw PR, bisecting the $\angle AOC$, and through O draw QS, bisecting the $\angle COB$. § 231

With O as center and radius $\frac{1}{4}$ in. describe a ⊙ cutting PR in the points W and Y, and QS in the points X and Z.

Then W, X, Y, or Z is the point required. Q.E.F.

Proof. The Ⓢ is the locus of a point $\frac{1}{4}$ in. from O. § 160

PR and QS are the locus of a point equidistant from AB and CD. § 152

$\therefore$ W, X, Y, or Z is the point required, by § 254. Q.E.D.

Discussion. Since the circle will cut the bisector of each of the four angles formed by the lines AB and CD, it is always possible to find four points that will satisfy the given conditions.

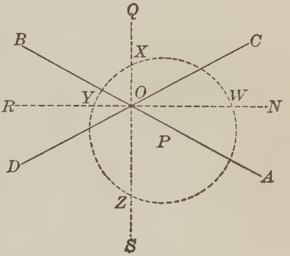
5. *To find a point that is $\frac{1}{2}$ in. from a given point and equidistant from two intersecting lines.*

Given the point P and the lines AB and CD, intersecting at O.

Required to find a point $\frac{1}{2}$ in. from P and equidistant from AB and CD.

Construction. Through O draw NR, bisecting the $\angle AOC$, and through O draw QS, bisecting the $\angle COB$. § 231

With P as center and radius $\frac{1}{2}$ in. describe a ⊙ cutting NR in the points W and Y, and QS in the points X and Z.



Then W, X, Y, or Z is the point required. Q.E.F.

Proof. The ⊙ is the locus of a point $\frac{1}{2}$ in. from P. § 160

NR and QS are the locus of a point equidistant from AB and CD. § 152

Hence W, X, Y, and Z, the intersections of these loci, satisfy the two given conditions, by § 254. Q.E.D.

Discussion. If $\frac{1}{2}$ in. is greater than the distance from P to each bisector, there are four points that satisfy the given conditions.

If $\frac{1}{2}$ in. is greater than the distance from P to one bisector, and $\frac{1}{2}$ in. is equal to the distance from P to the other bisector, then there are three points that satisfy the given conditions.

If $\frac{1}{2}$ in. is greater than the distance from P to one bisector, and $\frac{1}{2}$ in. is less than the distance from P to the other bisector, then there are two points that satisfy the given conditions.

If $\frac{1}{2}$ in. is equal to the distance from P to each bisector, then there are two points that satisfy the given conditions.

If $\frac{1}{2}$ in. is less than the distance from P to one bisector, and $\frac{1}{2}$ in. is equal to the distance from P to the other bisector, then there is one point that satisfies the given conditions.

If $\frac{1}{2}$ in. is less than the distance from P to each bisector, then there is no point that satisfies the given conditions.

6. *To find a point that is $\frac{1}{2}$ in. from a given point and equidistant from two parallel lines.*

Given the point P and the parallel lines AB and CD.

Required to find a point $\frac{1}{2}$ in. from P and equidistant from AB and CD.

Construction. From any point E of AB draw $EF \perp$ to CD. § 227

Bisect EF and call the point of bisection G. § 229

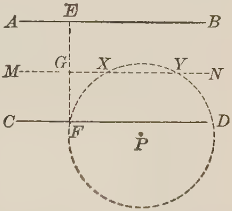
Through G draw $MN \perp$ to EF. § 228

With P as center and $\frac{1}{2}$ in. as radius describe a ⊙ cutting MN in the points X and Y.

Then X or Y is the point required. Q.E.F.

Proof. EF is $\perp$ to CD and also to MN. Const.

$\therefore$ EF is $\perp$ to AB. § 97



Since G, any point of MN, is equidistant from AB and CD, therefore every point of MN is equidistant from AB and CD. § 128

$\therefore$ MN is the locus of a point equidistant from AB and CD. § 148

Again, the ⊙ is the locus of a point $\frac{1}{2}$ in. from P. § 160

Hence X and Y, the intersections of the two loci, satisfy the two given conditions, by § 254. Q.E.D.

Discussion. If the distance from P to $MN < \frac{1}{2}$ in., there are two points that satisfy the given conditions.

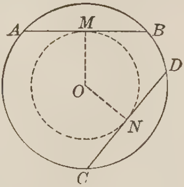
If the distance from P to $MN = \frac{1}{2}$ in., there is one point that satisfies the given conditions, since MN is then tangent to the circle.

If the distance from P to $MN > \frac{1}{2}$ in., there is no point that satisfies the given conditions, since MN will not cut or touch the circle.

7. *Find the locus of the mid-point of a chord of a given length that can be drawn in a given circle.*

Given AB and CD, chords of the circle with center O, and of the given length.

Required to find the locus of the mid-point of AB.



Construction. Draw $OM \perp$ to AB. § 229

With center O and OM as radius describe a ⊙.

Then this ⊙ is the locus required. Q.E.F.

Proof. Draw $ON \perp$ to CD. § 229

Now $AM = MB$, and $CN = ND$. § 174

Also $OM = ON$. § 178

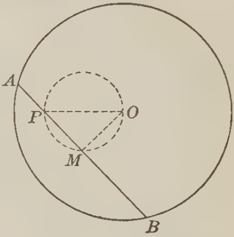
Hence the mid-point of every chord of length AB lies at the distance OM from the center O.

Therefore the ⊙ through M and N is the required locus, by § 160. Q.E.D.

8. *Find the locus of the mid-point of a chord drawn through a given point within a given circle.*

Given the point P within the circle with center O, and AB any chord drawn through P.

Required to find the locus of the mid-point of AB.



Construction. Draw OP.

On OP as a diameter describe a ⊙.

This ⊙ is the locus required. Q.E.F.

Proof. Let M be the point of intersection of the small ⊙ and AB.

Draw OM.

Then $\angle OMP = 1$ rt. $\angle$. § 215

That is, OM is $\perp$ to AB.

$\therefore MB = MA$. § 174

Similarly, every chord drawn through P is bisected by this small circle.

Therefore the circle on OP as a diameter is the locus of the mid-point of a chord drawn through P, by § 148. Q.E.D.

9. *To describe a circle that shall pass through a given point and cut equal chords of a given length from two parallels.*

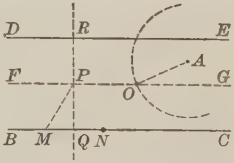
Given the point A, the parallel lines BC and DE, and the length MN.

Required to describe a ⊙ that shall pass through A and cut chords of length MN from BC and DE.

Analysis. Assume the problem solved and O the center of the required circle.

Since the ⊙ cuts equal chords from BC and DE, the center must be equidistant from BC and DE.

Hence one locus for O is the line $FG \parallel$ to BC and equidistant from BC and DE.



Construction. Bisect MN. § 229

At the point of bisection, Q, draw a line $\perp$ to MN, cutting DE in R. § 228

Bisect QR. § 229

At the point of bisection, P, draw $FG \perp$ to QR. § 228

Draw PM.

Then with A as center and PM as radius describe a ⊙.

Let this ⊙ cut FG in O.

Then O is the center and PM the radius of the required ⊙. Q.E.F.

Proof. Every ⊙ with center on FG and radius PM cuts chords each equal to MN from the ∥s BC and DE. Const.

Every ⊙ with center on the ⊙ described with A as center and the same radius will pass through A. § 160

Therefore the ⊙ with center O is the required ⊙. Q.E.D.

Discussion. The problem is impossible if the distance from A to FG is greater than PM.

10. *To describe a circle that shall be tangent to each of two given intersecting lines.*

Given two intersecting lines.

Required to describe a ⊙ that shall be tangent to each line.

Construction. Draw the bisectors of the ∠s formed by the two lines. § 231

Then with any point in either bisector as center and the distance from that point to either of the given lines as a radius describe a ⊙.

That ⊙ is the ⊙ required. Q.E.F.

Proof. Every point in either bisector is equidistant from the intersecting lines. § 152

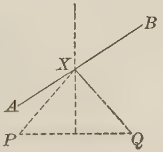
∴ each of the intersecting lines is tangent to the circle, by § 184. Q.E.D.

Discussion. This is an indeterminate problem, since any point on either bisector may be chosen as center, and thus an infinite number of circles may be drawn that satisfy the conditions.

11. *To find in a given line a point that is equidistant from two given points.*

Given the line AB and the points P and Q.

Required to find a point in AB equidistant from P and Q.



Construction. Draw PQ.

Bisect PQ. § 229

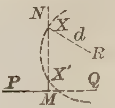
At the mid-point of PQ erect a line ⊥ to PQ, § 228 cutting the line AB at X.

Then X is the point required. Q.E.F.

Proof. Draw PX and QX.

$PX = QX$, by § 150. Q.E.D.

12. *To find a point that is equidistant from two given points and at a given distance from a third given point.*



Given the points P, Q, and R, and the distance d.

Required to find a point equidistant from P and Q, and at a distance d from R.

Construction. Draw PQ.

Draw MN, the ⊥ bisector of PQ. § 229

With R as center and radius d describe a circle cutting MN at X and X'.

Then X or X' is the point required. Q. E. F.

Proof. X is equidistant from P and Q. § 150

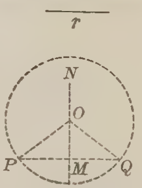
X is at the distance d from R, by § 162. Q. E. D.

Discussion. Let d' denote the distance from R to MN. Then the number of solutions is 2, 1, or 0, according as $d >, =,$ or $< d'$.

13. *To describe a circle that has a given radius and passes through two given points.*

Given the points P and Q and the distance r.

Required to describe a ⊙ that shall have radius r and shall pass through P and Q.



Construction. Draw PQ.

Draw MN, the ⊥ bisector of PQ. §§ 229, 228

With P as center and radius r describe an arc cutting MN at O.

With center O and radius r describe a ⊙.

This ⊙ is the circle required. Q. E. F.

Proof. The ⊙ has r for radius, Const.

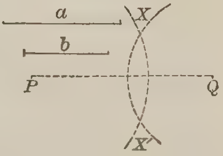
and the center O is equidistant from P and Q. § 150

∴ O is at the distance r from both P and Q. Q. E. D.

14. *To find a point at given distances from two given points.*

Given the points P and Q and the lines a and b.

Required to find a point at a distance a from P, and at a distance b from Q.



Construction. With P as center and radius a describe an arc.

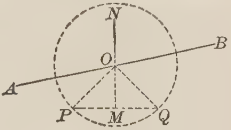
With Q as center and radius b describe another arc cutting the first arc at X and X'.

Then X or X' is the point required. Q. E. F.

Proof. X is, by construction, at the distance a from P, and at the distance b from Q. Q. E. D.

Discussion. If $PQ = a + b$, there is one solution; namely, a point in the line PQ where the two arcs touch each other. If $PQ > a + b$, or $< a - b$, the problem is impossible; for the arcs cannot cut or touch each other. If PQ has any value between $a + b$ and $a - b$, there are two solutions (as in the figure).

15. *To describe a circle that has its center in a given line and passes through two given points.*



Given the line AB and the points P and Q.

Required to describe a circle that has its center in AB and passes through P and Q.

Construction. Draw PQ.

Draw MN, the $\perp$ bisector of PQ. §§ 229, 228

Let MN cut AB at O.

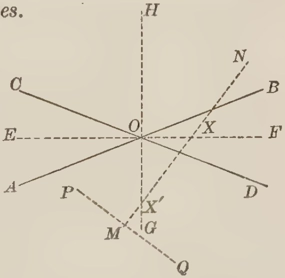
With center O and radius OP describe a $\odot$.

This $\odot$ is the $\odot$ required. Q.E.F.

Proof. The center O is in AB. Const.

Also $OP = OQ$, by § 150. Q.E.D.

16. *To find a point that is equidistant from two given points and also equidistant from two given intersecting lines.*



Given the two points P and Q and the lines AB and CD, intersecting at O.

Required to find a point equidistant from P and Q, and also equidistant from AB and CD.

Construction. Draw EF, the bisector of $\angle COA$, and GH, the bisector of $\angle BOC$. § 231

Draw PQ.

Draw MN, the $\perp$ bisector of PQ, cutting EF and GH in the points X and X' respectively. §§ 229, 228

Then X or X' is the point required. Q.E.F.

Proof. EF and GH are the locus of a point equidistant from AB and CD. § 152

MN is the locus of a point equidistant from P and Q. § 150

$\therefore$ X or X' is the point required, by § 254. Q.E.D.

Discussion. In general, two solutions may be obtained.

If MN is parallel to one of the bisectors of the angle, only one solution is possible.

If MN coincides with one of the bisectors, the problem is indeterminate, since any point on that bisector will satisfy the conditions.

17. *To find a point that is equidistant from two given points and also equidistant from two given parallel lines.*

Given the points P and Q and the ∥ lines AB and CD.

Required to find a point equidistant from P and Q, and also equidistant from AB and CD.

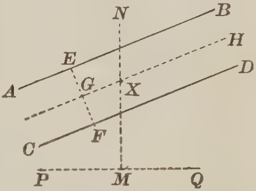
Construction. Draw PQ.

Draw MN, the ⊥ bisector of PQ. §§ 229, 228

From any point E on AB draw EF ⊥ to CD. § 227

Draw GH, the ⊥ bisector of EF, cutting MN in X. §§ 229, 228

Then X is the point required. Q.E.F.



Proof. Since EF is ⊥ to CD, Const.

∴ EF is ⊥ to AB. § 97

Again, EF is ⊥ to GH. Const.

∴ GH is ∥ to AB and ∥ to CD. § 95

Also $GE = GF$. Const.

∴ any point on GH is equidistant from AB and CD. § 128

MN is the locus of a point equidistant from P and Q. § 150

∴ X is the point required, by § 254. Q.E.D.

Discussion. In general, there is one solution.

If PQ is ⊥ to the given lines, there is no solution unless MN coincides with GH, in which case the problem is indeterminate.

18. *To find a point that is equidistant from two given intersecting lines and at a given distance from a given point.*

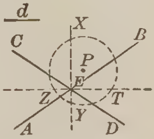
Given AB and CD, two lines intersecting at E, the point P, and the distance d.

Required to find a point equidistant from AB and CD, and at a distance d from P.

Construction. Draw ZT bisecting ∠ CEA, and XY bisecting ∠ BEC. § 231

With P as center and radius d describe a ⊙ cutting the bisectors in X, Y, Z, and T.

Then X, Y, Z, or T is the point required. Q.E.F.



Proof. XY and ZT are the locus of a point equidistant from AB and CD. § 152

The ⊙ XZY is the locus of a point at the distance d from point P. § 160

∴ X, Y, Z, or T is the point required, by § 254. Q.E.D.

Discussion. If d is greater than the distance from P to each bisector, there are four solutions.

If d is greater than the distance from P to one bisector and equal to the distance from P to the other, there are three solutions.

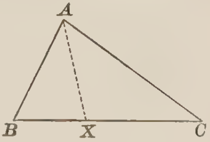
If d is equal to the distance from P to each bisector, there are two solutions.

If d is less than the distance from P to one bisector and greater than the distance from P to the other, there are two solutions.

If d is equal to the distance from P to one bisector and less than the distance from P to the other, there is one solution.

If d is less than the distance from P to each bisector, there is no solution.

19. *To find a point that lies in one side of a given triangle and is equidistant from the other two sides.*



Given the $\triangle ABC$.

Required to find a point lying in BC, and equidistant from AB and AC.

Construction. Bisect $\angle A$. § 231

Let the bisector meet BC in X.

Then X is the point required. Q.E.F.

Proof. AX is the locus of a point equidistant from the sides of $\angle BAC$. § 152

$\therefore$ X is equidistant from AB and AC. Q.E.D.

Exercise 39. Page 145

1. *To draw a common tangent to two given circles.*

Given two ⓢ with centers O and O'.

Required to draw a common tangent to these two ⓢ.

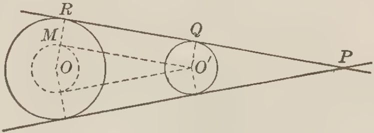


Fig. 1

Case 1. Considering a common external tangent (Fig. 1).

Analysis. Suppose the problem solved, and RQ the required tangent.

Draw the radii OR and $O'Q$.

Then OR and $O'Q$ are $\perp$ to RQ, § 185

and OR is $\parallel$ to $O'Q$. § 95

Through O' draw $MO' \parallel$ to RQ.

Then $MR = O'Q$. § 127

$\therefore OM = OR - O'Q$.

But MO' is $\perp$ to OM. § 97

$\therefore MO'$ is a tangent, drawn through O', the center of the smaller ⊙, to a ⊙ with center O and a radius equal to the difference of the radii of the two given Ⓢ.

Construction. With center O and a radius equal to the difference of the radii of the given Ⓢ draw a ⊙.

Through O' draw a tangent MO' to this ⊙. § 246

Through M, the point of contact, draw the radius OR.

Through O' draw the radius $O'Q \parallel$ to OR. § 233

Draw RQ.

Then the line RQ is the required tangent. Q. E. F.

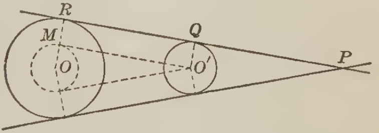


FIG. 1

Proof. Since MO' is $\perp$ to OR, § 185

and OR is $\parallel$ to $O'Q$, Const.

$\therefore MO'$ is $\perp$ to $O'Q$. § 97

Now $MR = O'Q$. Const.

$\therefore RQ$ is $\parallel$ to MO'. § 130

$\therefore RQ$ is $\perp$ to OR and $O'Q$. § 97

$\therefore RQ$ is a common tangent to the given Ⓢ, by § 184. Q. E. D.

CASE 2. Considering a common internal tangent (Fig. 2).

Analysis. Suppose the problem solved, and RQ the required tangent.

Draw the radii OR and $O'Q$.

Then OR and $O'Q$ are $\perp$ to RQ, § 185

and OR is $\parallel$ to $O'Q$. § 95

Through O' draw $MO' \parallel$ to RQ, cutting OR produced in M.

Then $MR = O'Q$. § 127

$\therefore OM = OR + O'Q$.

But MO' is $\perp$ to OM. § 97

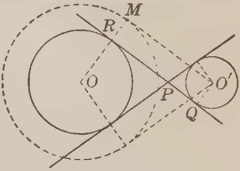


FIG. 2

$\therefore MO'$ is a tangent drawn through O', the center of the smaller ⊙, to a ⊙ with center O and a radius equal to the sum of the radii of the two given Ⓢ.

Construction. With center O and a radius equal to the sum of the radii of the given Ⓢ draw a ⊙.

Through O' draw a tangent MO' to this ⊙. § 246

Through M, the point of contact, draw the radius OM, cutting the given concentric ⊙ in R.

Through O' draw the radius $O'Q \parallel$ to OM. § 233

Then the line RQ is the required tangent. Q.E.F.

Proof. Since MO' is $\perp$ to OM, § 185

and OM is $\parallel$ to $O'Q$, Const.

$\therefore$ MO' is $\perp$ to $O'Q$. § 97

Now $MR = O'Q$. Const.

$\therefore$ RQ is $\parallel$ to MO'. § 130

$\therefore$ RQ is $\perp$ to OR and $O'Q$. § 97

$\therefore$ RQ is a common tangent to the given ⓢ, by § 184. Q.E.D.

Discussion. Let d denote the line segment joining the centers of the given circles, and r and r' their radii respectively.

If $d > r + r'$, two common external and two common internal tangents may be drawn.

If $d = r + r'$, one internal and two external tangents may be drawn.

If $d < r + r'$ but $> r - r'$, two external tangents may be drawn.

If $d = r - r'$, one external tangent may be drawn.

If $d < r - r'$, no solution is possible.

If $r = r'$, MO' coincides with OO' in Case 1. The radii OR and $O'Q$ have then simply to be drawn as perpendiculars to OO'.

2. *To draw a common tangent to two given circles, using the following figures.*

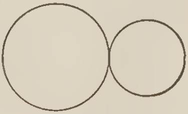


FIG. 1

FIG. 1. By using the method given in Ex. 1, Case 1, two external tangents may be drawn.

One common internal tangent may be drawn by drawing the line of centers of the given circles, and drawing through the point of contact of the circles a perpendicular to that line.

This perpendicular is a common tangent, by §§ 197, 184.

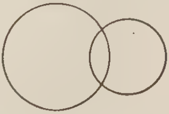


FIG. 2

FIG. 2. By using the method given in Ex. 1, Case 1, two external tangents may be drawn.

No internal tangent is possible.

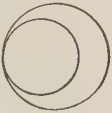


FIG. 3

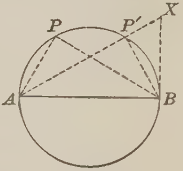
FIG. 3. Only one external tangent can be drawn by drawing the line of centers of the given circles, and drawing through the point of contact of the circles a perpendicular to that line.

This perpendicular is a common tangent, by §§ 197, 184.

3. *The locus of the vertex of a right triangle, having a given hypotenuse as its base, is the circle described upon the given hypotenuse as a diameter.*

Given the line AB and the ⊙ described upon it as a diameter.

To prove that this ⊙ is the locus of the vertex of a right triangle, having AB as its base.



Proof. Let P be any point on the ⊙, and X any point not on the ⊙. Draw AP, BP, AX, BX.

Since arc BPA is a semicircle, Const.

$\therefore \angle APB = 1$ rt. $\angle$. § 215

Let AX or AX produced cut the arc BPA in P', and draw BP'.

Again, $\angle AP'B = 1$ rt. $\angle$. § 215

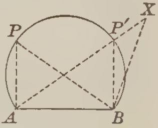
$\therefore \angle BP'X = 1$ rt. $\angle$. § 43

In the $\triangle P'BX$, $\angle P'XB$ is not a rt. $\angle$. § 109

∴ the ⊙ is the required locus, by § 148. Q.E.D.

4. *The locus of the vertex of a triangle, having a given base and a given angle at the vertex, is the arc which forms with the base a segment in which the given angle may be inscribed.*

Given the line AB and an arc forming with this line a segment in which a given $\angle C$ may be inscribed.



To prove that the arc is the locus of the vertex of a triangle having AB as base and $\angle C$ at the vertex.

Proof. Let P be any point on the arc, and X any point not on the arc. Draw AP, BP, AX, BX.

Then $\angle APB = \angle C$. § 217

Let AX or AX produced cut the arc in P', and draw BP'.

Again, $\angle AP'B = \angle C$. § 217

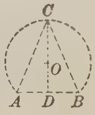
Now $\angle AP'B = \angle P'XB + \angle XBP'$. § 111

$\therefore \angle AP'B > \angle P'XB$. Ax. 11

That is, $\angle AXB < \angle C$. Ax. 8

∴ the arc BPA is the required locus, by § 148. Q.E.D.

5. *To construct an isosceles triangle, having given the base and the angle at the vertex.*



Given the line AB and the $\angle C$.

Required to construct an isosceles $\triangle$ with AB as base and the $\angle C$ at the vertex.

Construction. Upon AB as chord describe the segment of a ⊙ in which $\angle C$ may be inscribed.

Draw the $\perp$ bisector DC of AB, cutting the arc in point C. §§ 229, 228.

Draw AC and BC.

Then the $\triangle ABC$ is the $\triangle$ required. Q. E. F.

Proof. Since arc BCA is the segment upon AB in which $\angle C$ may be inscribed, Const.

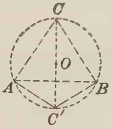
$$\therefore \angle ACB = \angle C.$$

Again, since DC is the $\perp$ bisector of AB, Const.

$$\therefore AC = BC. \qquad \text{§ 150}$$

$\therefore$ the $\triangle ABC$ is the required $\triangle$. Q. E. D.

6. *To construct an isosceles triangle, having given the base and the radius of the circumscribed circle.*



Given the line AB and the radius r.

Required to construct an isosceles $\triangle$ with AB as base, and with the radius of the circumscribed $\odot$ equal to r.

Construction. Describe a $\odot$ with the radius r.

With any point A of the $\odot$ as center and with a radius equal to AB describe an arc cutting the $\odot$ in point B.

Draw AB, and draw the $\perp$ bisector of AB, cutting the $\odot$ in the points C and C'. §§ 229, 228.

Draw AC and BC, and AC' and BC'.

Then the $\triangle ABC$ or $\triangle ABC'$ is the required $\triangle$. Q. E. F.

Proof. Since CC' is the $\perp$ bisector of AB, Const.

$$\therefore AC = BC, \text{ and } AC' = BC'. \qquad \text{§ 150}$$

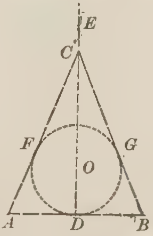
Also, the $\triangle$s ABC and ABC' have the base AB and the circumscribed circle with the radius r. Const.

$\therefore$ $\triangle$s ABC and ABC' satisfy the conditions. Q. E. D.

Discussion. In general, two solutions may be had.

If $AB > 2r$, no solution is possible.

7. *To construct an isosceles triangle, having given the base and the radius of the inscribed circle.*



Given the line AB and the radius r.

Required to construct an isosceles $\triangle$ with the base AB, and with the radius of the inscribed $\odot$ equal to r.

Construction. Draw the $\perp$ bisector DE of AB.

On DE take $DO = r$. §§ 229, 228

With O as center and radius r describe a $\odot$.

Through A and B draw two tangents, AF and BG, to the $\odot$ and let them intersect at C. § 246

Then the $\triangle ABC$ is the required $\triangle$. Q. E. F.

Proof. Let the points of contact of the sides AC and BC with the ⊙ be F and G respectively.

Then $AF = AD$, and $BG = BD$. § 192

But $AD = BD$. Const.

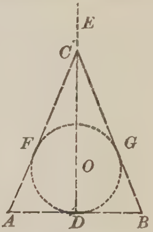
$\therefore AF = BG$. Ax. 8

Now $CF = CG$. § 192

$\therefore AC = BC$. Ax. 1

The $\triangle ABC$ also has the base AB, and the radius of the inscribed ⊙ equal to r. Const.

$\therefore \triangle ABC$ satisfies the conditions. Q.E.D.

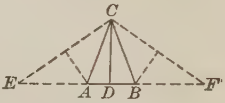


Discussion. If AB is equal or less than $2r$, no solution is possible.

8. *To construct an isosceles triangle, having given the perimeter and the altitude.*

Given the lines EF and DC.

Required to construct an isosceles $\triangle$ with the perimeter equal to EF and the altitude DC.



Analysis. Suppose the problem solved, $\triangle ABC$ the required $\triangle$, and CD the given altitude.

Produce BA to E, making $AE = AC$; and AB to F, making $BF = BC$.

The ⊿ EAC and BFC are isosceles,

and $EF = AC + AB + BF$.

Since $\triangle ABC$ is isosceles and CD the ⊥ through the vertex,

$\therefore AD = DB$.

Also $AE = BF$. Ax. 8

$\therefore ED = DF$. Ax. 1

$\therefore CD$ is the ⊥ bisector of EF.

Construction. Draw the ⊥ bisector of EF and take DC equal to the given altitude. §§ 229, 228

Draw EC and FC.

Draw the ⊥ bisectors of EC and FC, cutting EF in A and B. §§ 229, 228

Draw AC and BC.

Then $\triangle ABC$ is the required $\triangle$. Q.E.F.

Proof. Since A and B are points in the ⊥ bisectors of EC and FC, Const.

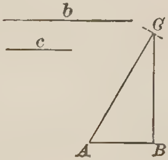
$\therefore EA = AC$, and $BF = BC$. § 150

Now $EF = EA + AB + BF$. Ax. 11

$\therefore EF = AC + AB + BC$. Ax. 9

Since CD is the $\perp$ bisector of EF, Const.
$\therefore EC = FC$, § 150
and $\angle FEC = \angle CFE$. § 74
Also $\angle AEC = \angle ECA$,
and $\angle CFB = \angle BCF$. § 74
$\therefore \angle ECA = \angle BCF$. Ax. 8
$\therefore$ the $\triangle EAC$ is congruent to $\triangle BFC$. § 72
$\therefore AC = BC$. § 76
$\therefore$ the $\triangle ABC$ satisfies the conditions. Q.E.D.

9. *To construct a right triangle, having given the hypotenuse and one side.*



Given the lines b and c.

Required to construct a right $\triangle$ with the hypotenuse equal to b and one side equal to c.

Construction. Draw $AB = c$.

Through B draw a $\perp$ to AB. § 228

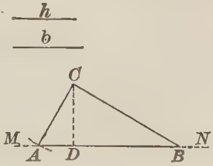
With A as center and b as radius draw an arc cutting the $\perp$ in C.

Draw AC.

Then the $\triangle ABC$ is the required $\triangle$. Q.E.F.

Discussion. In order to make a solution possible, b must be larger than c.

10. *To construct a right triangle, having given one side and the altitude upon the hypotenuse.*



Given the lines b and h.

Required to construct a right triangle with one side equal to b and the altitude upon the hypotenuse equal to h.

Construction. In point D of any line MN erect a $\perp$ to MN. § 228

Upon this $\perp$ take $DC = h$.

With C as center and a radius equal to b describe an arc cutting MN in A.

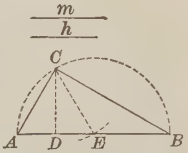
Draw AC.

Through C draw a $\perp$ to AC, cutting MN in B. § 228

Then the $\triangle ABC$ is the required $\triangle$. Q.E.F.

Discussion. In order to make a solution possible, b must be larger than h.

11. *To construct a right triangle, having given the median and the altitude upon the hypotenuse.*



Given the lines m and h.

Required to construct a right △ with the median and the altitude upon the hypotenuse equal to m and h respectively.

Analysis. Suppose the problem solved and △ ABC the required △ with the altitude CD and the median CE drawn.

Since $\angle ACB$ is a rt. ∠, it may be inscribed in a semicircle upon AB as diameter and E as center. § 215

$\therefore CE = AE = EB.$ § 162

Construction. Construct a right ∠, taking one side $DC = h$. § 228

With C as center and m as radius draw an arc cutting the other side in E.

With E as center and the same radius draw an arc cutting DE produced in A and B.

Draw AC and BC.

Then the △ ABC is the required △. Q. E. F.

Proof. The altitude $CD = h$,

and the median $CE = m$. Const.

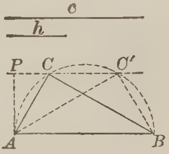
Also, since $\angle ACB$ is inscribed in a semicircle, Const.

$\therefore \angle ACB = 1$ rt. ∠. § 215

∴ the △ ABC satisfies the conditions. Q. E. D.

Discussion. If $m < h$, the problem is impossible.

12. *To construct a right triangle, having given the hypotenuse and the altitude upon the hypotenuse.*



Given the lines c and h.

Required to construct a right △ with c as hypotenuse and h as altitude upon c.

Construction. Draw $AB = c$.

Bisect AB. § 229

Upon AB as diameter draw a semicircle.

Through A draw a ⊥ to AB. § 228

Take $AP = h$.

Through P draw a ∥ to AB, cutting the semicircle in C and C'. § 233

Draw AC, BC, and AC', BC'.

Then the △ ABC or △ ABC' is the required △. Q. E. F

Proof. $AB = c$. Const.

PC' is $\parallel$ to AB. Const.

$\therefore$ the distance from C or C' to AB is equal to AP, that is, equal to h. § 128

Since the $\angle$s ACB and $AC'B$ are inscribed in a semicircle, Const.

$\therefore \angle ACB = \angle AC'B = 1$ rt. $\angle$. § 215

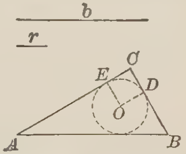
$\therefore$ the $\triangle$s ABC and ABC' satisfy the conditions. Q. E. D.

Discussion. Since $\triangle ABC$ is congruent to $\triangle ABC'$, there is really only one solution.

If $h = \frac{1}{2} AB$, PC' will be a tangent and the $\triangle$ will be isosceles.

If $h > \frac{1}{2} AB$, no solution is possible.

13. *To construct a right triangle, having given the radius of the inscribed circle and one side.*



Given the lines b and r.

Required to construct a right $\triangle$ with one side equal to b, and the radius of the inscribed $\odot$ equal to r.

Analysis. Suppose the problem solved and $\triangle ABC$ the required $\triangle$, with the inscribed $\odot$, and the $\perp$ radii OE and OD upon the sides AC and BC drawn.

Then OD is $\parallel$ to EC, and OE is $\parallel$ to DC. § 95

$\therefore OD = EC$, and $OE = DC$. § 125

Since $EC = DC$, § 192

$\therefore EC = OE = r$. Ax. 8

Construction. Draw the line $AC = b$.

On AC take $CE = r$.

Through E draw a $\perp$ to AC, taking $EO = r$. § 228

With O as center and radius r describe a $\odot$.

Through A draw a tangent to the $\odot$. § 246

Through C draw a $\perp$ to AC, cutting the tangent in B. § 228

Then the $\triangle ABC$ is the required $\triangle$. Q. E. F.

Proof. $AC = b$. Const.

$\angle ACB = 1$ rt. $\angle$. Const.

Draw $OD \perp$ to BC.

Then OD is $\parallel$ to EC. § 95

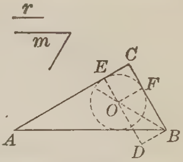
Also OE is $\parallel$ to DC. § 95

$\therefore OD = EC = r$. § 125

$\therefore BC$ is tangent to the $\odot$. § 184

$\therefore$ the $\triangle ABC$ satisfies the conditions. Q. E. D

14. *To construct a right triangle, having given the radius of the inscribed circle and an acute angle.*



Given the line r and the $\angle m$.

Required to construct a right $\triangle$ with the radius of the inscribed $\odot$ equal to r, and one acute $\angle$ equal to $\angle m$.

Analysis. Suppose the problem solved and $\triangle ABC$ the required $\triangle$, with the inscribed $\odot$ drawn, and $\angle CBA = \angle m$.

Since the $\odot$ is tangent to both sides of $\angle CBA$, its center O must lie in the bisector of that $\angle$. § 152

Since the distance from O to BC is equal to r, O must also lie in the $\parallel$ that is drawn at a distance r to BC.

Construction. Construct $\angle B = \angle m$. § 232

Bisect $\angle B$. § 231

Through B draw a $\perp$ to one side of $\angle B$, taking $BD = r$. § 228

Through D draw a $\parallel$ to that side, cutting the bisector in O. § 233

With O as center and radius r draw a $\odot$.

Let DO produced cut this $\odot$ in E.

Through E draw a $\perp$ to DE, cutting the sides of the $\angle B$ in C and A.

Then the $\triangle ABC$ is the required $\triangle$. Q. E. F.

Proof. Draw $OF \perp$ to BC.

Then $OF = DB = r$. § 128

Since O lies on the bisector of $\angle CBA$, Const.

$\therefore$ O is also at a distance r from AB. § 152

$\therefore$ AB and BC are tangent to the $\odot$. § 184

Since AC is $\perp$ to OE, Const.

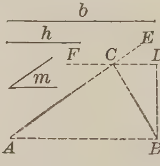
$\therefore$ AC is tangent to the $\odot$. § 184

Since DE is $\parallel$ to BC, Const.

$\therefore$ $\angle ACB = 1$ rt. $\angle$. § 97

$\therefore$ the $\triangle ABC$ satisfies the conditions. Q. E. D.

15. *To construct a triangle, having given the base, the altitude, and an angle at the base.*



Given the lines c and h, and the $\angle m$.

Required to construct a $\triangle$ with c as base, h as altitude, and one $\angle$ at the base equal to $\angle m$.

Construction. Draw AB equal to c.

With AB as one side and A as vertex draw $\angle BAE = \angle m$. § 232

Through B draw a $\perp$ to AB, taking $BD = h$. § 228

Through D draw $DF \parallel$ to BA, cutting AE in C. § 233

Draw BC.

Then $\triangle ABC$ is the $\triangle$ required. Q.E.F.

Proof. $AB = c$. Const.

$\angle BAC = \angle m$. Const.

Since DF is $\parallel$ to BA, Const.

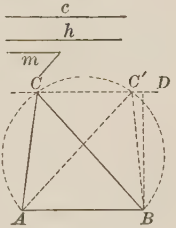
$\therefore$ the distance from C to AB is equal to BD; that is, equal to h. § 128

$\therefore$ the $\triangle ABC$ satisfies the conditions. Q.E.D.

16. *To construct a triangle, having given the base, the altitude, and the angle at the vertex.*

Given the lines c and h, and the $\angle m$.

Required to construct a $\triangle$ with c as base, h as altitude, and the $\angle m$ as $\angle$ at the vertex.



Analysis. Suppose the problem solved, and $\triangle ABC$ the required $\triangle$.

Since the base AB and the $\angle C$ at the vertex are given, one locus for the vertex is an arc forming with AB a segment in which the $\angle m$ may be inscribed.

Another locus for the vertex is a parallel to AB at a distance from it equal to h, the given altitude.

Construction. Draw $AB = c$.

Upon AB as chord describe a segment of a circle in which $\angle m$ may be inscribed. § 247

Through B draw a $\perp$ to AB, taking $BD = h$. § 228

Through D draw a $\parallel$ to AB, cutting the arc in C and C'. § 233

Draw AC, BC, and AC', BC'.

Then $\triangle ABC$ or $\triangle ABC'$ is the required $\triangle$. Q.E.F.

Proof. $AB = c$. Const.

Since the arc BCA forms with AB as chord a segment in which $\angle m$ may be inscribed, Const.

$\therefore \angle ACB = \angle m$. § 217

Since DC' is $\parallel$ to AB and at a distance h from it, Const.

$\therefore$ the distance from C to the base AB is equal to h. § 128

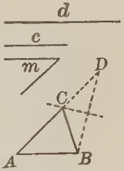
$\therefore$ the $\triangle ABC$ satisfies the conditions. Q.E.D.

Discussion. Since $\triangle ABC$ is congruent to $\triangle ABC'$, there is really only one solution.

If h is equal to the distance from the highest point of the arc to AB, the $\triangle$ will be an isosceles $\triangle$.

If h is larger than the distance from the highest point of the arc to AB, no solution is possible.

17. *To construct a triangle, having given one side, an adjacent angle, and the sum of the other sides.*



Given the lines d and c, and the $\angle m$.

Required to construct a $\triangle$ with c as base, one adjacent $\angle$ equal to $\angle m$, and the sum of the other sides equal to d.

Analysis. Suppose the problem solved and $\triangle ABC$ the required $\triangle$, with AB the given base and $\angle A$ the given $\angle$.

Produce AC, making $CD = CB$, and draw BD.

Then $AD = AC + CB$.

Also the $\triangle BDC$ is isosceles.

$\therefore$ C lies on the $\perp$ bisector of BD.

Construction. Draw $AB = c$.

With AB as one side and A as vertex draw the $\angle A = \angle m$. § 232

Take $AD = d$.

Draw BD.

Draw the $\perp$ bisector of BD, cutting AD in C. §§ 229, 228

Draw BC.

Then $\triangle ABC$ is the required $\triangle$. Q.E.F.

Proof. $AB = c$, and $\angle BAC = \angle m$, by construction.

Since C lies on the $\perp$ bisector of BD, Const.

$\therefore BC = CD$. § 150

Now $AD = AC + CD$. Ax. 11

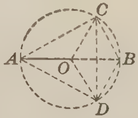
$\therefore AD = AC + BC$. Ax. 9

That is, $AC + BC = d$.

$\therefore$ the $\triangle ABC$ satisfies the conditions. Q.E.D.

Discussion. In order to make a solution possible, d must be larger than c.

18. *To construct an equilateral triangle, having given the radius of the circumscribed circle.*



Given the radius r.

Required to construct an equilateral $\triangle$ with the radius of the circumscribed circle equal to r.

Construction. Describe a $\odot$ with center O and the radius r.

Draw any diameter AOB.

With center B and radius r draw an arc cutting the $\odot$ in C and D.

Draw AC, AD, and CD.

Then the $\triangle ADC$ is the required $\triangle$. Q.E.F

Proof. Draw OC, OD, BC, and BD.

$OB = OC$. § 162

$BC = OB$. Const.

$\therefore$ the $\triangle OBC$ is equilateral.

$\therefore \angle BOC = 60°$. § 107

Similarly, $\angle DOB = 60°$.

$\therefore \angle DOC = 120°$.

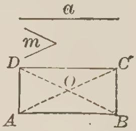
Now $\angle COA = \angle AOD = 120°$. § 58

$\therefore$ arc AC = arc AD = arc DC. § 166

$\therefore AC = AD = DC$. § 170

$\therefore$ the $\triangle ADC$ satisfies the conditions. Q. E. D.

19. *To construct a rectangle, having given one side and the angle between the diagonals.*



Given the line a and the $\angle m$.

Required to construct a rectangle with one side equal to a and the $\angle$ between the diagonals equal to $\angle m$.

Analysis. Suppose the problem solved, and the rectangle $ABCD$ the required rectangle with the diagonals AC and BD drawn intersecting at O, the given $\angle$ being $\angle BOC$, and AB the given side.

In the rt. $\triangle$s ABD and ABC,

$AB = AB$, Iden.

$AD = BC$. § 125

$\therefore$ the rt. $\triangle$s ABD and ABC are congruent. § 68

$\therefore AC = BD$. § 67

Since the diagonals bisect each other, § 131

$\therefore AO = BO$. Ax. 4

$\therefore \angle OBA = \angle BAO$. § 74

Now $\angle OBA + \angle BAO = \angle BOC$. § 111

$\therefore \angle OBA = \frac{1}{2} \angle BOC = \frac{1}{2} \angle m$.

$\therefore$ the isosceles $\triangle ABO$ is determined.

Construction. Draw $AB = a$.

Upon AB as one side and A and B as vertices construct two $\angle$s, each equal to $\frac{1}{2} \angle m$. §§ 231, 232

Let the sides of these $\angle$s meet at O.

On AO produced take $OC = AO$.

On BO produced take $OD = BO$.

Draw BC, CD, and DA.

Then the quadrilateral $ABCD$ is the required rectangle. Q. E. F

Proof. $AB = a.$ Const.

$\angle BOC = \angle BAO + \angle OBA.$ § 111

$\therefore \angle BOC = \frac{1}{2}\angle m + \frac{1}{2}\angle m = m.$

Now $AO = BO,$ § 76

and $AO = OC.$ Const.

$\therefore BO = OC.$ Ax. 8

$\therefore \angle CBO = \angle OCB.$ § 74

Now $\angle BOC = 2 \text{ rt.} \angle s - (\angle CBO + \angle OCB),$ § 107

or $\angle BOC = 2 \text{ rt.} \angle s - 2\angle CBO.$ Ax. 9

Also $\angle BOC = \angle BAO + \angle OBA,$ § 111

or $\angle BOC = 2\angle OBA.$ Ax. 9

$\therefore 2\, OBA = 2 \text{ rt.} \angle s - 2\angle CBO,$ Ax. 8

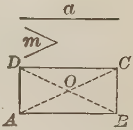
or $\angle OBA + \angle CBO = 1 \text{ rt.} \angle.$ Ax. 4

That is, $\angle CBA = 1 \text{ rt.} \angle.$

Similarly $\angle BAD = 1 \text{ rt.} \angle.$

$\angle ADC = 1 \text{ rt.} \angle.$

$\angle DCB = 1 \text{ rt.} \angle.$



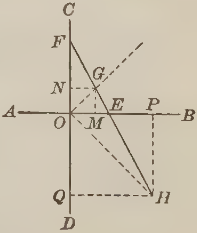
$\therefore$ the quadrilateral $ABCD$ is a rectangle. § 119

$\therefore$ the rectangle $ABCD$ satisfies the conditions. Q. E. D.

Discussion. There are two solutions possible, according to whether $\angle BOC$ or $\angle AOB$ is to be the given angle.

In the latter case $\angle BAO = \frac{1}{2}(2 \text{ rt.} \angle s - \angle AOB)$, and the construction is the same.

20. *Given two perpendiculars, AB and CD, intersecting in O, and a line intersecting these perpendiculars in E and F; to construct a square, one of whose angles shall coincide with one of the right angles at O, and the vertex of the opposite angle of the square shall lie in EF. (Two solutions.)*



Analysis. Suppose the problem solved, and the quadrilateral $OMGN$ one of the required squares.

Draw the diagonal OG.

In the rt. $\triangle s$ OMG and OGN,

$OM = ON,$

and $GM = GN.$ § 65

$\therefore$ the rt. $\triangle s$ OMG and OGN are congruent, § 68

$\therefore \angle MOG = \angle GON.$ § 67

That is, the point G lies on the bisector of $\angle BOC$.

Construction. Bisect the $\angle s$ BOC and DOB. § 231

Let these bisectors meet EF and EF produced in G and H.

From G and H let fall $\perp$s upon the given lines AB and CD.

Let the intersecting points of the $\perp$s and the lines AB and CD be M, P, N, Q.

Then the quadrilaterals $OMGN$ and $OQHP$ are the required squares. Q. E. F.

Proof. In the quadrilateral $OMGN$,

$\angle MON = 1$ rt. $\angle$. Hyp.

Also $\angle GMO = \angle ONG = 1$ rt. $\angle$. Const.

Now $\angle NGM = 4$ rt. $\angle$s $- (\angle MON + \angle GMO + \angle ONG)$. § 144

$\therefore \angle NGM = 4$ rt. $\angle$s $- 3$ rt. $\angle$s. Ax. 8

$\angle NGM = 1$ rt. $\angle$.

Also ON is $\parallel$ to GM,

and OM is $\parallel$ to GN. § 95

$\therefore GM = ON$. § 127

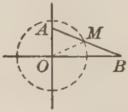
$GM = GN$. § 152

$\therefore ON = GN$. Ax. 8

$\therefore$ the quadrilateral $OMGN$ is a square.

Similarly, the quadrilateral $OQHP$ is a square. Q. E. D.

21. *A straight rod moves so that its ends constantly touch two fixed rods perpendicular to each other. Find the locus of its mid-point.*



Given two $\perp$ rods bisecting each other at O, and one rod AB which moves so that its ends constantly touch the other rods.

Required to find the locus of M, the mid-point of AB.

Construction. With O as center and a radius equal to $\frac{1}{2} AB$ draw a $\odot$.

This $\odot$ is the required locus. Q. E. F.

Proof. Draw the moving rod AB in any of its positions.

Draw OM.

Since $\angle BOA = 1$ rt. $\angle$, Given

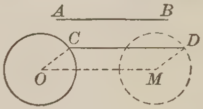
$\therefore$ the $\angle BOA$ may be inscribed in a semicircle with AB as diameter and consequently with M as center and $\frac{1}{2} AB$ as radius. § 215

$\therefore OM = \frac{1}{2} AB$.

That is, the mid-point M will always be at a distance equal to $\frac{1}{2} AB$ from O.

$\therefore$ the locus of the point M is a $\odot$ with O as center and a radius equal to $\frac{1}{2} AB$, by § 160. Q. E. D.

22. *A line moves so that it remains parallel to a given line, and so that one end lies on a given circle. Find the locus of the other end.*



Given the fixed line AB, and the line CD which moves so that it remains ∥ to AB, and so that C lies on the ⊙ with center O.

Required to find the locus of point D.

Construction. Through O draw a ∥ to AB, taking $OM = CD$. § 233

Describe a ⊙ with M as center and a radius equal to the radius of the given ⊙.

This ⊙ with center M is the required locus. Q.E.F.

Proof. Draw CD in any of its positions.

Draw OM ∥ to AB, making $OM = CD$.

Draw OC and MD.

Now OM is ∥ to CD, § 96

and $OM = CD$. Hyp.

∴ the quadrilateral $OMDC$ is a ▱. § 130

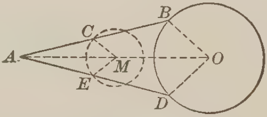
∴ $MD = OC$. § 125

That is, the point D will always be at a distance equal to OC from point M.

∴ the locus of point D is a ⊙ with M as center and with a radius equal to OC, the radius of the given ⊙. Q.E.D.

23. *Find the locus of the mid-point of a line-segment that is drawn from a given external point to a given circle.*

Given the point A, the ⊙ with center O, and the line-segment drawn from A to the ⊙.



Required to find the locus of the mid-point of the line-segment.

Construction. Draw AO.

Bisect AO at M. § 229

Describe a ⊙ with M as center and a radius equal to one half the radius of the given ⊙.

This ⊙ with center M is the required locus. Q.E.F.

Proof. Draw AB, any one of the line-segments.

Bisect AB at C.

Draw MC and OB.

Then $MC = \frac{1}{2} OB$. § 136

Draw AD, any other of the line-segments, with E, the point of bisection.

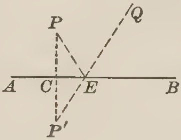
Draw ME and OD.

Then again, $ME = \frac{1}{2} OD = \frac{1}{2} OB$. § 136

That is, the mid-point of the line-segment is always at a distance of $\frac{1}{2} OB$ from point M.

$\therefore$ the locus of the mid-point of the line-segment drawn from A to the given ⊙ is the ⊙ with M as center and with a radius equal to one half the radius of the given ⊙, by § 160. Q. E. D.

24. *To draw lines from two given points P and Q which shall meet on a given line AB and make equal angles with AB.*



Given the line AB and the points P and Q.

Required to draw lines from P and Q which shall meet in the line AB and make equal $\angle$s with AB.

Construction. From P draw $PC \perp$ to AB. § 227

Produce PC and take $CP' = PC$.

Draw $P'Q$, cutting AB in E.

Draw PE.

Then PE and EQ are the required lines. Q. E. F.

Proof. In the right $\triangle$s CEP and $P'EC$,

$PC = CP'$, Const.

and $CE = CE$. Iden.

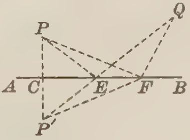
$\therefore$ the $\triangle$s CEP and $P'EC$ are congruent. § 68

$\therefore \angle PEC = \angle CEP'$. § 67

Now $\angle CEP' = \angle BEQ$. § 60

$\therefore \angle PEC = \angle BEQ$, by Ax. 8. Q. E. D.

25. *To find the shortest path from a point P to a line AB and thence to a point Q.*



Given the line AB and the points P and Q.

Required to draw the shortest path from P to AB and thence to Q.

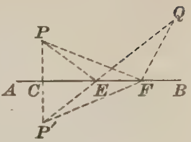
Construction. Draw $PC \perp$ to AB. § 227

Produce PC, taking $CP' = PC$.

Draw $P'Q$, cutting AB in point E.

Draw PE.

Then PE and EQ represent the shortest path from P to AB and thence to Q. Q. E. F.



Proof. Take any other point F on AB.

Draw PF, QF, and $P'F$.

In the rt. △ CEP and $P'EC$,

$PC = CP'$, Const.

and $CE = CE$. Iden.

∴ the △ CEP and $P'EC$ are congruent. § 68

∴ $PE = P'E$. § 67

Similarly $PF = P'F$.

In the △ $P'FQ$, $P'Q < P'F + FQ$. § 112

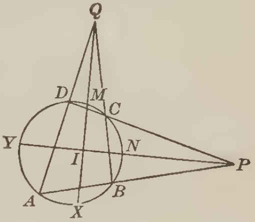
Now $P'Q = P'E + EQ$.

∴ $P'Q = PE + EQ$. Ax. 9

∴ $PE + EQ < PF + FQ$, by Ax. 9. Q. E. D.

Discussion. The shortest path from a point to a line and thence to another point also makes equal angles with the line (Ex. 24).

26. *The bisectors of the angles included by the opposite sides (produced) of an inscribed quadrilateral intersect at right angles.*



Given $ABCD$, the inscribed quadrilateral, with the sides AB and CD produced meeting at P, and with the sides AD and BC produced meeting at Q, and the bisectors of the ∠s P and Q meeting at I.

To prove that PI is ⊥ to QI.

Proof. Let the bisector of ∠ P cut the circle in N and Y, and let the bisector of ∠ Q cut the ⊙ in M and X.

Now $\angle AQX$ is measured by $\frac{1}{2}$(arc AX − arc DM),

and $\angle XQB$ is measured by $\frac{1}{2}$(arc XB − arc MC). § 221

Since $\angle AQX = \angle XQB$, Given

∴ $\frac{1}{2}$(arc AX − arc DM) = $\frac{1}{2}$(arc XB − arc MC). Ax. 8

∴ arc AX − arc DM = arc XB − arc MC. Ax. 3

Similarly, arc YA − arc BN = arc DY − arc CN.

∴ arc AX + arc YA − arc DM − arc BN

= arc XB + arc DY − arc MC − arc CN. Ax. 1

∴ arc YX + arc MN = arc YM + arc XN.

Now $\angle YIX$ is measured by $\frac{1}{2}$(arc YX + arc MN),

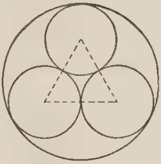
and $\angle MIY$ is measured by $\frac{1}{2}$(arc YM + arc XN). § 219

∴ $\angle YIX = \angle MIY$. Ax. 8

∴ $\angle YIX$ = 1 rt. ∠. § 26

That is, PI is ⊥ to QI. Q. E. D.

27. *Construct this design, making the figure twice this size.*



Construct the equilateral △.

Then draw the small circles with the vertices of the △ as centers and one half its side as radius.

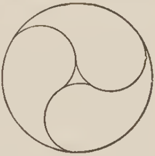
Draw the median from one vertex.

Draw the median from another vertex, cutting the first median drawn.

Produce one of the medians beyond its vertex, cutting one of the circles.

The distance from this point of intersection to the point of intersection of the medians is the radius of the circumscribed circle, whose center is the point of bisection of the medians.

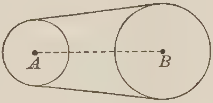
28. *A circular window in a church has a design similar to the accompanying figure. Draw it, making the figure twice this size.*



Construct a figure similar to the one given in Ex. 27.

Erase those lines that are not shown in the adjacent figure.

29. *Two wheels of radii* 1 *ft.* 6 *in. and* 2 *ft.* 3 *in. respectively are connected by a belt, drawn straight between the points of tangency. The centers being* 6 *ft. apart, draw the figure mathematically. Use the scale of* 1 *in. to the foot.*



On the plan 1 ft. will be represented by 1 in.

Draw a straight line $AB = 6$ in.

With A as center and a radius equal to $1\frac{1}{2}$ in. describe a ⊙.

With B as center and a radius equal to $2\frac{1}{4}$ in. draw another ⊙.

Using the method given in Ex. 1, draw the two common external tangents to the two ⓢ.

30. *A water wheel is broken and all but a fragment is lost. A workman wishes to restore the wheel. Make a drawing showing how he can construct a wheel the size of the original.*

Given an arc.

Required to find the radius of the ⊙ of which this arc is a fragment.

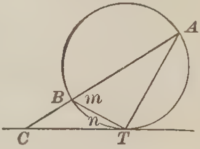
Construction. Draw two chords in the given arc.

Draw the $\perp$ bisectors of these chords. § 229, 228

Then O, the point of intersection of these bisectors, is the center of the original $\odot$. § 177

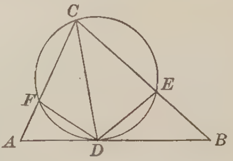
Any radius drawn from O to the arc is the required radius. Q. E. F.

31. *In this figure $\angle m = 62°$, and $\angle n = 28°$. Find the number of degrees in each of the other angles, and determine whether AB is a diameter.*



$\angle m = 62°$, and $\angle n = 28°$. Given
$\angle n$ is measured by $\frac{1}{2}$ arc BT. § 220
$\angle A$ is measured by $\frac{1}{2}$ arc BT. § 214
$\therefore \angle A = 28°$. Ax. 8
$\angle ATB = 180° - (\angle m + \angle A)$. § 107
$\therefore \angle ATB = 180° - 90° = 90°$.
$\angle CBT = 180° - \angle m$. § 43
$\therefore \angle CBT = 118°$.
$\angle ATC = \angle n + \angle ATB$.
$\therefore \angle ATC = 28° + 90° = 118°$.
$\angle TCA = 180° - (\angle ATC + \angle A)$. § 107
$\therefore \angle TCA = 180° - 146° = 34°$.
Since $\angle ATB = 90°$, it is inscribed in a semicircle. § 215
$\therefore AB$ is a diameter.

32. *In this figure $\angle B = 41°$, $\angle A = 65°$, and $\angle BDC = 97°$. Find the number of degrees in each of the other angles, and determine whether CD is a diameter.*



Let BC meet the $\odot$ in E.
Let AC meet the $\odot$ in F.
$\angle B = 41°$, $\angle A = 65°$,
and $\angle BDC = 97°$. Given
$\angle DFC$ is measured by $\frac{1}{2}$ arc DEC. § 214
$\angle BDC$ is measured by $\frac{1}{2}$ arc DEC. § 220
$\therefore \angle DFC = 97°$. Ax. 8
$\angle CDA = 180° - \angle BDC$. § 43
$\therefore \angle CDA = 180° - 97° = 83°$.
$\angle CED = \angle CDA$. §§ 214, 220
$\therefore \angle CED = 83°$.
$\angle AFD = 180° - \angle DFC$. § 43
$\therefore \angle AFD = 180° - 97° = 83°$.
$\angle FDA = 180° - (\angle A + \angle AFD)$. § 107

$\therefore \angle FDA = 180° - 148° = 32°.$

$\angle FCD = \angle FDA.$ § 220

$\therefore \angle FCD = 32°.$

$\angle ACB = 180° - (\angle A + \angle B).$ § 107

$\therefore \angle ACB = 180° - 106° = 74°.$

$\angle DCB = \angle ACB - \angle ACD.$

$\therefore \angle DCB = 74° - 32° = 42°.$

$\angle EDB = \angle DCB.$ § 220

$\therefore \angle EDB = 42°.$

$\angle DEB = 180° - \angle CED.$ § 43

$\therefore \angle DEB = 180° - 83° = 97°.$

Since AB is tangent to the circle, and CD is not $\perp$ to AB, CD does not pass through the center and therefore is not a diameter.

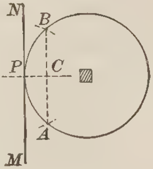
33. *Construct or explain why it is impossible to construct a triangle with sides 3 in., 2 in., 6 in.; also one with sides 5 in., 7 in., 12 in.; also one with sides 2 in., 1 in., $1\frac{1}{2}$ in.*

CASE 1. It is impossible to construct a triangle with sides 3 in., 2 in., 6 in., since 6 in. $>$ 3 in. + 2 in. § 112

CASE 2. It is impossible to construct a triangle with sides 5 in., 7 in., 12 in., since 12 in. = 5 in. + 7 in. § 112

CASE 3. Construct a triangle with sides 2 in., 1 in., $1\frac{1}{2}$ in. § 237

34. *Show how to draw a tangent to this circle at the point P, the center of the circle not being accessible.*



Given point P on a ⊙ whose center is not accessible.

Required to draw a tangent to the ⊙ through point P.

Construction. With P as center and a convenient radius draw an arc cutting the ⊙ in A and B.

Draw AB.

Bisect AB in C. § 229

Draw PC.

Through P draw $MN \perp$ to PC. § 228

Then MN is the required tangent. Q.E.F.

Proof. Since P and C are equidistant from A and B, Const.

$\therefore PC$ is the $\perp$ bisector of AB, § 151

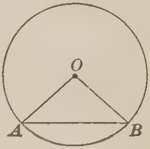
and PC goes through the center of the ⊙. § 174

Now MN is $\perp$ to PC. Const.

$\therefore MN$ is tangent to the ⊙, by § 184. Q.E.D.

Exercise 40. Page 149

1. *In a circle whose center is O the chord AB is drawn so that* $\angle BAO = 27^\circ$. *How many degrees are there in* $\angle AOB$*?*



Given in the $\odot$ whose center is O the chord AB, making with the radius OA the $\angle BAO$ equal to 27°, and the radius OB.

Required to find the number of degrees in the $\angle AOB$.

Solution. In the $\triangle AOB$, $OA = OB$. § 162

$\therefore$ the $\triangle AOB$ is an isosceles $\triangle$. § 62

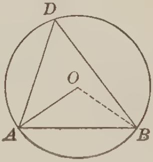
$\therefore \angle BAO = \angle OBA$. § 74

$\therefore \angle BAO + \angle OBA = 2\angle BAO = 2 \times 27^\circ$, or 54°. Ax. 9

But $\angle BAO + \angle OBA + \angle AOB = 180^\circ$. § 107

$\therefore \angle AOB = 180^\circ - 54^\circ = 126^\circ$.

2. *In a circle whose center is O the chord AB is drawn so that* $\angle BAO = 25^\circ$. *On the circle, and on the same side of AB as the center O, the point D is taken and is joined to A and B. How many degrees are there in* $\angle ADB$*?*



Given in the $\odot$ whose center is O the chord AB, making with the radius OA the $\angle BAO$ equal to 25°; point D on the $\odot$ and on the same side of AB as the center O; and the chords AD and BD.

Required to find the number of degrees in the $\angle ADB$.

Solution. Draw the radius OB.

In the $\triangle AOB$, $OA = OB$. § 162

$\therefore$ the $\triangle AOB$ is an isosceles $\triangle$. § 62

$\therefore \angle BAO = \angle OBA$. § 74

$\therefore \angle BAO + \angle OBA = 2\angle BAO = 2 \times 25^\circ$, or 50°. Ax. 9

But $\angle BAO + \angle OBA + \angle AOB = 180^\circ$. § 107

$\therefore \angle AOB = 180^\circ - 50^\circ = 130^\circ$. Ax. 2

$\therefore$ arc $AB = 130^\circ$. § 213

Now $\angle ADB$ is measured by $\frac{1}{2}$ arc AB. § 214

$\therefore \angle ADB = \frac{1}{2}$ of $130^\circ = 65^\circ$.

3. *What is the locus of the mid-point of a chord of a circle formed by secants drawn from a given external point?*

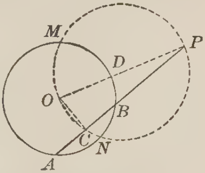
Given any $\odot$ whose center is O, and P an external point; also any secant PBA to the $\odot$ from P.

Required to find the locus of the mid-point of the chord AB.

Construction. Draw PO and bisect it in D. § 229

With D as a center and OD as a radius describe a ⊙ cutting the given ⊙ in M and N, and the chord AB in C.

Then the arc MN is the locus required. Q.E.F.



Proof. Draw OC.

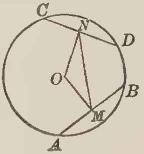
Then OC is $\perp$ to AB. § 215

$\therefore OC$ bisects AB. § 174

That is, the mid-point of chord AB lies in the arc MN.

Therefore arc MN is the locus of the mid-point of the chord AB. Q.E.D.

4. *In a circle whose center is O two perpendiculars OM and ON are drawn to the chords AB and CD respectively, and it is known that $\angle NMO = \angle ONM$. Prove that $AB = CD$.*



Given in the ⊙ whose center is O the ⊥s OM and ON to the chords AB and CD respectively, and the line MN, the $\angle NMO$ being equal to the $\angle ONM$.

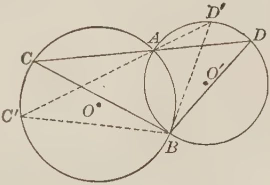
To prove that $AB = CD$.

Proof. In the $\triangle NOM$, $\angle NMO = \angle ONM$. Given

$\therefore OM = ON$. § 76

$\therefore AB = CD$, by § 178. Q.E.D.

5. *Two circles intersect at the points A and B. Through A a variable secant is drawn, cutting the circles at C and D. Prove that the angle DBC is constant.*



Given two Ⓢ intersecting in the points A and B; the variable secant CD terminated by the circumferences; and the chords BC and BD.

To prove that the $\angle DBC$ is constant.

Proof. Let $C'AD'$ be any other position of the given secant CAD.

Draw the chords BC' and BD'.

The ∠s CAC' and DAD' are equal. § 60

$\therefore$ the arcs CC' and DD' have the same number of degrees. § 214

But the ∠s CBC' and DBD' are measured by the arcs CC' and DD' respectively. § 214

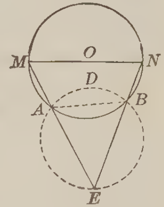
$\therefore$ the ∠s CBC' and DBD' are equal. Ax. 8

$\therefore \angle D'BC + \angle DBD' = \angle D'BC + \angle CBC'$. Ax. 1

That is, $\angle DBC = \angle D'BC'$. Ax. 11

Therefore $\angle DBC$ is constant. Q.E.D.

6. *Let A and B be two fixed points on a given circle, and M and N be the extremities of a rotating diameter of the same circle. Find the locus of the point of intersection of the lines AM and BN.*



Given A and B, two fixed points on the ⊙ whose center is O; MN, a rotating diameter of the given ⊙; and the chords AM and BN.

Required to find the locus of the point of intersection of the chords AM and BN.

Construction. Draw the chord AB.

Produce MA and NB to intersect in E.

Circumscribe the ⊙ ABE about the △ ABE. § 240

Then the ⊙ ABE is the locus required. Q. E. F.

Proof. Case 1. When the lines AM and BN intersect without the given ⊙.

The ∠ E is constant, being always measured by half a semicircumference less half the arc AB. § 221

Therefore the vertex of ∠ E must lie on the arc AEB of the ⊙ constructed.

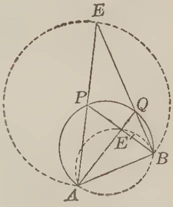
Case 2. When the lines AM and BN intersect within the given ⊙.

The ∠ E is constant, being always measured by half the sum of the arc AB and a semicircumference. § 219

Therefore the vertex of ∠ E must lie on the arc ADB of the ⊙ constructed.

Hence the ⊙ ABE is the locus of the point of intersection of the lines AM and BN. Q.E.D.

7. *Upon a line AB a segment of a circle containing 240° is constructed, and in the segment any chord PQ subtending an arc of 60° is drawn. Find the locus of the point of intersection of AP and BQ; also of AQ and BP.*



Given the segment $APQB$ of a ⊙ containing 240° described on the line AB, and the chord PQ subtending on the segment $APQB$ the arc PQ equal to 60°; also the lines AP and BQ, and the lines AQ and BP.

Required to find (1) the locus of the point of intersection of the lines AP and BQ; (2) the locus of the point of intersection of the lines AQ and BP.

Construction. (1) Produce AP and BQ to intersect in E.

Circumscribe a ⊙ about the △ EAB. § 240

Then the arc BEA of the ⊙ EAB is the locus required.

(2) Let AQ and BP intersect in E'.

Circumscribe a $\odot$ about the $\triangle E'AB$. § 240

Then the arc $BE'A$ of the $\odot E'AB$ is the locus required. Q. E. F.

Proof. (1) Complete the $\odot APQB$.

The $\angle E$ is constant, being always measured by half the arc AB less half the arc PQ. § 221

But the arcs AB and PQ equal $360° - 240°$, or $120°$, and $60°$ respectively. Given

$\therefore \angle E = \frac{1}{2}(120° - 60°) = 30°$. § 221

Therefore the vertex of the $\angle E$ must lie on the arc AEB.

Hence the arc AEB of the $\odot EAB$ is the locus of the intersection of the lines AP and BQ.

(2) The $\angle E'$ is constant, being always measured by half the arc AB greater than half the arc PQ. § 219

But the arcs AB and PQ equal $120°$ and $60°$ respectively. Given

$\therefore \angle E' = \frac{1}{2}(120° + 60°) = 90°$. § 219

Therefore the vertex of the $\angle E'$ must lie on the arc $AE'B$.

Hence the arc $AE'B$ of the $\odot E'AB$ is the locus of the intersection of the lines AQ and BP. Q. E. D.

8. *To construct a square, given the sum of the diagonal and one side.*

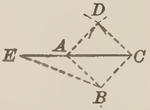
Given line EC, the sum of the diagonal and one side of a square.

Required to construct the square.

Construction. Construct the $\angle ECB = 45°$. §§ 228, 231

Construct the $\angle BEC = \frac{1}{2}$ of $45°$, or $22° 30'$. § 231

Produce CB and EB to intersect in B.



With B as center and BC as radius describe an arc cutting EC in A.

With A and C as centers and BC as radius describe arcs intersecting at D.

Then draw AB, AD, and CD.

Then the quadrilateral $ABCD$ is the square required. Q. E. F.

Proof. The quadrilateral $ABCD$ is equilateral. Const.

$\therefore ABCD$ is a ▱. § 129

$\angle BAC = \angle ACB$. § 74

But $\angle ACB = 45°$. Const.

$\therefore \angle BAC = 45°$. Ax. 9

$\therefore \angle CBA = 90°$. § 107

$\therefore \angle ADC = 90°$. § 124

$\therefore$ $\angle$s BAD and DCB each equal $90°$. § 124

Therefore the ▱ $ABCD$ is a square, by § 65. Q. E. D.

BOOK III

Exercise 42. Page 156

1. *Prove that* $a : b = ma : mb$.

Given the ratio $a : b$.

To prove that $a : b = ma : mb$.

Proof. $a : b = a : b$. Iden.

That is, $\frac{a}{b} = \frac{a}{b}$. § 257

$$\therefore \frac{a}{b} = \frac{ma}{mb}.$$

That is, $a : b = ma : mb$, by § 257. Q. E. D.

2. *If* $a : b = c : d$, *and* $m : n = p : q$, *prove that* $am : bn = cp : dq$.

Given $a : b = c : d$, and $m : n = p : q$.

To prove that $am : bn = cp : dq$.

Proof. $\frac{a}{b} = \frac{c}{d}$,

and $\frac{m}{n} = \frac{p}{q}$. § 257

$$\therefore \frac{am}{bn} = \frac{cp}{dq}. \qquad \text{Ax. 3}$$

That is, $am : bn = cp : dq$, by § 257. Q. E. D.

3. *If* $a : b = c : d$, *prove that* $a : d = bc : d^2$.

Given $a : b = c : d$.

To prove that $a : d = bc : d^2$.

Proof. $\frac{a}{b} = \frac{c}{d}$. § 257

Multiply each member by $\frac{b}{d}$.

Then $\frac{a}{d} = \frac{bc}{d^2}$. Ax. 3

That is, $a : d = bc : d^2$, by § 257. Q. E. D.

4. *If* $a : b = c : d$, *prove that* $1 : b = c : ad$.

Given $a : b = c : d$.

To prove that $1 : b = c : ad$.

Proof. $\frac{a}{b} = \frac{c}{d}$. § 257

Divide each member by a.

Then $\frac{1}{b} = \frac{c}{ad}$. Ax. 4

That is, $1 : b = c : ad$, by § 257. Q. E. D.

5. *If* $a : b = c : d$, *prove that* $ad : b = c : 1$.

Given $a : b = c : d$.

To prove that $ad : b = c : 1$.

Proof. $\frac{a}{b} = \frac{c}{d}$. § 257

Multiply each member by d.

Then $\frac{ad}{b} = \frac{c}{1}$. Ax. 3

That is, $ad : b = c : 1$.

6. *If* $a : b = c : d$, *prove that* $ma : b = mc : d$.

Given $a : b = c : d$.

To prove that $ma : b = mc : d$.

Proof. $\frac{a}{b} = \frac{c}{d}$. § 257

Multiply each member by m.

Then $\frac{ma}{b} = \frac{mc}{d}$. Ax. 3

That is, $ma : b = mc : d$, by § 257. Q. E. D.

7. *If* $a : b = c : d$, *prove that* $ma : nb = mc : nd$.

Given $a : b = c : d$.

To prove that $ma : nb = mc : nd$.

Proof. $\frac{a}{b} = \frac{c}{d}$. § 257

Multiply each member by $\frac{m}{n}$.

Then $\frac{ma}{nb} = \frac{mc}{nd}$. Ax. 3

That is, $ma : nb = mc : nd$, by § 257. Q. E. D.

8. *If* $a : b = c : d$, *prove that* $a - 1 : b = bc - d : bd$.

Given $a : b = c : d$.

To prove that $a - 1 : b = bc - d : bd$.

Proof. $ad = bc$. § 261

Multiply each member by b.

Then $abd = b^2c$. Ax. 3

Subtract bd from each member.

Then $abd - bd = b^2c - bd$. Ax. 2

Factoring, $bd(a - 1) = b(bc - d)$.

$\therefore a - 1 : b = bc - d : bd$, by § 264. Q.E.D.

9. *If* $a : b = c : d$, *prove that* $a + 1 : 1 = bc + d : d$.

Given $a : b = c : d$.

To prove that $a + 1 : 1 = bc + d : d$.

Proof. $ad = bc$. § 261

Add d to each member.

Then $ad + d = bc + d$.

$\therefore d(a + 1) = bc + d$.

Divide each member by d.

Then $a + 1 = \frac{bc + d}{d}$. Ax. 4

$\therefore a + 1 : 1 = bc + d : d$, by § 257. Q.E.D.

10. *If* $a : b = c : d$, *prove that* $1 : bc = 1 : ad$.

Given $a : b = c : d$.

To prove that $1 : bc = 1 : ad$.

Proof. $bc = ad$, § 261

and $1 = 1$. Iden.

$\therefore \frac{1}{bc} = \frac{1}{ad}$. Ax. 4

That is, $1 : bc = 1 : ad$, by § 257. Q.E.D.

11. *If* $a : b = c : d$, prove that $a + b : a - b = c + d : c - d$.

Given $a : b = c : d$.

To prove that $a + b : a - b = c + d : c - d$.

Proof. $a + b : b = c + d : d$, § 267

and $a - b : b = c - d : d$. § 268

That is, $\frac{a + b}{b} = \frac{c + d}{d}$,

and $\frac{a - b}{b} = \frac{c - d}{d}$. § 257

Divide the first members and the second members.

Then $\frac{a+b}{a-b} = \frac{c+d}{c-d}$. Ax. 4

That is, $a+b : a-b = c+d : c-d$, by § 257. Q.E.D.

12. *If* $a : b = b : c$, *prove that* $c : b = b : a$.

Given $a : b = b : c$.
To prove that $c : b = b : a$.

Proof. $ac = b^2$. § 261

$\therefore\ c : b = b : a$, by § 264. Q.E.D.

13. *If* $a : b = b : c$, *prove that* $a : c = b^2 : c^2$.

Given $a : b = b : c$.
To prove that $a : c = b^2 : c^2$.

Proof. $ac = b^2$. § 261

Divide each member by c^2.

Then $\frac{a}{c} = \frac{b^2}{c^2}$. Ax. 4

That is, $a : c = b^2 : c^2$, by § 257. Q.E.D.

14. *If* $a : b = b : c$, *prove that* $(b + \sqrt{ac})(b - \sqrt{ac}) = 0$.

Given $a : b = b : c$.
To prove that $(b + \sqrt{ac})(b - \sqrt{ac}) = 0$.

Proof. $b^2 = ac$. § 261

Extract the square root of each member.

Then $b = \pm\sqrt{ac}$.

$\therefore\ b + \sqrt{ac} = 0$,

and $b - \sqrt{ac} = 0$. Ax. 2

Multiply the right members and the left members.

Then $(b + \sqrt{ac})(b - \sqrt{ac}) = 0$, by Ax. 3. Q.E.D.

15. *If* $a : b = b : c$, *prove that* $ac - 1 : b - 1 = b + 1 : 1$.

Given $a : b = b : c$.
To prove that $ac - 1 : b - 1 = b + 1 : 1$.

Proof. $ac = b^2$. § 261

Subtract 1 from each member.

Then $ac - 1 = b^2 - 1$. Ax. 2

Factoring, $ac - 1 = (b+1)(b-1)$.

$\therefore\ ac - 1 : b - 1 = b + 1 : 1$, by § 264. Q.E.D.

16. *If* $2:7=3:x$, *show that* $2x=21$, *and* $x=10\frac{1}{2}$.

Given $2:7=3:x$.

To prove that $x=10\frac{1}{2}$.

Proof. $2x=21$. § 261

$\therefore x=10\frac{1}{2}$, by Ax. 3. Q.E.D.

17. *Find the value of* x *in the proportion* $1:7=3:x$.

Given $1:7=3:x$.

Required to find the value of x.

Solution. $x=21$, by § 261. Q.E.F.

18. *Find the value of* x *in the proportion* $2:9=5:x$.

Given $2:9=5:x$.

Required to find the value of x.

Solution. $2x=45$. § 261

$\therefore x=22\frac{1}{2}$, by Ax. 4. Q.E.F.

19. *Find the value of* x *in the proportion* $4:28=3:x$.

Given $4:28=3:x$.

Required to find the value of x.

Solution. $4x=84$. § 261

$\therefore x=21$, by Ax. 4. Q.E.F.

20. *Find the value of* x *in the proportion* $2:8=x:12$.

Given $2:8=x:12$.

Required to find the value of x.

Solution. $8x=24$. § 261

$\therefore x=3$, by Ax. 4. Q.E.F

21. *Find the value of* x *in the proportion* $3:5=x:9$.

Given $3:5=x:9$.

Required to find the value of x.

Solution. $5x=27$. § 261

$\therefore x=5\frac{2}{5}$, by Ax. 4. Q.E.F.

22. *Find the value of* x *in the proportion* $7:21=x:5$.

Given $7:21=x:5$.

Required to find the value of x.

Solution. $21x=35$. § 261

$\therefore x=1\frac{2}{3}$, by Ax. 4. Q.E.F.

23. *Find the value of* x *in the proportion* $3 : 5 = x + 1 : 10$.

Given $3 : 5 = x + 1 : 10$.

Required to find the value of x.

Solution. $5x + 5 = 30$. § 261

$\therefore 5x = 25$. Ax. 2

$\therefore x = 5$, by Ax. 4. Q. E. F.

24. *Find the value of* x *in the proportion* $8 : 15 = 2x + 3 : 45$.

Given $8 : 15 = 2x + 3 : 45$.

Required to find the value of x.

Solution. $15(2x + 3) = 8 \times 45$. § 261

$\therefore 2x + 3 = 24$. Ax. 4

$\therefore 2x = 21$. Ax. 2

$\therefore x = 10\frac{1}{2}$, by Ax. 4. Q. E. F.

25. *Find the value of* x *in the proportion* $0.8 : x = 4 : 9$.

Given $0.8 : x = 4 : 9$.

Required to find the value of x.

Solution. $4x = 7.2$. § 261

$\therefore x = 1.8$, by Ax. 4. Q. E. F.

26. *Find the value of* x *in the proportion* $0.7 : x = 21 : 15$.

Given $0.7 : x = 21 : 15$.

Required to find the value of x.

Solution. $21x = 10.5$. § 261

$\therefore x = 0.5$, by Ax. 4. Q. E. F.

27. *Find the value of* x *in the proportion* $0.25 : x = 5 : 8$.

Given $0.25 : x = 5 : 8$.

Required to find the value of x.

Solution. $5x = 2$. § 261

$\therefore x = 0.4$, by Ax. 4. Q. E. F.

28. *Find the value of* x *in the proportion* $x : 1.3 = 4 : 0.26$.

Given $x : 1.3 = 4 : 0.26$.

Required to find the value of x.

Solution. $0.26x = 5.2$. § 261

$\therefore x = 20$, by Ax. 4. Q. E. F

29. *Find the value of x in the proportion $x : 2.7 = 7 : 5.4$.*

Given $x : 2.7 = 7 : 5.4$.

Required to find the value of x.

Solution. $5.4x = 7 \times 2.7$. § 261

$\therefore x = 3\frac{1}{2}$, by Ax. 4. Q.E.F.

30. *Find the value of x in the proportion $x : 8.1 = 0.3 : 0.9$.*

Given $x : 8.1 = 0.3 : 0.9$.

Required to find the value of x.

Solution. $0.9x = 2.43$. § 261

$\therefore x = 2.7$, by Ax. 4. Q.E.F.

31. *Find the value of x in the proportion $2 : x = x : 32$.*

Given $2 : x = x : 32$.

Required to find the value of x.

Solution. $x^2 = 64$. § 261

$\therefore x = \pm 8$, by Ax. 5. Q.E.F.

32. *Find the value of x in the proportion $7 : x = x : 28$.*

Given $7 : x = x : 28$.

Required to find the value of x.

Solution. $x^2 = 196$. § 261

$\therefore x = \pm 14$, by Ax. 5. Q.E.F.

33. *Find the value of x in the proportion $1 : 1 + x = x - 1 : 3$.*

Given $1 : 1 + x = x - 1 : 3$.

Required to find the value of x.

Solution. $x^2 - 1 = 3$. § 261

$\therefore x^2 = 4$. Ax. 1

$\therefore x = \pm 2$, by Ax. 5. Q.E.F.

34. *Find the value of x in the proportion $5 : x - 2 = x + 2 : 1$.*

Given $5 : x - 2 = x + 2 : 1$.

Required to find the value of x.

Solution. $x^2 - 4 = 5$. § 261

$\therefore x^2 = 9$. Ax. 1

$\therefore x = \pm 3$, by Ax. 5. Q.E.F.

35. *Find the value of x in the proportion $x^2 : 2a = 3a : 6$.*

Given $x^2 : 2a = 3a : 6$.

Required to find the value of x.

Solution. $6x^2 = 6a^2$. § 261

$\therefore x^2 = a^2$. Ax. 4

$\therefore x = \pm a$, by Ax. 5. Q. E. F.

36. *Find the value of x in the proportion $x : 4a = 2a^2 : x^2$.*

Given $x : 4a = 2a^2 : x^2$.

Required to find the value of x.

Solution. $x^3 = 8a^3$. § 261

$\therefore x = 2a$, or $a(-1 \pm \sqrt{-3})$, by Ax. 5. Q. E. F.

37. *Find the value of x in the proportion $a : 1 = x - 1 : 7$.*

Given $a : 1 = x - 1 : 7$.

Required to find the value of x.

Solution. $x - 1 = 7a$. § 261

$\therefore x = 7a + 1$, by Ax. 1. Q. E. F.

38. *Find the value of x in the proportion $x + 1 : x - 1 = 3 : 2$.*

Given $x + 1 : x - 1 = 3 : 2$.

Required to find the value of x.

Solution. $3x - 3 = 2x + 2$. § 261

$\therefore x = 5$, by Ax. 1. Q. E. F.

39. *Find the value of x in the proportion $3 : x + 4 = x - 4 : 3$.*

Given $3 : x + 4 = x - 4 : 3$.

Required to find the value of x.

Solution. $x^2 - 16 = 9$. § 261

$\therefore x^2 = 25$. Ax. 1

$\therefore x = \pm 5$, by Ax. 5. Q. E. F.

40. *Find the value of x in the proportion $ab : b = b - cx : bc - x$.*

Given $ab : b = b - cx : bc - x$.

Required to find the value of x.

Solution. $ab(bc - x) = b(b - cx)$. § 261

$\therefore a(bc - x) = b - cx$. Ax. 4

$\therefore abc - ax = b - cx$.

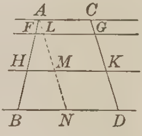
$\therefore ax - cx = abc - b$. Ax. 1

$\therefore x(a - c) = b(ac - 1)$.

$\therefore x = \dfrac{b(ac - 1)}{a - c}$, by Ax. 4. Q. E. F.

Exercise 43. Page 159

1. *In the figure of § 275 suppose $AH = 5$ in., $AF = 2$ in., and $CK = 6$ in. Find the length of CG.*



Given the ∥ lines AC, FG, HK, and BD, cut by the transversals AB and CD in points A, F, H, B, and C, G, K, D respectively; also $AH = 5$ in., $AF = 2$ in., and $CK = 6$ in.

Required to find CG.

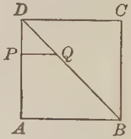
Solution. $AF : CG = AH : CK$. § 275

$\therefore$ 2 in. $: CG = 5 : 6$. Ax. 9

$\therefore 5\,CG = 12$ in. § 261

$\therefore CG = 2\frac{2}{5}$ in., by Ax. 4. Q.E.F.

2. *In this square PQ is ∥ to AB. If a side of the square is 10 in., $DB = 14.14$ in. If $DP = 3$ in., what is the length of DQ?*



Given DB, the diagonal of the square $ABCD$, and PQ, a line joining AD and BD and ∥ to AB; also $AD = 10$ in., $DB = 14.14$ in., and $DP = 3$ in.

Required to find DQ.

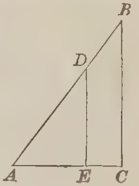
Solution. $DA : DB = DP : DQ$. § 273

$\therefore 10 : 14.14 = 3$ in. $: DQ$. Ax. 9

$\therefore 10\,DQ = 42.42$ in. § 261

$\therefore DQ = 4.242$ in., by Ax. 4. Q.E.F.

3. *The sides of a triangle are respectively 3 in., 4 in., and 5 in. A line is drawn parallel to the 4-inch side, cutting the 3-inch side 1 in. from the vertex of the largest angle. Find the length of the two segments cut from the longest side.*



Given the $\triangle ABC$, with sides AB, BC, and AC equal to 5 in., 4 in., and 3 in. respectively; also DE drawn ∥ to BC, so that $EC = 1$ in.

Required to find AD and DB.

Solution. $AC : AE = AB : AD$. § 273

That is, $AC : AC - EC = AB : AD$.

Then $3 : 2 = 5$ in. $: AD$. Ax. 9

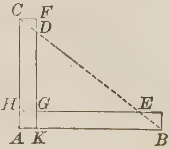
$\therefore 3\,AD = 10$ in. § 261

$\therefore AD = 3\frac{1}{3}$ in. Ax. 4

Since $DB = AB - AD$,

$\therefore DB = 5$ in. $- 3\frac{1}{3}$ in. $= 1\frac{2}{3}$ in., by Ax. 9. Q.E.F.

4. *Two pieces of timber 1 ft. wide are fitted together at right angles, as here shown. AB is 8 ft. long, AC 6 ft. long, and the distance BC, along the dotted line, is 10 ft. A carpenter finds it necessary to saw along the dotted line. Find the length of the slanting cut across the upright piece; across the horizontal piece.*



Given the rectangles AF and BG joined so that $\angle EGF$ is a rt. $\angle$; also CB cutting FG in D, and GE in E. $AB = 8$ ft., $AC = 6$ ft., $BC = 10$ ft., and $AK = 1$ ft.

Required to find CD and BE.

Solution. Produce EG to meet AC at H.

Since KF is $\parallel$ to AC, Given

$\therefore CD : CB = AK : AB.$ § 274

$\therefore CD : 10 \text{ ft.} = 1 : 8.$ Ax. 9

$\therefore 8\,CD = 10 \text{ ft.}$ § 261

$\therefore CD = 1\frac{1}{4} \text{ ft.}$ Ax. 4

Similarly $BE : BC = AH : AC.$ § 274

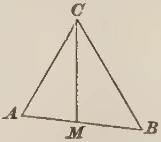
$\therefore BE : 10 \text{ ft.} = 1 : 6.$ Ax. 9

$\therefore 6\,BE = 10 \text{ ft.}$ § 261

$\therefore BE = 1\frac{2}{3}$ ft., by Ax. 4. Q.E.F.

Exercise 44. Page 164

1. *In a triangle ABC, $AB = 6.5$, $CA = 6$, $BC = 7$. Find the segments of AB made by the bisector of the angle C.*



Given, in the $\triangle ABC$, $AB = 6.5$, $CA = 6$, and $BC = 7$; also CM, the bisector of $\angle C$, cutting AB at M.

Required to find AM and BM.

Solution. $AM : BM = AC : BC.$ § 279

Then $AM + BM : AM = AC + BC : AC.$ § 267

Or $AB : AM = AC + BC : AC.$

$\therefore 6.5 : AM = 13 : 6.$ Ax. 9

$\therefore 13\,AM = 39.$ § 261

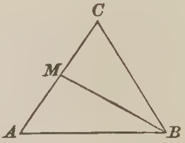
$\therefore AM = 3.$ Ax. 4

But $BM = AB - AM.$

$\therefore BM = 6.5 - 3$ Ax. 9

$= 3.5.$ Q.E.F.

2. *In a triangle ABC, $CA = 7.5$, $BC = 7$, $AB = 8$. Find the segments of CA made by the bisector of the angle B.*



Given, in the $\triangle ABC$, $CA = 7.5$, $BC = 7$, and $AB = 8$; also BM, the bisector of $\angle B$, cutting AC at M.

Required to find AM and CM.

Solution. $AM : CM = AB : BC.$ § 279

Then $AM + CM : AM = AB + BC : AB.$ § 267

Or $AC : AM = AB + BC : AB.$

$\therefore 7.5 : AM = 15 : 8.$ Ax. 9

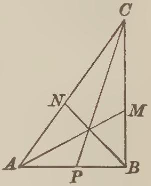
$\therefore 15\,AM = 60.$ § 261

$\therefore AM = 4.$ Ax. 4

But $CM = AC - AM.$ Ax. 2

$\therefore CM = 7.5 - 4$, or 3.5, by Ax. 9. Q. E. F.

3. *The sides of a triangle are* 12, 16, 20. *Find the segments of the sides made by bisecting the angles.*



Given, in the $\triangle ABC$, $AB = 12$, $BC = 16$, and $AC = 20$; also lines AM, BN, and CP, the bisectors of $\angle$s A, B, and C respectively, cutting BC, AC, and AB in points M, N, and P respectively.

Required to find BM and CM, AN and CN, and AP and BP.

Solution. $BM : CM = AB : AC.$ § 279

Then $BM + CM : BM = AB + AC : AB.$ § 267

Or $BC : BM = AB + AC : AB.$

$\therefore 16 : BM = 32 : 12.$ Ax. 9

$\therefore 32\,BM = 192.$ § 261

$\therefore BM = 6.$ Ax. 4

But $CM = BC - BM.$ Ax. 2

$\therefore CM = 16 - 6$, or 10. Ax. 9

Similarly $AN : CN = AB : BC.$ § 279

Then $AN + CN : AN = AB + BC : AB.$ § 267

Or $AC : AN = AB + BC : AB.$

$\therefore 20 : AN = 28 : 12.$ Ax. 9

$\therefore 28\,AN = 240.$ § 261

$\therefore AN = 8\frac{4}{7}.$ Ax. 4

But $CN = AC - AN.$ Ax. 2

$\therefore CN = 20 - 8\frac{4}{7}$, or $11\frac{3}{7}$. Ax. 9

Again, $AP : BP = AC : BC.$ § 279

Then $AP + BP : AP = AC + BC : AC.$ § 267

Or $AB : AP = AC + BC : AC.$

$\therefore 12 : AP = 36 : 20.$ Ax. 9

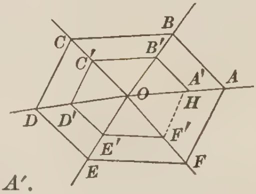
$\therefore 36\,AP = 240.$ § 261

$\therefore AP = 6\tfrac{2}{3}.$ Ax. 4

But $BP = AB - AP.$ Ax. 2

$\therefore BP = 12 - 6\tfrac{2}{3}$, or $5\tfrac{1}{3}$, by Ax. 9. Q.E.F.

4. *If a spider, in making its web, makes $A'B'$ ∥ to AB, $B'C'$ ∥ to BC, $C'D'$ ∥ to CD, $D'E'$ ∥ to DE, and $E'F'$ ∥ to EF, and then runs a line from F' ∥ to FA, will it strike the point A'? Prove it.*



Given the lines $A'B'$, $B'C'$, $C'D'$, $D'E'$, $E'F'$ ∥ to AB, BC, CD, DE, EF respectively in the spider web shown in the figure; also a line $F'H$ drawn from F' ∥ to FA.

To prove that $F'H$ passes through point A'.

Proof.

$$\frac{OA'}{OA} = \frac{OB'}{OB};\ \frac{OB'}{OB} = \frac{OC'}{OC};\ \frac{OC'}{OC} = \frac{OD'}{OD};$$

$$\frac{OD'}{OD} = \frac{OE'}{OE};\ \frac{OE'}{OE} = \frac{OF'}{OF};\ \frac{OF'}{OF} = \frac{OH}{OA}. \qquad \text{§ 273}$$

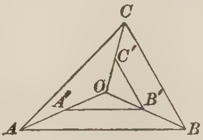
$$\therefore \frac{OA'}{OA} = \frac{OH}{OA}. \qquad \text{Ax. 8}$$

$$\therefore OA' = OH. \qquad \text{§ 263}$$

That is, H coincides with A'.

Therefore $F'H$ passes through A', by Post. 1. Q.E.D.

5. *From any point O within the triangle ABC the lines OA, OB, OC are drawn and are bisected respectively by A', B', and C'. Prove that $\angle CBA = \angle C'B'A'$.*



Given the lines OA, OB, and OC drawn from O, any point within the $\triangle ABC$, to A, B, and C respectively; A', B', and C', the mid-points of OA, OB, and OC respectively; also $A'B'$ and $C'B'$ including the $\angle C'B'A'$.

To prove that $\angle CBA = \angle C'B'A'$.

Proof. $A'B'$ is ∥ to AB,

and $B'C'$ is ∥ to BC. § 136

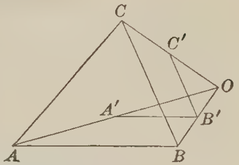
$\therefore \angle OBA = \angle OB'A'$,

and $\angle CBO = \angle C'B'O.$ § 102

$\therefore \angle OBA + \angle CBO = \angle OB'A' + \angle C'B'O.$ Ax. 1

That is, $\angle CBA = \angle C'B'A'$, by Ax. 11. Q.E.D.

6. *Prove Ex. 5 if the point O is outside the triangle.*

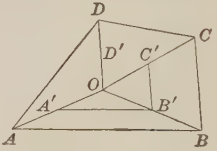


Given the lines OA, OB, and OC drawn from O, any point outside the $\triangle ABC$, to A, B, and C respectively; A', B', and C', the mid-points of OA, OB, and OC respectively; also $A'B'$ and $C'B'$, including the $\angle C'B'A'$.

To prove that $\angle CBA = \angle C'B'A'$.

Proof. $A'B'$ is $\parallel$ to AB,
and $B'C'$ is $\parallel$ to BC. § 136
$\therefore \angle OBA = \angle OB'A'$,
and $\angle OBC = \angle OB'C'$. § 102
$\therefore \angle OBA - \angle OBC = \angle OB'A' - \angle OB'C'$. Ax. 2
That is, $\angle CBA = \angle C'B'A'$. Q.E.D.

7. *From any point O within the quadrilateral ABCD lines are drawn to the vertices A, B, C, D, and are bisected by A', B', C', D'. Prove that* $\angle CBA = \angle C'B'A'$.

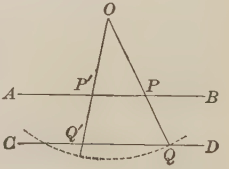


Given the lines OA, OB, OC, and OD drawn from any point O within the quadrilateral $ABCD$ to the vertices A, B, C, and D respectively; also A', B', C', and D', the mid-points of OA, OB, OC, and OD respectively; also $A'B'$ and $C'B'$, including the $\angle C'B'A'$.

To prove that $\angle CBA = \angle C'B'A'$.

Proof. $A'B'$ is $\parallel$ to AB,
and $B'C'$ is $\parallel$ to BC. § 136
$\therefore \angle CBO = \angle C'B'O$,
and $\angle OBA = \angle OB'A'$. § 102
$\therefore \angle CBO + \angle OBA = \angle C'B'O + \angle OB'A'$. Ax. 1
That is, $\angle CBA = \angle C'B'A'$, by Ax. 11. Q.E.D.

8. *If a pendulum swings at the point O, cutting two parallel lines at P and Q respectively, the ratio OP : OQ is constant.*



Given OQ and OQ', any two positions of the swinging pendulum which cuts the line AB in P and P', and CD in Q and Q', AB and CD being $\parallel$.

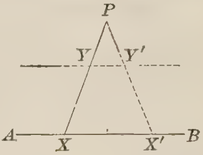
To prove that $\frac{OP}{OQ}$ is a constant ratio.

Proof. $\frac{OP}{OQ} = \frac{OP'}{OQ'}$. § 273

Therefore the value of the ratio $\frac{OP}{OQ}$ does not change for any position of the pendulum OQ.

$\therefore$ the ratio $\frac{OP}{OQ}$ is constant. Q. E. D.

9. *Through a fixed point P a line is drawn cutting a fixed line at X. PX is then divided at Y so that the ratio $PY : YX$ is constant. Find the locus of the point Y as X moves along the fixed line.*



Given the line PX drawn to a fixed line AB at X from a fixed external point P, divided so that $PY : YX$ is the constant ratio $m : n$.

Required to find the locus of the point Y as X moves along the fixed line AB.

Construction. Through Y draw $YY' \parallel$ to AB. § 233

Then YY' is the required locus. Q. E. F.

Proof. Draw any other line PX' from P to AB, cutting AB at X' and YY' at Y'.

Then $PY : YX = PY' : Y'X'$. § 273

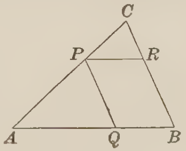
Since $PY : YX = m : n$, Given

$\therefore$ YY' divides PX' in the given ratio $m : n$. Ax. 8

Since YY' divides any line drawn from P to the fixed line AB in the ratio $m : n$, it divides every line from P to AB in the ratio $m : n$.

$\therefore$ YY', a line drawn through $Y \parallel$ to AB, is the locus of the point Y as X moves along the fixed line AB. Q. E. D.

10. *From the point P on the side CA of the triangle ABC parallels to the other sides are drawn meeting AB in Q and BC in R. Prove that $AQ : QB = BR : RC$.*



Given the lines PQ and PR drawn $\parallel$ to CB and AB respectively from any point P in the side CA of $\triangle ABC$, cutting AB and CB in Q and R respectively.

To prove that $AQ : QB = BR : RC$.

Proof. PQ is $\parallel$ to CB. Given

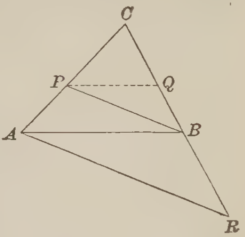
$\therefore AQ : QB = AP : PC$. § 273

Also PR is $\parallel$ to AB. Given

$\therefore AP : PC = BR : RC$. § 273

$\therefore AQ : QB = BR : RC$, by Ax. 8. Q. E. D.

11. *In the triangle ABC, P and Q are taken on the sides CA and BC so that $AP : PC = BQ : QC$. AR is then drawn parallel to PB, meeting CB produced in R. Prove that CB is the mean proportional between CQ and CR.*



Given, in the $\triangle ABC$ points P and Q, taken on the sides CA and BC respectively, so that $AP : PC = BQ : QC$; also line PB, and AR drawn ∥ to PB, meeting CB produced in R.

To prove that $CQ : CB = CB : CR$.

Proof. Draw the line PQ.

Since $AP : PC = BQ : QC$, Given

$\therefore$ PQ is ∥ to AB. § 276

$\therefore$ $CQ : CB = CP : CA$. § 274

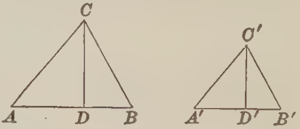
Also PB is ∥ to AR. Given

$\therefore$ $CB : CR = CP : CA$. § 274

$\therefore$ $CQ : CB = CB : CR$, by Ax. 8. Q.E.D.

Exercise 45. Page 170

1. *The corresponding altitudes of two similar triangles have the same ratio as any two corresponding sides.*



Given CD and $C'D'$, corresponding altitudes of the two similar △ ABC and $A'B'C'$.

To prove that $\dfrac{CD}{C'D'} = \dfrac{CA}{C'A'} = \dfrac{CB}{C'B'} = \dfrac{AB}{A'B'}$.

Proof. $\angle A = \angle A'$. § 282

$\therefore$ rt. △ ADC and $A'D'C'$ are similar. § 287

$$\therefore \frac{CD}{C'D'} = \frac{CA}{C'A'}. \qquad \S\ 282$$

Now, in the similar △ ABC and $A'B'C'$, Given

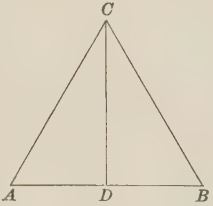
$$\frac{CA}{C'A'} = \frac{CB}{C'B'} = \frac{AB}{A'B'}. \qquad \S\ 282$$

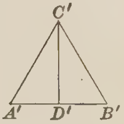
$$\therefore \frac{CD}{C'D'} = \frac{CA}{C'A'} = \frac{CB}{C'B'} = \frac{AB}{A'B'}, \text{ by Ax. 8.} \qquad \text{Q.E.D.}$$

2. *The base and altitude of a triangle are* 15 *in. and* 7 *in. respectively. The corresponding base of a similar triangle is* 3.75 *in. Find the corresponding altitude.*

Given CD and $C'D'$, corresponding altitudes of the two similar △ ABC and $A'B'C'$, AB and $A'B'$ being corresponding bases, $AB = 15$ in., $CD = 7$ in., and $A'B' = 3.75$ in.

Required to find $C'D'$.





Solution. $\angle A = \angle A'$. § 282

$\therefore$ rt. △ ADC and $A'D'C'$ are similar. § 287

$\therefore CD : C'D' = AD : A'D'$. § 282

Likewise $\angle B = \angle B'$. § 282

$\therefore$ rt. △ BDC and $B'D'C'$ are similar. § 287

$\therefore CD : C'D' = BD : B'D'$. § 282

$\therefore AD : A'D' = BD : B'D'$. Ax. 8

$\therefore AD : BD = A'D' : B'D'$. § 265

$$\therefore \frac{AD + BD}{AD} = \frac{A'D' + B'D'}{A'D'}. \qquad \text{§ 267}$$

That is, $$\frac{AB}{AD} = \frac{A'B'}{A'D'}. \qquad \text{Ax. 11}$$

Then $AB : A'B' = AD : A'D'$. § 265

But $CD : C'D' = AD : A'D'$. Proved

$\therefore AB : A'B' = CD : C'D'$. Ax. 8

$\therefore 15 : 3.75 = 7 : C'D'$. Ax. 9

$\therefore 15\, C'D' = 26.25$. § 261

$\therefore C'D' = 1.75$. Ax. 4

Therefore the length of $C'D'$ is 1.75 in. Q.E.F.

3. *If two parallels are cut by three concurrent transversals, the corresponding segments of the parallels are proportional.*

Given the two ∥ lines AE and BF cut by the three concurrent transversals OA, OC, and OE, in the points A, C, E, and B, D, F respectively.

To prove that $AC : CE = BD : DF$.

Proof. FIG. 1. In △ OCA and ODB,

$\angle ACO = \angle BDO$,

and $\angle OAC = \angle OBD$. § 100

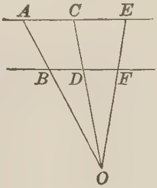


FIG. 1

$\therefore \triangle OCA$ and ODB are similar. § 286

$\therefore AC : BD = OC : OD.$ § 282

Similarly, in $\triangle OCE$ and ODF,

$\angle OCE = \angle ODF,$

and $\angle CEO = \angle DFO.$ § 102

$\therefore \triangle OCE$ and ODF are similar. § 286

$\therefore CE : DF = OC : OD.$ § 282

$\therefore AC : BD = CE : DF.$ Ax. 8

Or $AC : CE = BD : DF$, by § 265. Q. E. D.

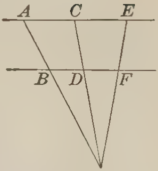


FIG. 1

Proof. FIG. 2. In $\triangle OCA$ and ODB,

$\angle COA = \angle DOB,$ § 60

and $\angle ACO = \angle BDO.$ § 100

$\therefore \triangle OCA$ and ODB are similar. § 286

$\therefore AC : BD = OC : OD.$ § 282

Similarly, in $\triangle OCE$ and ODF,

$\angle EOC = \angle FOD,$ § 60

and $\angle OCE = \angle ODF.$ § 100

$\therefore \triangle OCE$ and ODF are similar. § 286

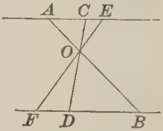


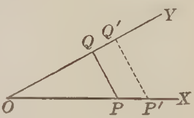
FIG. 2

$\therefore CE : DF = OC : OD.$ § 282

$\therefore AC : BD = CE : DF.$ Ax. 8

Or $AC : CE = BD : DF$, by § 265. Q. E. D.

4. *The point P is any point on the side OX of the angle XOY. From P a perpendicular PQ is let fall on OY. Prove that for any position of P on OX the ratio $OP : PQ$ is constant, and the ratio $PQ : OQ$ is constant.*



Given PQ a line drawn from P, any point on side OX of $\angle XOY$, $\perp$ to OY, cutting OY in Q.

To prove that the ratio $OP : PQ$ is constant; also that the ratio $PQ : OQ$ is constant.

Proof. Take P' any other position of P on OX, and draw $P'Q' \perp$ to OY, cutting OY in Q'.

The rt. $\triangle OPQ$ and $OP'Q'$ are similar. § 287

$\therefore OP : OP' = PQ : P'Q'.$ § 282

$\therefore OP : PQ = OP' : P'Q'.$ § 265

$\therefore$ the value of the ratio $\dfrac{OP}{PQ}$ does not change for any position of P.

$\therefore$ the ratio $\dfrac{OP}{PQ}$ is constant.

From the similar $\triangle OPQ$ and $OP'Q'$,

$PQ : P'Q' = OQ : OQ'.$ § 282

$\therefore PQ : OQ = P'Q' : OQ'.$ § 265

$\therefore$ the value of the ratio $\frac{PQ}{OQ}$ does not change for any position of P.

The ratio $\frac{PQ}{OQ}$ is therefore constant. Q.E.D.

5. *In drawing a map of a triangular field with sides 75 rd., 60 rd., and 50 rd. respectively, the longest side is drawn 1 in. long. How long are the other two sides drawn?*

Given the triangular field ABC, represented by the map designated $A'B'C'$; also $AB = 75$ rd., $BC = 60$ rd., $CA = 50$ rd., and $A'B' = 1$ in. Required to find $B'C'$ and $C'A'$.

Solution. $\triangle ABC$ and $A'B'C'$ are similar. Given

$\therefore AB : A'B' = BC : B'C'$. § 282

$\therefore AB : BC = A'B' : B'C'$. § 265

$\therefore 75 : 60 = 1 : B'C'$. Ax. 9

$\therefore 75\ B'C' = 60$. § 261

$\therefore B'C' = \frac{4}{5}$. Ax. 4

Again, $AB : A'B' = AC : A'C'$. § 282

$\therefore AB : AC = A'B' : A'C'$. § 265

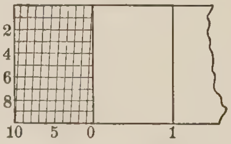
$\therefore 75 : 50 = 1 : A'C'$. Ax. 9

$\therefore 75\ A'C' = 50$. § 261

$\therefore A'C' = \frac{2}{3}$. Ax. 4

Therefore the lengths of $B'C'$ and $A'C'$ are $\frac{4}{5}$ in. and $\frac{2}{3}$ in. respectively. Q.E.F.

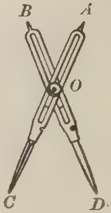
6. *This figure represents part of a diagonal scale used by draftsmen. The distance from 0 to 10 is 1 centimeter, or 10 millimeters. Show how to measure 5 mm.; 1 mm.; 0.9 mm.; 0.5 mm.; 1.5 mm. On what proposition does this depend?*



The divisions on the diagonal scale are millimeters, as here shown. Hence to measure 5 mm. move the diagonal scale to the right or left five divisions, or 5 mm.; to measure 1 mm. move the scale to the right or left one division. In the ninth parallel to the base line (counting from the top), the first segment on the right represents a length of 0.9 mm. The first segment on the right of the fifth parallel represents a length of 0.5 mm. The first two segments together on the same parallel represent a length of 1.5 mm.

This problem depends on Prop. IX, § 273.

7. *This figure represents a pair of proportional compasses used by draftsmen. By adjusting the screw at O the lengths OA and OC, and the corresponding lengths OB and OD, may be varied proportionally. Prove that △ OAB is always similar to △ OCD. If OA = 3 in. and OC = 5 in., then AB is what part of CD?*



Given the figure $ABCD$ representing a pair of proportional compasses; also lines AB and CD.

To prove that $\triangle OAB$ is similar to $\triangle OCD$.

Proof. $\angle AOB = \angle COD$. § 60

Also $OA : OB = OC : OD$. Given

$\therefore$ △s OAB and OCD are similar, by § 288. Q. E. D.

Again, given $OA = 3$ in. and $OC = 5$ in.

To find AB as expressed by a part of CD.

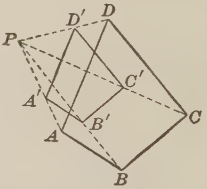
Solution. $OA : OC = AB : CD$. Given

$\therefore 3 : 5 = AB : CD$. Ax. 9

$\therefore 5\,AB = 3\,CD$. § 261

$\therefore AB = \frac{3}{5}\,CD$, by Ax. 4. Q. E. F.

8. *ABCD is any polygon and P is any point. On AP any point A′ is taken and A′B′ is drawn parallel to AB as shown. Then B′C′ and C′D′ are drawn parallel to BC and CD. Is D′A′ parallel to DA? Is A′B′C′D′ similar to ABCD? Prove it.*



Given PA, PB, PC, and PD, lines drawn to the vertices A, B, C, and D of the polygon $ABCD$ from any point P; also $A'B'$ drawn from any point A' in AP ∥ to AB, meeting PB in B'; and lines $B'C'$ and $C'D'$, as shown in the figure, ∥ to BC and CD respectively, and $A'D'$ joining A' to D'.

To prove that 1. $D'A'$ is ∥ to DA,

and that 2. $A'B'C'D'$ is similar to $ABCD$.

Proof. 1. $A'B'$ is ∥ to AB. Given

$\therefore PA' : PA = PB' : PB$. § 273

Also $B'C'$ is ∥ to BC. Given

$\therefore PB' : PB = PC' : PC$. § 273

Again, $C'D'$ is ∥ to CD. Given

$\therefore PC' : PC = PD' : PD$. § 273

Therefore $$\frac{PA'}{PA} = \frac{PB'}{PB} = \frac{PC'}{PC} = \frac{PD'}{PD}.$$ Ax. 8

Or $$\frac{PA'}{PA} = \frac{PD'}{PD}.$$ Ax. 8

$\therefore D'A'$ is ∥ to DA, by § 276. Q. E. D.

Proof. 2. Since $PA' : PA = PB' : PB$, § 273
and $\angle A'PB'$ is $\angle APB$. Iden.
$\therefore$ △ $PA'B'$ and PAB are similar. § 288
In like manner, △ $PB'C'$ and PBC are similar,
△ $PC'D'$ and PCD are similar,
and △ $PA'D'$ and PAD are similar.

Then $\frac{A'B'}{AB} = \frac{PB'}{PB} = \frac{B'C'}{BC} = \frac{PC'}{PC} = \frac{C'D'}{CD} = \frac{PD'}{PD} = \frac{A'D'}{AD}$. § 282

That is, $\frac{A'B'}{AB} = \frac{B'C'}{BC} = \frac{C'D'}{CD} = \frac{A'D'}{AD}$. Ax. 8

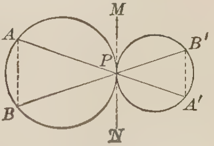
Then polygons $A'B'C'D'$ and $ABCD$ have their corresponding sides proportional.

Furthermore $\angle PB'A' = \angle PBA$,
and $\angle C'B'P = \angle CBP$. § 102
$\therefore \angle PB'A' + \angle C'B'P = \angle PBA + \angle CBP$. Ax. 1
That is, $\angle C'B'A' = \angle CBA$. Ax. 11
Similarly, it may be shown that
$\angle D'C'B' = \angle DCB$.
Again, $\angle PD'C' = \angle PDC$,
and $\angle PD'A' = \angle PDA$. § 102
$\therefore \angle PD'C' - \angle PD'A' = \angle PDC - \angle PDA$. Ax. 2
That is, $\angle A'D'C' = \angle ADC$.
Similarly, it may be shown that
$\angle B'A'D' = \angle BAD$.

Then polygons $A'B'C'D'$ and $ABCD$ have their corresponding angles equal.

$\therefore A'B'C'D'$ is similar to $ABCD$, by § 282. Q. E. D.

9. *If two circles are tangent externally, the corresponding segments of two lines drawn through the point of contact and terminated by the circles are proportional.*

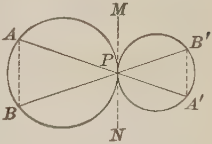


Given the lines AA' and BB' drawn through P, the point of contact of the ⓢ $PB'A'$ and PBA tangent externally, and terminated by the circumferences.

To prove that $PA : PA' = PB : PB'$.

Proof. Draw AB and $A'B'$.
Also draw the common internal tangent MN.
Now $\angle BAP$ is measured by $\frac{1}{2}$ arc BP, § 214
and $\angle BPN$ is measured by $\frac{1}{2}$ arc BP. § 220
$\therefore \angle BAP = \angle BPN$. Ax. 8

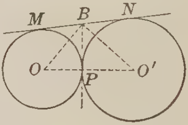
Similarly $\angle B'A'P$ is measured by $\frac{1}{2}$ arc $B'P$, § 214
and $\angle B'PM$ is measured by $\frac{1}{2}$ arc $B'P$. § 220
$\therefore \angle B'A'P = \angle B'PM$. Ax. 8
But $\angle BPN = \angle B'PM$. § 60
$\therefore \angle BAP = \angle B'A'P$. Ax. 8
Again, $\angle APB = \angle A'PB'$. § 60
Therefore the $\triangle$s BPA and $B'PA'$ are similar. § 286
Hence $PA : PA' = PB : PB'$, by § 282. Q. E. D.



10. *If two circles are tangent externally, their common external tangent is the mean proportional between their diameters.*

Given d and d', the diameters of the two ⊙s whose centers are O and O' respectively, tangent externally at point P; also MN, their common external tangent.

To prove that $d : MN = MN : d'$.



Proof. Draw OO', the line of centers.

Also draw PB, the common internal tangent, cutting MN in B.

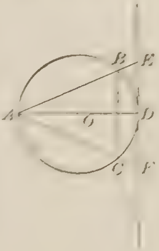
Then draw BO and BO'.

Now $\angle MBO = \angle OBP$,
and $\angle O'BN = \angle PBO'$. § 192
$\therefore \angle MBO + \angle O'BN = \angle OBP + \angle PBO'$. Ax. 1
But $\angle MBO + \angle O'BN + \angle OBP + \angle PBO' = 2$ rt. $\angle$s. § 43
$\therefore \angle OBP + \angle PBO' = \frac{1}{2}$ of 2 rt. $\angle$s, or 90°. Ax. 4
$\therefore \angle OBO' = 1$ rt. $\angle$. Ax. 11
Now OO' passes through P. § 197
$\therefore$ $\triangle$s BPO and $O'PB$ are rt. $\triangle$s. § 185
$\therefore \angle POB$ is the complement of $\angle BO'P$.
Also $\angle POB$ is the complement of $\angle OBP$. § 107
$\therefore \angle BO'P = \angle OBP$. § 58
$\therefore$ rt. $\triangle$s OPB and $O'PB$ are similar. § 287
$\therefore OP : PB = PB : O'P$. § 282
But $PB = MB = NB = \frac{1}{2} MN$. § 192
$\therefore OP : \frac{1}{2} MN = \frac{1}{2} MN : O'P$. Ax. 9
$\therefore 2\,OP : MN = MN : 2\,O'P$. Ax. 3
Now $d = 2\,OP$, and $d' = 2\,O'P$. § 163
$\therefore d : MN = MN : d'$, by Ax. 9. Q. E. D.

11. *AB and AC are chords drawn from any point A on a circle, and AD is a diameter. If the tangent at D intersects AB and AC at E and F, the triangles ABC and AEF are similar.*

Given AD, the diameter of the ○ whose center is O; AB and AC, two chords drawn from A, meeting the tangent to the ○ at D in points E and F respectively; also the line BC.

To prove that △s ABC and AEF are similar.



Proof. $\angle CAB = \angle CAB$. Iden.

$\angle BCA$ is measured by $\frac{1}{2}$ arc AB. § 214

That is, $\angle BCA$ is measured by $\frac{1}{2}$(arc ABD − arc BD). Ax. 9

$\angle E$ is measured by $\frac{1}{2}$(arc ACD − arc BD). § 221

But arc ABD = arc ACD. § 175

∴ arc ABD − arc BD = arc ACD − arc BD. Ax. 2

Then $\angle BCA = \angle E$. Ax. 8

△s ABC and AEF are similar, by § 286. Q.E.D.

12. *If AD and BE are two altitudes of the triangle ABC, the triangles DEC and ABC are similar.*

Given AD and BE, two altitudes of the △ ABC; also the line DE.

To prove that △s DEC and ABC are similar.

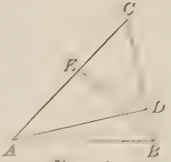


FIG. 1

Proof. FIG. 1. In the rt. △s ADC and BEC, Given

$\angle C = \angle C$. Iden.

∴ rt. △s ADC and BEC are similar. § 287

∴ $AC : BC = DC : EC$. § 282

Then in the △s ABC and DEC,

we have $\angle C = \angle C$, Iden.

and $AC : BC = DC : EC$. Proved

∴ △s ABC and DEC are similar, by § 288. Q.E.D.

FIG. 2. In the rt. △s ADC and BEC, Given

$\angle DCA = \angle BCE$. § 60

∴ rt. △s ADC and BEC are similar. § 287

∴ $AC : BC = DC : EC$. § 282

Now $\angle ACB = \angle ECD$. § 60

∴ △s ABC and DEC are similar, by § 288. Q.E.D.

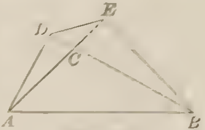


FIG. 2

Exercise 46. Page 176

1. *The perimeters of two similar polygons are 18 in. and 14 in. If a side of the first is 3 in., find the corresponding side of the second.*

Given AB and $A'B'$, two corresponding sides of the two similar polygons M and M', whose perimeters are p and p' respectively; $p = 18$ in., $p' = 14$ in., and $AB = 3$ in.

Required to find $A'B'$.

Solution. $p : p' = AB : A'B'.$ § 291

$\therefore\ 18 : 14 = 3 : A'B'.$ Ax. 9

$\therefore\ 18\,A'B' = 42.$ § 261

$\therefore\ A'B' = 2\frac{1}{3}.$ Ax. 4

Therefore $A'B' = 2\frac{1}{3}$ in. Q. E. F.

2. *In two similar triangles ABC and $A'B'C'$, $AB = 6$ in., $BC = 7$ in., $CA = 8$ in., and $A'B' = 9$ in. Find $B'C'$ and $C'A'$.*

Given, in the two similar $\triangle$ ABC and $A'B'C'$, $AB = 6$ in., $BC = 7$ in., $CA = 8$ in., and $A'B' = 9$ in.

Required to find $B'C'$ and $C'A'$.

Solution. From the similar $\triangle$ ABC and $A'B'C'$,

$AB : A'B' = BC : B'C'.$ § 282

$\therefore\ 6 : 9 = 7 : B'C'.$ Ax. 9

$\therefore\ 6\,B'C' = 63.$ § 261

$\therefore\ B'C' = 10.5.$ Ax. 4

Again, from the similar $\triangle$ ABC and $A'B'C'$,

$AB : A'B' = CA : C'A'.$ § 282

$\therefore\ 6 : 9 = 8 : C'A'.$ Ax. 9

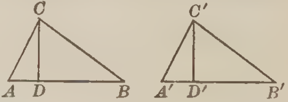
$\therefore\ 6\,C'A' = 72.$ § 261

$\therefore\ C'A' = 12.$ Ax. 4

$\therefore\ B'C' = 10.5$ in., and $C'A' = 12$ in. Q. E. F.

3. *The corresponding bases of two similar triangles are 11 in. and 13 in. The altitude of the first is 6 in. Find the corresponding altitude of the second.*

Given CD and $C'D'$, corresponding altitudes of the two similar $\triangle$ ABC and $A'B'C'$, AB and $A'B'$ being corresponding bases; $AB = 11$ in., $A'B' = 13$ in., and $CD = 6$ in.



Required to find $C'D'$.

Solution. $\angle A = \angle A'.$ § 282

$\therefore$ rt. $\triangle$ ADC and $A'D'C'$ are similar. § 287

$\therefore\ CD : C'D' = AD : A'D'.$ § 282

Likewise $\angle B = \angle B'.$ § 282

$\therefore$ rt. $\triangle$ BDC and $B'D'C'$ are similar. § 287

$\therefore\ CD : C'D' = BD : B'D'.$ § 282

$\therefore\ AD : A'D' = BD : B'D'.$ Ax. 8

$\therefore\ AD : BD = A'D' : B'D'.$ § 265

$\therefore\ \dfrac{AD + BD}{AD} = \dfrac{A'D' + B'D'}{A'D'}.$ § 267

That is, $AB : AD = A'B' : A'D'.$ Ax. 11

Then $AB : A'B' = AD : A'D'$. § 265

But $CD : C'D' = AD : A'D'$. Proved

$\therefore AB : A'B' = CD : C'D'$. Ax. 8

$\therefore 11 : 13 = 6 : C'D'$. Ax. 9

$\therefore 11\,C'D' = 78$. § 261

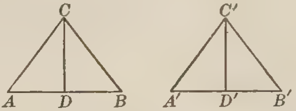
$\therefore C'D' = 7\frac{1}{11}$. Ax. 4

$\therefore C'D' = 7\frac{1}{11}$ in. Q.E.F.

4. *The perimeter of an equilateral triangle is* 51 *in. Find the side of an equilateral triangle of half the altitude.*

Given, in the equilateral $\triangle$s ABC and $A'B'C'$, $C'D'$, the altitude of $\triangle A'B'C'$, equal to half CD, the altitude of $\triangle ABC$; also $AB + BC + CA = 51$ in.

Required to find $A'C'$.



Solution. $\angle A = \angle A'$. § 145

$\therefore$ rt. $\triangle$s ADC and $A'D'C'$ are similar. § 287

$AC : A'C' = CD : C'D'$. § 282

But $C'D' = \frac{1}{2}CD$. Given

$\therefore CD : C'D' = 2 : 1$.

$\therefore AC : A'C' = 2 : 1$. Ax. 9

In $\triangle ABC$, $AB = AC = BC$. Given

$\therefore AC = \frac{1}{3}(AB + AC + BC) = 17$.

$\therefore 17 : A'C' = 2 : 1$. Ax. 9

$\therefore 2\,A'C' = 17$. § 261

$\therefore A'C' = 8.5$. Ax. 4

$\therefore A'C' = 8.5$ in. Q.E.F.

5. *The sides of a polygon are* 2 *in.*, $2\frac{1}{2}$ *in.*, $3\frac{1}{4}$ *in.*, 3 *in.*, *and* 5 *in. Find the perimeter of a similar polygon whose longest side is* 7 *in.*

Given the similar polygons AC and $A'C'$, with side $A'B'$ corresponding to side AB. $AB = 5$ in., $BC = 3$ in., $CD = 3\frac{1}{4}$ in., $DE = 2\frac{1}{2}$ in., $AE = 2$ in., and $A'B' = 7$ in. Denote the perimeters of polygons AC and $A'C'$ by p and p' respectively.

Required to find p'.

Solution. $\frac{p}{p'} = \frac{AB}{A'B'}$. § 291

But $p = 5 + 3 + 3\frac{1}{4} + 2\frac{1}{2} + 2 = 15\frac{3}{4}$. Ax. 11

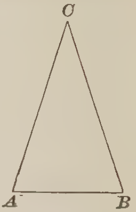
$\therefore 15\frac{3}{4} : p' = 5 : 7$. Ax. 9

$\therefore 5\,p' = 110\frac{1}{4}$. § 261

$\therefore p' = 22\frac{1}{20}$. Ax. 4

$\therefore p' = 22\frac{1}{20}$ in. Q.E.F.

6. *The perimeter of an isosceles triangle is* 13, *and the ratio of one of the equal sides to the base is* $1\frac{2}{3}$. *Find the three sides.*

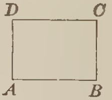


Given AB, the base of the isosceles $\triangle ABC$ whose perimeter is 13, the ratio of side AC to base AB being $1\frac{2}{3}$.

Required to find AB, AC, and BC.

Solution. $AC : AB = 1\frac{2}{3} : 1.$ Given
$\therefore 1\frac{2}{3} AB = AC.$ § 261
$\therefore AB = \frac{3}{5} AC.$ Ax. 4
But $AC = BC.$ § 62
Then $AB + AC + BC = 2\frac{3}{5} AC.$ Ax. 1
$\therefore 2\frac{3}{5} AC = 13.$ Ax. 9
$\therefore AC = 5.$ Ax. 4
Then $BC = 5,$ Ax. 9
and $AB = 3.$ Ax. 9
$\therefore AB = 3$, $AC = 5$, and $BC = 5$. Q.E.F.

7. *The perimeter of a rectangle is* 48 *in., and the ratio of two of the sides is* $\frac{5}{7}$. *Find the sides.*



Given the rectangle $ABCD$ whose perimeter is 48 in., the ratio of AD to DC being $\frac{5}{7}$.

Required to find the sides of rectangle $ABCD$.

Solution. $AD = BC$, and $AB = CD.$ § 125
Since $AD : DC = 5 : 7,$ Given
$AD = \frac{5}{7} DC.$ § 261
Then $BC = \frac{5}{7} DC.$ Ax. 9
$\therefore AD + BC + AB + DC = 3\frac{3}{7} DC.$ Ax. 1
$\therefore 3\frac{3}{7} DC = 48.$ Ax. 9
$\therefore DC = 14.$ Ax. 4
Then $AD = 10$, by Ax. 3. Q.E.F

8. *In drawing a map to the scale* $\frac{1}{100000}$, *what length will represent the sides of a county that is a rectangle* 25 *mi. long and* 10 *mi. wide? Answer to the nearest tenth of an inch.*

Solution. The length of the county, expressed in inches, is $25 \times 5280 \times 12$, or 1,584,000 in.; the width of the county is $10 \times 5280 \times 12$, or 633,600 in. Since the map is drawn to the scale $\frac{1}{100000}$, 15.8 in. will represent on the map the length of the county, and 6.3 in. the width. Q.E.F.

9. *Two circles touch at P. Through P three lines are drawn, meeting one circle in A, B, C, and the other in A′, B′, C′ respectively. Prove the triangles ABC, A′B′C′ similar.*

Given (Fig. 1) two ⓢ whose centers are O and O' tangent externally at P, through which lines AA', BB', and CC' are drawn, having their extremities on the circles; also $\triangle ABC$ and $A'B'C'$.

To prove that $\triangle ABC$ and $A'B'C'$ are similar.

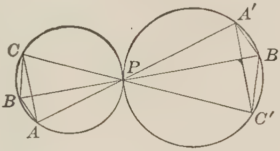


FIG. 1

Proof. $\angle BCA = \angle BPA$,

and $\angle B'C'A' = \angle B'PA'$. § 217

But $\angle BPA = \angle B'PA'$. § 60

$\therefore \angle BCA = \angle B'C'A'$. Ax. 8

Likewise, it may be shown that

$$\angle CAB = \angle C'A'B'.$$

Therefore $\triangle ABC$ and $A'B'C'$ are similar, by § 286. Q.E.D.

Given (Fig. 2) two ⓢ whose centers are O and O' tangent internally at P, through which lines AA', BB', and CC' are drawn, cutting the inner circle at A, B, C respectively, and terminating in the outer circle at A', B', C', respectively; also $\triangle ABC$ and $A'B'C'$.

To prove that $\triangle ABC$ and $A'B'C'$ are similar.

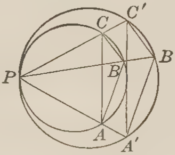


FIG. 2

Proof. $\angle ACB = \angle APB$,

and $\angle A'C'B' = \angle A'PB'$. § 217

But $\angle APB = \angle A'PB'$. Iden.

$\therefore \angle ACB = \angle A'C'B'$. Ax. 8

Likewise, it may be shown that

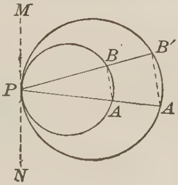
$$\angle BAC = \angle B'A'C'.$$

Therefore $\triangle ABC$ and $A'B'C'$ are similar, by § 286. Q.E.D.

10. *If two circles are tangent internally, all chords of the greater circle drawn from the point of contact are divided proportionally by the smaller circle.*

Given the ⓢ whose centers are O and O' tangent internally at P; also PA' and PB', any two chords of the larger circle cut at A and B respectively by the smaller circle.

To prove that $PA' : A'A = PB' : B'B$.



Proof. Draw AB and $A'B'$.

Draw the common tangent MN through P.

Then $\angle PBA$ is measured by $\frac{1}{2}$ arc PA. § 214

Also $\angle NPA$ is measured by $\frac{1}{2}$ arc PA. § 220

$\therefore \angle PBA = \angle NPA$. Ax. 8

Again, $\angle PB'A'$ is measured by $\frac{1}{2}$ arc PA'. § 214

Also $\angle NPA'$ is measured by $\frac{1}{2}$ arc PA'. § 220

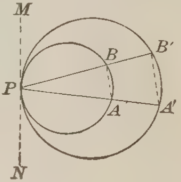
$\therefore \angle PB'A' = \angle NPA'$. Ax. 8

But $\angle NPA$ is $\angle NPA'$. Iden.

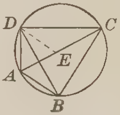
$\therefore \angle PBA = \angle PB'A'$. Ax. 8

$\therefore AB$ is $\parallel$ to $A'B'$. § 103

$\therefore PA' : A'A = PB' : B'B$, by § 273. Q. E. D.



11. *In an inscribed quadrilateral the product of the diagonals is equal to the sum of the products of the opposite sides.*



Given AC and BD, the diagonals of the quadrilateral $ABCD$ inscribed in the $\odot ABC$.

To prove that $AC \times BD = AB \times DC + AD \times BC$.

Proof. Draw DE, making

$\angle EDC = \angle ADB$.

Then in the $\triangle$ ABD and ECD,

$\angle ADB = \angle EDC$. Hyp.

Also $\angle DBA = \angle DCE$. § 217

$\therefore \triangle$ ABD and ECD are similar. § 286

$\therefore DC : DB = EC : AB$. § 282

$\therefore DC \times AB = DB \times EC$. § 261

Furthermore in the $\triangle$ AED and BCD,

$\angle CAD = \angle CBD$. § 217

Also, since $\angle ADB = \angle EDC$, Hyp.

$\therefore \angle ADB + \angle BDE = \angle EDC + \angle BDE$. Ax. 1

Or $\angle ADE = \angle BDC$. Ax. 11

$\therefore \triangle$ AED and BCD are similar. § 286

$\therefore AD : BD = AE : BC$. § 282

$\therefore BD \times AE = AD \times BC$. § 261

But $DB \times EC = DC \times AB$. Proved

$\therefore DB \times EC + BD \times AE = DC \times AB + AD \times BC$. Ax. 1

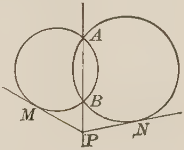
$\therefore BD(EC + AE) = DC \times AB + AD \times BC$.

Since $EC + AE = AC$, Ax. 11

$\therefore BD \times AC = DC \times AB + AD \times BC$, by Ax. 9. Q. E. D.

Exercise 47. Page 181

1. *The tangents to two intersecting circles, drawn from any point in their common chord produced, are equal.*



Given AB, the common chord of the two intersecting Ⓢ; also tangents PM and PN drawn from any point P in AB produced to the Ⓢ.

To prove that $PM = PN$.

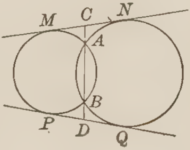
Proof. $\overline{PM}^2 = PA \times PB$,

and $\overline{PN}^2 = PA \times PB$. § 302

$\therefore \overline{PM}^2 = \overline{PN}^2$. Ax. 8

$\therefore PM = PN$, by Ax. 5. Q.E.D.

2. *The common chord of two intersecting circles, if produced, bisects their common tangents.*



Given MN and PQ, the common tangents of the intersecting Ⓢ; also AB, their common chord, cutting MN in C and PQ in D.

To prove that $CM = CN$; also $DP = DQ$.

Proof. $\overline{CM}^2 = CB \times AC$

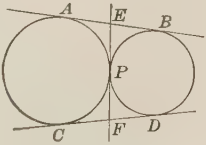
and $\overline{CN}^2 = CB \times AC$. § 302

$\therefore \overline{CM}^2 = \overline{CN}^2$. Ax. 8

$\therefore CM = CN$. Ax. 5

Similarly $DP = DQ$. Q.E.D.

3. *If two circles are tangent externally, the common internal tangent bisects the two common external tangents.*



Given AB and CD, the common external tangents of the Ⓢ tangent externally at P; also EF, the common internal tangent, cutting AB and CD in E and F respectively.

To prove that $EA = EB$, and $FC = FD$.

Proof. $EA = EP$,

and $EB = EP$. § 192

$\therefore EA = EB$. Ax. 8

Similarly, it may be shown that

$FC = FD$. Q.E.D.

4. *If a line drawn from a vertex of a triangle divides the opposite side into segments proportional to the adjacent sides, the line bisects the angle at the vertex.*

Given, in $\triangle ABC$, line CM drawn, cutting AB in M so that $AM : MB = AC : BC$.

To prove that CM bisects $\angle C$.

Proof. The bisector of the $\angle ACB$ must pass through M. § 279

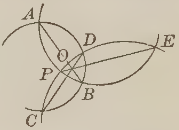
Hence CM has the two points C and M in common with the bisector of the $\angle ACB$.

Therefore CM must coincide with that bisector. § 54

$\therefore$ CM bisects the $\angle ACB$. Q. E. D.

5. *If three circles intersect one another, the common chords are concurrent.*

Case 1. When a portion of the plane is common to the three intersecting circles.



Given AB, CD, and EP, the common chords of three intersecting ⓢ, inclosing the area DPB common to the three ⓢ.

To prove that AB, CD, and EP are concurrent.

Proof. Let two of the chords, AB and CD, meet at O.

Join the point of intersection E to O, and suppose that EO produced meets the arc ABE in P, and the arc EDC in Q.

Then $EO \times OP = AO \times BO$;

also $EO \times OQ = DO \times CO$. § 299

But $AO \times BO = DO \times CO$. § 299

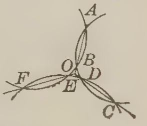
$\therefore EO \times OP = EO \times OQ$. Ax. 8

$\therefore OP = OQ$. Ax. 4

That is, point Q coincides with point P.

$\therefore$ AB, CD, and EP are concurrent, by § 154. Q. E. D.

Case 2. When the three intersecting circles inclose a portion of the plane not included by any one of the circles.



Given AB, CD, and EF, the common chords of three intersecting ⓢ, inclosing EBD, a portion of the plane not found within any of the three ⓢ.

To prove that AB, CD, and EF are concurrent.

Proof. Let two of the chords, CD and EF, intersect at O.

Draw OB, and produce it to meet the arcs CDB and EBA.

Suppose that OB cuts the arc CDB in G, and the arc EBA in A.

Then $OB \times OG = OD \times OC$,

and $OB \times OA = OE \times OF$. § 303

But $OD \times OC = OE \times OF$. § 303

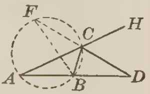
$\therefore OB \times OG = OB \times OA$. Ax. 8

$\therefore OG = OA$. Ax. 4

That is, point G coincides with point A.

$\therefore AB$, CD, and EF are concurrent, by § 154. Q. E. D.

6. *The square on the bisector of an exterior angle of a triangle is equal to the product of the segments determined by this bisector upon the opposite side, diminished by the product of the other two sides.*



Given the $\triangle ABC$; also CD, the bisector of the exterior $\angle BCH$, meeting AB produced in D.

To prove that $\overline{CD}^2 = AD \times BD - AC \times BC$.

Proof. Circumscribe the $\odot ABC$ about the $\triangle ABC$.

Produce DC to meet the $\odot$ again at F, and draw BF.

Then $\angle A = \angle F$. § 217

Also $\angle DCH = \angle FCA$. § 60

But $\angle BCD = \angle DCH$. Given

$\therefore \angle BCD = \angle FCA$. Ax. 8

$\therefore \angle BCD + \angle ACB = \angle FCA + \angle ACB$. Ax. 1

That is, $\angle ACD = \angle FCB$. Ax. 11

$\therefore$ the $\triangle$s ADC and FBC are similar. § 286

$\therefore FC : AC = BC : CD$. § 282

$\therefore FC \times CD = AC \times BC$. § 261

Now $AD \times BD = FD \times CD$ § 303

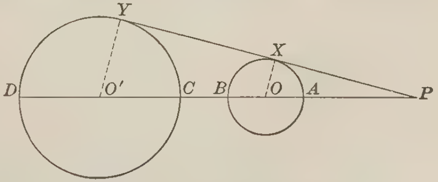
$= (FC + CD) \times CD$ Ax. 11

$= FC \times CD + \overline{CD}^2$.

$\therefore AD \times BD = AC \times BC + \overline{CD}^2$. Ax. 9

$\therefore \overline{CD}^2 = AD \times BD - AC \times BC$, by Ax. 2. Q. E. D.

7. *If the line of centers of two circles meets the circles at the consecutive points A, B, C, D, and meets the common external tangent at P, then $PA \times PD = PB \times PC$.*



Given XY, the common external tangent of the Ⓢ whose centers are O and O'; also the line of centers OO', cutting the two Ⓢ at the consecutive points A, B, C, D, and YX produced at P.

To prove that $PA \times PD = PB \times PC$.

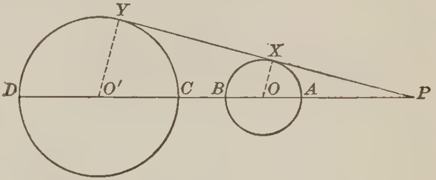
Proof. Draw the radii OX and $O'Y$.

Then the $\angle$s PXO and PYO' are rt. $\angle$s. § 185

$\therefore$ the $\triangle$s POX and $PO'Y$ are similar. § 287

$\therefore PO : PO' = OX : O'Y.$ § 282

$\therefore PO : OX = PO' : O'Y.$ § 265



$\therefore PO - OX : OX = PO' - O'Y : O'Y.$ § 268

But $OX = OA$, and $O'Y = O'C.$ § 162

$\therefore PO - OA : OX = PO' - O'C : O'Y.$ Ax. 9

That is, $PA : OX = PC : O'Y.$

$\therefore PA : PC = OX : O'Y.$ § 265

Since $PO : OX = PO' : O'Y,$ Proved

$\therefore PO + OX : OX = PO' + O'Y : O'Y.$ § 267

But $OX = OB$, and $O'Y = O'D.$ § 162

$\therefore PO + OB : OX = PO' + O'D : O'Y.$ Ax. 9

That is, $PB : OX = PD : O'Y.$ Ax. 11

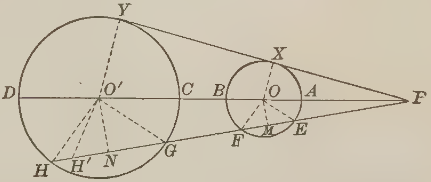
$\therefore PB : PD = OX : O'Y.$ § 265

$\therefore PA : PC = PB : PD.$ Ax. 8

$\therefore PD \times PA = PB \times PC$, by § 261. Q.E.D.

8. *The line of centers of two circles meets the common external tangent at P, and a secant is drawn from P, cutting the circles at the consecutive points E, F, G, H. Prove that* $PE \times PH = PF \times PG$.

Given P, the point of intersection of the common external tangent YX of the ⓢ whose centers are O and O', with the line of centers OO', which cuts the ⓢ at the consecutive points A, B, C, D; also PH, a secant that cuts the ⓢ at the consecutive points E, F, G, H.



To prove that $PE \times PH = PF \times PG.$

Proof. Draw the radii OX, OE, OF, and $O'Y$, $O'G$, $O'H$.

Draw OM and $O'N \perp$ to PH.

Since the $\angle O'PN$ is common,

$\therefore$ the rt. $\triangle$s OMP and $O'NP$ are similar. § 287

$\therefore PO : PO' = OM : O'N.$ § 282

Now the $\angle$s OXP and $O'YP$ are rt. $\angle$s. § 185

Since the $\angle YPO'$ is common,

$\therefore$ the rt. $\triangle$s XOP and $YO'P$ are similar. § 287

$\therefore PO : PO' = OX : O'Y$. § 282

$\therefore OM : O'N = OX : O'Y$. Ax. 8

But $OX = OF$, and $O'Y = O'H$. § 162

$\therefore OM : O'N = OF : O'H$. Ax. 8

Draw $O'H' \parallel$ to OF.

Then $\angle MFO = \angle NH'O'$. § 102

$\therefore$ the rt. $\triangle$s FMO and $H'NO'$ are similar. § 287

$\therefore OM : O'N = OF : O'H'$. § 282

But $OM : O'N = OF : O'H$. Proved

$\therefore O'H = O'H'$, and coincides with it.

Since $O'H'$ is $\parallel$ to OF, Hyp.

$\therefore O'H$ is $\parallel$ to OF.

$\therefore PO : PO' = PF : PH$. § 273

In the same way it may be proved that

$PO : PO' = PE : PG$.

$\therefore PE : PG = PF : PH$. Ax. 8

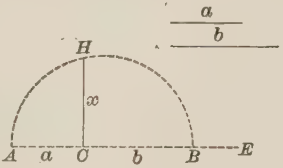
$\therefore PE \times PH = PF \times PG$, by § 261. Q. E. D.

Exercise 48. Page 187

1. *If a and b are two given lines, construct a line equal to x, where $x = \sqrt{ab}$. Consider the special case of $a = 2$, $b = 3$.*

Given the two lines a and b.

Required to construct x, where $x = \sqrt{ab}$.



Construction. Draw any line AE, and on AE take $AC = a$, and $CB = b$.

On AB as a diameter describe a semicircle.

At C erect the $\perp CH$, meeting the $\odot$ at H. § 228

Then CH is the required line x. Q. E. F.

Proof. $AC : CH = CH : CB$. § 297

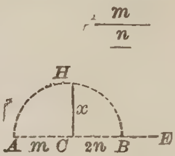
$\therefore a : x = x : b$. Ax. 9

$\therefore x^2 = ab$. § 261

$\therefore x = \sqrt{ab}$, by Ax. 5. Q. E. D.

In the special case, where $a = 2$, and $b = 3$, construct a equal to two times, and b equal to three times, some unit of measure, and proceed as in the above. The required line x is here equal to $\sqrt{6}$.

2. *If m and n are two given lines, construct a line equal to x, where* $x = \sqrt{2mn}$.



Given the two lines m and n.

Required to construct x, where $x = \sqrt{2mn}$.

Construction. Draw any line AE, and on AE take $AC = m$, and $CB = 2n$.

On AB as a diameter describe a semicircle.

At C erect the $\perp CH$, meeting the $\odot$ at H. § 228

Then CH is the required line x. Q. E. F.

Proof. $AC : CH = CH : CB$. § 297

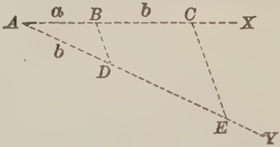
$\therefore m : x = x : 2n$. Ax. 9

$\therefore x^2 = 2mn$. § 261

$\therefore x = \sqrt{2mn}$, by Ax. 5. Q. E. D.

3. *Determine both by geometric construction and arithmetically the third proportional to the lines $1\frac{1}{2}$ in. and 2 in.*

Given the two lines a and b of length $1\frac{1}{2}$ in. and 2 in. respectively.



Required to construct the third proportional to a and b.

Construction. Draw two lines AX and AY containing any convenient angle.

On AX take $AB = a$, and $BC = b$.

On AY take $AD = b$, and draw BD.

From C draw $CE \parallel$ to BD, meeting AY at E. § 233

Then DE is the third proportional required. Q. E. F.

Proof. $AB : BC = AD : DE$. § 273

$\therefore a : b = b : DE$. Ax. 9

$\therefore DE$ is the third proportional to a and b, by § 260. Q. E. D.

Solution. $a : b = b : DE$. Proved

$\therefore 1\frac{1}{2} : 2 = 2 : DE$. Ax. 9

$1\frac{1}{2} \times DE = 2 \times 2$. § 261

$\therefore DE = 2\frac{2}{3}$. Ax. 4

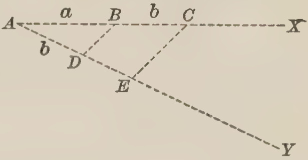
$\therefore DE = 2\frac{2}{3}$ in. Q. E. F.

4. *Determine both by geometrical construction and arithmetically the third proportional to the lines 4 in. and 3 in.*

Given the two lines a and b of length 4 in. and 3 in. respectively.

Required to construct the third proportional to a and b.

Construction. Draw two lines AX and AY containing any convenient angle.



On AX take AB equal to a, and BC equal to b.

On AY take AD equal to b, and draw BD.

From C draw $CE \parallel$ to BD, meeting AY at E. § 233

Then DE is the third proportional required. Q. E. F.

Proof. $AB : BC = AD : DE.$ § 273

$\therefore a : b = b : DE.$ Ax. 9

$\therefore$ DE is the third proportional to a and b, by § 260. Q. E. D.

Solution. $a : b = b : DE.$ Proved

$\therefore 4 : 3 = 3 : DE.$ Ax. 9

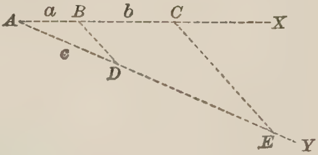
$\therefore 4\,DE = 3 \times 3.$ § 261

$\therefore DE = 2\frac{1}{4}.$ Ax. 4

$\therefore DE = 2\frac{1}{4}$ in. Q. E. F.

5. *Determine both by geometric construction and arithmetically the fourth proportional to the lines* $1\frac{1}{4}$ *in.,* 2 *in.,* $2\frac{1}{4}$ *in.*

Given the three lines a, b, and c of length $1\frac{1}{4}$ in., 2 in., and $2\frac{1}{4}$ in. respectively.



Required to construct the fourth proportional to a, b, and c.

Construction. Draw the two lines AX and AY containing any convenient angle.

On AX take $AB = a$, and $BC = b$.

On AY take $AD = c$, and draw BD.

From C draw $CE \parallel$ to BD, meeting AY at E. § 233

Then DE is the fourth proportional required. Q. E. F.

Proof. $AB : BC = AD : DE.$ § 273

$\therefore a : b = c : DE.$ Ax. 9

$\therefore$ DE is the fourth proportional to a, b, and c, by § 259. Q. E. D.

Solution. $a : b = c : DE.$ Proved

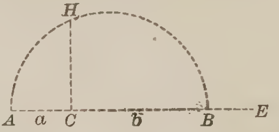
$\therefore 1\frac{1}{4} : 2 = 2\frac{1}{4} : DE.$ Ax. 9

$\therefore 1\frac{1}{4} \times DE = 2 \times 2\frac{1}{4}.$ § 261

$\therefore DE = 3\frac{3}{5}.$ Ax. 4

$\therefore DE = 3\frac{3}{5}$ in. Q. E. F.

6. *Determine both by geometrical construction and arithmetically the mean proportional between the lines* 1.2 *in. and* 2.7 *in.*



Given the two lines a and b of length 1.2 in. and 2.7 in. respectively.

Required to construct the mean proportional between a and b.

Construction. Draw any line AE.

On AE take $AC = a$, and $CB = b$.

On AB as a diameter describe a semicircle.

At C erect the $\perp CH$, meeting the circle at H. § 228

Then CH is the mean proportional required. Q.E.F.

Proof. $AC : CH = CH : CB$. § 297

$\therefore a : CH = CH : b$. Ax. 9

$\therefore CH$ is the mean proportional between a and b, by § 260. Q.E.D.

Solution. $a : CH = CH : b$. Proved

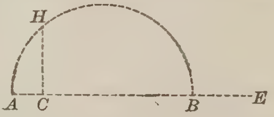
$\therefore 1.2 : CH = CH : 2.7$. Ax. 9

$\therefore \overline{CH}^2 = 1.2 \times 2.7$. § 261

$\therefore CH = 1.8$. Ax. 5

$\therefore CH = 1.8$ in. Q.E.F.

7. *Find geometrically the square root of* 5. *Measure the line and thus determine the approximate arithmetical value.*



Given the line a of length one unit.

Required to construct the $\sqrt{5}$.

Construction. Draw any line AE.

On AE take $AC = 1$, and $CB = 5$.

On AB as a diameter describe a semicircle.

At C erect the $\perp CH$, meeting the ⊙ at H. § 228

Then CH is the $\sqrt{5}$ required. Q.E.F.

Proof. $AC : CH = CH : CB$. § 297

$\therefore 1 : CH = CH : 5$. Ax. 9

$\therefore \overline{CH}^2 = 5$. § 261

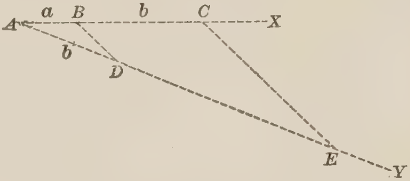
$\therefore CH = \sqrt{5}$, by Ax. 5. Q.E.D.

On measuring CH we find its approximate length to be 2.2.

8. *A map is drawn to the scale of* 1 *in. to* 50 *mi. How far apart are two places that are* $2\frac{3}{32}$ *in. apart on the map?*

Solution. Since 1 in. on the map represents a distance of 50 mi., $\therefore 2\frac{3}{32}$ in. represents a distance of $2\frac{3}{32} \times 50$ mi., or $104\frac{1}{16}$ mi.

9. *Find by geometric construction and arithmetically the third proportional to the two lines* $1\frac{3}{16}$ *in. and* $2\frac{5}{8}$ *in.*



Given the two lines a and b of length $1\frac{3}{16}$ in. and $2\frac{5}{8}$ in. respectively.

Required to construct the third proportional to a and b.

Construction. Draw two lines AX and AY containing any convenient angle.

On AX take $AB = a$, and $BC = b$.

On AY take $AD = b$, and draw BD.

From C draw $CE \parallel$ to BD, meeting AY at E. § 233

Then DE is the third proportional required. Q. E. F.

Proof. $AB : BC = AD : DE$. § 273

$\therefore a : b = b : DE$. Ax. 9

$\therefore$ DE is the third proportional to a and b, by § 260. Q. E. D.

Solution. $a : b = b : DE$. Proved

$1\frac{3}{26} : 2\frac{5}{8} = 2\frac{5}{8} : DE$. Ax. 9

$\therefore 1\frac{3}{16} \times DE = 2\frac{5}{8} \times 2\frac{5}{8}$. § 261

$\therefore DE = 5\frac{61}{76}$.

$\therefore DE = 5\frac{61}{76}$ in. Q. E. F.

10. *Divide a line* 1 *in. long in extreme and mean ratio. Measure the two segments and determine their lengths to the nearest sixteenth of an inch.*

Given the line AB, 1 in. long.

Required to divide AB in extreme and mean ratio.

Construction. Divide AB internally at C and externally at C' in extreme and mean ratio. § 311

Then $AB : AC = AC : CB$, and $AB : AC' = AC' : C'B$. Q. E. F.

Solution. Let $x =$ the number of inches in AC.

Then $1 - x =$ the number of inches in CB.

Since $AB : AC = AC : CB$, § 310

$\therefore 1 : x = x : 1 - x$. Ax. 9

$\therefore x^2 = 1 - x$. § 261

$\therefore x^2 + x = 1$. Ax. 1

$\therefore x^2 + x + \frac{1}{4} = \frac{5}{4}$. Ax. 1

$\therefore x + \frac{1}{2} = \pm \frac{1}{2}\sqrt{5}$. Ax. 5

$\therefore x = \frac{1}{2}(-1 \pm \sqrt{5})$. Ax. 2

$$\therefore x = \frac{\pm\sqrt{5} - 1}{2}.$$

Hence $$AC = \frac{8(\sqrt{5}-1)}{16}\text{ in.} = \frac{10}{16}\text{ in., approximately,}$$

and $$CB = 1\text{ in.} - \tfrac{10}{16}\text{ in.} = \tfrac{6}{16}\text{ in., approximately.}$$

Let $y =$ the number of inches in AC'.

Then $1 + y =$ the number of inches in $C'B$.

Since $$AB : AC' = AC' : C'B, \qquad \text{§ 310}$$

$$\therefore 1 : y = y : 1 + y. \qquad \text{Ax. 9}$$

$$\therefore y^2 = 1 + y. \qquad \text{§ 261}$$

$$\therefore y^2 - y = 1. \qquad \text{Ax. 2}$$

$$\therefore y^2 - y + \tfrac{1}{4} = \tfrac{5}{4}. \qquad \text{Ax. 1}$$

$$\therefore y - \tfrac{1}{2} = \pm \tfrac{1}{2}\sqrt{5}. \qquad \text{Ax. 5}$$

$$\therefore y = \tfrac{1}{2}(1 \pm \sqrt{5}). \qquad \text{Ax. 1}$$

$$\therefore y = \frac{1 \pm \sqrt{5}}{2}.$$

$$\therefore AC' = \frac{8(1+\sqrt{5})}{16}\text{ in.} = \frac{26}{16}\text{ in.} = 1\tfrac{10}{16}\text{ in., approximately,}$$

and $$C'B = 1\text{ in.} + 1\tfrac{10}{16}\text{ in.} = 2\tfrac{10}{16}\text{ in., approximately.} \qquad \text{Q.E.F.}$$

11. *Divide a line 5 in. long in extreme and mean ratio. Measure the two segments and determine their lengths to the nearest sixteenth of an inch.*

Given the line AB, 5 in. long.

Required to divide AB in extreme and mean ratio.

Construction. Divide AB internally at C and externally at C' in extreme and mean ratio. § 311

Then $AB : AC = AC : CB$, and $AB : AC' = AC' : C'B$. Q.E.F.

Solution. Let $x =$ the number of inches in AC.

Then $5 - x =$ the number of inches in CB.

Since $$AB : AC = AC : CB, \qquad \text{§ 310}$$

$$\therefore 5 : x = x : 5 - x. \qquad \text{Ax. 9}$$

$$\therefore x^2 = 25 - 5x. \qquad \text{§ 261}$$

$$\therefore x^2 + 5x + \tfrac{25}{4} = \tfrac{125}{4}. \qquad \text{Ax. 1}$$

$$\therefore x + \tfrac{5}{2} = \pm \tfrac{5}{2}\sqrt{5}. \qquad \text{Ax. 5}$$

$$\therefore x = \tfrac{5}{2}(-1 \pm \sqrt{5}). \qquad \text{Ax. 2}$$

$$\therefore x = \frac{5(\pm\sqrt{5}-1)}{2}.$$

Hence $$AC = \frac{40(\sqrt{5}-1)}{16}\text{ in.} = \frac{49}{16}\text{ in.} = 3\tfrac{1}{16}\text{ in., approximately,}$$

and $$CB = 5\text{ in.} - 3\tfrac{1}{16}\text{ in.} = 1\tfrac{15}{16}\text{ in., approximately.}$$

Let $y =$ the number of inches in AC'.

Then $5 + y =$ the number of inches in $C'B$.

Since $$AB : AC' = AC' : C'B, \qquad \text{§ 310}$$

$$\therefore 5 : y = y : 5 + y. \qquad \text{Ax. 9}$$

$$\therefore y^2 = 25 + 5y. \qquad \S\ 261$$

$$\therefore y^2 - 5y = 25. \qquad \text{Ax. 2}$$

$$\therefore y^2 - 5y + \tfrac{25}{4} = \tfrac{125}{4}. \qquad \text{Ax. 1}$$

$$\therefore y - \tfrac{5}{2} = \pm \tfrac{5}{2}\sqrt{5}. \qquad \text{Ax. 5}$$

$$\therefore y = \pm \tfrac{5}{2}\sqrt{5} + \tfrac{5}{2}. \qquad \text{Ax. 2}$$

$$\therefore y = \tfrac{5}{2}(\pm\sqrt{5} + 1).$$

Hence $AC' = \dfrac{40(\sqrt{5}+1)}{16}$ in. $= \dfrac{129}{16}$ in. $= 8\tfrac{1}{16}$ in., approximately,

and $C'B = 5$ in. $+ 8\tfrac{1}{16}$ in. $= 13\tfrac{1}{16}$ in., approximately. Q.E.F.

12. *Divide a line 6 in. long in extreme and mean ratio. Measure the two segments and determine their lengths to the nearest sixteenth of an inch.*

Given the line AB, 6 in. long.

Required to divide AB in extreme and mean ratio.

Construction. Divide AB internally at C and externally at C' in extreme and mean ratio. § 311

Then $AB : AC = AC : CB$, and $AB : AC' = AC' : C'B$. Q.E.F.

Solution. Let $x =$ the number of inches in AC.

Then $6 - x =$ the number of inches in CB.

Since $AB : AC = AC : CB$, § 310

$$\therefore 6 : x = x : 6 - x. \qquad \text{Ax. 9}$$

$$\therefore x^2 = 36 - 6x. \qquad \S\ 261$$

$$\therefore x^2 + 6x + 9 = 45. \qquad \text{Ax. 1}$$

$$\therefore x + 3 = \pm 3\sqrt{5}. \qquad \text{Ax. 5}$$

$$\therefore x = \pm 3\sqrt{5} - 3. \qquad \text{Ax. 2}$$

$$\therefore x = 3(\pm\sqrt{5} - 1).$$

Hence $AC = \dfrac{48(\sqrt{5}-1)}{16}$ in. $= \dfrac{59}{16}$ in. $= 3\tfrac{11}{16}$ in., approximately,

and $CB = 6$ in. $- 3\tfrac{11}{16}$ in. $= 2\tfrac{5}{16}$ in., approximately.

Let $y =$ the number of inches in AC'.

Then $6 + y =$ the number of inches in $C'B$.

Since $AB : AC' = AC' : C'B$, § 310

$$\therefore 6 : y = y : 6 + y. \qquad \text{Ax. 9}$$

$$\therefore y^2 = 36 + 6y. \qquad \S\ 261$$

$$\therefore y^2 - 6y = 36. \qquad \text{Ax. 2}$$

$$\therefore y^2 - 6y + 9 = 45. \qquad \text{Ax. 1}$$

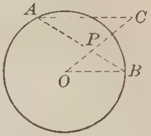
$$\therefore y - 3 = \pm 3\sqrt{5}. \qquad \text{Ax. 5}$$

$$\therefore y = 3(\pm\sqrt{5} + 1). \qquad \text{Ax. 2}$$

Hence $AC' = \dfrac{48(\sqrt{5}+1)}{16}$ in. $= \dfrac{155}{16}$ in. $= 9\tfrac{11}{16}$ in., approximately,

and $C'B = 6$ in. $+ 9\tfrac{11}{16}$ in. $= 15\tfrac{11}{16}$ in., approximately. Q.E.F.

13. *Through a given point P within a given circle to draw a chord AB so that the ratio $AP : BP$ shall equal a given ratio $m : n$.*



Given P, any point within the $\odot$ whose center is O; also the ratio $m : n$.

Required to draw through P a chord AB, so that

$$AP : PB = m : n.$$

Construction. Draw OP and produce it to a point C, so that $OP : PC = n : m$. § 307

Find x, the fourth proportional to n, m, and the radius. § 307

With C as center and x as radius cut the $\odot$ at A.

Draw CA, and draw $OB \parallel$ to AC. § 233

Draw AP and BP.

Then APB is a straight line, and the chord required. Q. E. F.

Proof. $OP : PC = n : m = OB : CA$. Const.

$\therefore OP : OB = PC : CA$. § 265

Also $\angle BOP = \angle ACP$. § 100

$\therefore$ the $\triangle$s POB and PCA are similar. § 288

$\therefore \angle OPB = \angle CPA$. § 282

Since OPC is a straight line,

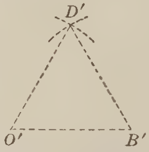
$\therefore APB$ is a straight line.

Also $AP : BP = PC : OP$. § 282

But $PC : OP = m : n$. Const.

$\therefore AP : BP = m : n$, by Ax. 8. Q. E. D.

14. *To draw two lines making an angle of 60°, and to construct all the circles of ½ in. radius that are tangent to both lines.*



Required to draw two lines making an angle of 60°, and to construct all the ⓢ of ½ in. radius that are tangent to both lines.

Construction. Draw a straight line $O'B'$ of convenient length.

With O' and B' as centers and radii $O'B'$ describe arcs intersecting at D'.

Draw $O'D'$ and $B'D'$.

Draw any line AB.

At any point O on AB draw OD, making $\angle BOD$ equal to $\angle B'O'D'$, and produce DO to E. § 232

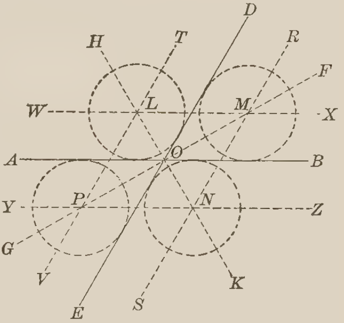
Draw FG and HK, the bisectors of $\angle$s BOD and DOA respectively. § 231

Then draw $RS \parallel$ to DE at the distance of ½ in. from DE, cutting FG and HK in M and N respectively. § 233

Also draw $VT \parallel$ to DE at the distance of $\frac{1}{2}$ in. from DE, cutting FG and HK in P and L. § 233

From M, N, P, and L as centers, with radii equal to $\frac{1}{2}$ in., describe ⓢ.

Then the ⓢ whose centers are M, N, P, and L are the required ⓢ. Q. E. F.



Proof. The $\triangle O'B'D'$ is equilateral. Const.

$\therefore$ the $\triangle O'B'D'$ is equiangular. § 75

$\therefore \angle B'O'D' = 60°$. § 107

$\angle BOD = \angle B'O'D'$. Const.

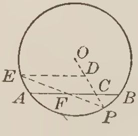
$\therefore \angle BOD = 60°$. Ax. 8

The points M, N, P, and L are each equally distant from lines AB and DE, and are each $\frac{1}{2}$ in. from the lines AB and DE. Const.

Hence the ⓢ whose centers are M, N, P, and L, described with $\frac{1}{2}$ in. as a radius, are tangent to lines AB and DE, by § 184. Q. E. D.

15. *To draw through a given point P in the arc subtended by a chord AB a chord which shall be bisected by AB.*

Given P, any point on the arc AB subtended by chord AB in the ⊙ whose center is O.



Required to draw through P a chord which shall be bisected by chord AB.

Construction. Draw the radius OP, cutting AB at C.

On CO take $CD = PC$.

Draw $DE \parallel$ to BA, meeting the ⊙ in E. § 233

Draw chord EP, cutting the chord AB in F.

Then EP is the required chord. Q. E. F.

Proof. In $\triangle PDE$,

CF bisects PD, and is $\parallel$ to DE. Const.

$\therefore$ CF bisects EP. § 135

That is, AB bisects chord EP. Q. E. D.

16. *To construct two circles of radii $\frac{1}{2}$ in. and 1 in. respectively, which shall be tangent externally; and to construct a third circle of radius 3 in., which shall be tangent to each of these two circles and inclose both of them.*

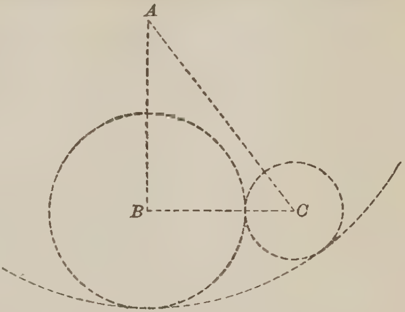
Construction. Construct $\triangle ABC$ so that $AB = 2$ in., $AC = 2\frac{1}{2}$ in., and $BC = 1\frac{1}{2}$ in. § 237

From C as center, with radius $\frac{1}{2}$ in., describe a ⊙.

From B as center, with radius 1 in., describe a ⊙.

Again, from A as center, with radius 3 in., describe a ⊙.

Then the Ⓢ whose centers are A, B, and C are the Ⓢ required. Q.E.F.



Proof. The radii of the Ⓢ whose centers are B and C respectively are 1 in. and $\frac{1}{2}$ in.

Since BC is $1\frac{1}{2}$ in., Const.

these two Ⓢ are tangent externally.

The radius of the ⊙ whose center is A is 3 in.

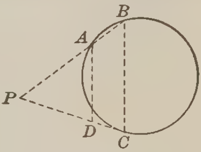
Since AB is 2 in., Const.

the ⊙ whose center is B is tangent internally to the ⊙ whose center is A.

Similarly, since AC is $2\frac{1}{2}$ in., Const.

the ⊙ whose center is C is tangent internally to the ⊙ whose center is A. Q.E.D.

17. *To draw through a given external point P a secant PAB to a given circle so that the ratio $PA : AB$ shall equal the given ratio $m : n$.*



Given P, any point outside the ⊙ ABC; also the ratio $m : n$.

Required to draw a secant PAB so that $PA : AB = m : n$.

Construction. Draw PC tangent to the ⊙. § 246

Divide PC at D so that the ratio $PD : DC$ is equal to $m : n$. § 306

Find k, the mean proportional between PD and PC. § 309

From P as center, with radius k, describe an arc cutting the ⊙ ABC at A.

Then draw PA, and produce it to meet the ⊙ again in B.

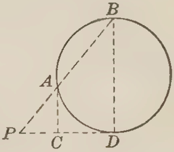
Then PAB is the required secant. Q.E.F.

Proof. Draw AD and BC.

$PD : PA = PA : PC$. Const.

But $PA : PC = PC : PB.$ § 302
$\therefore PD : PA = PC : PB.$ Ax. 8
$\therefore AD$ is $\parallel$ to $BC.$ § 276
$\therefore PA : AB = PD : DC.$ § 273
But $PD : DC = m : n.$ Const.
$\therefore PA : AB = m : n$, by Ax. 8. Q.E.D.

18. *To draw through a given external point P a secant PAB to a given circle so that $\overline{AB}^2 = PA \times PB$.*



Given P, any point outside the ⊙ ABD.

Required to draw a secant PAB so that

$$\overline{AB}^2 = PA \times PB.$$

Construction. Draw the tangent PD. § 246

Divide PD internally at C in extreme and mean ratio; § 311

that is, so that $PC : CD = CD : PD.$

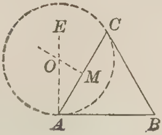
With P as center and CD as radius describe an arc cutting the ⊙ at A.

Draw PA, and produce it to meet the ⊙ again at B.

Then PAB is the secant required. Q.E.F.

Proof. Draw AC and BD.
$PC : CD = CD : PD.$ Const.
$\therefore PC : PA = PA : PD.$ Ax. 9
But $PA : PD = PD : PB.$ § 302
$\therefore PC : PA = PD : PB.$ Ax. 8
$\therefore AC$ is $\parallel$ to $BD.$ § 276
$\therefore PC : PA = CD : AB$ § 273
$= PA : AB.$ Ax. 9
But $PC : PA = PA : PD.$ Proved
$\therefore AB = PD.$
$\therefore PA : AB = AB : PB.$ Ax. 9
$\therefore \overline{AB}^2 = PA \times PB$, by § 261. Q.E.D.

19. *An equilateral triangle ABC is 2 in. on a side. To construct a circle which shall be tangent to AB at the point A and shall pass through the point C.*



Given the equilateral △ ABC, whose side is 2 in.

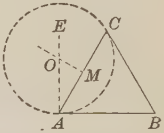
Required to construct a ⊙ which shall be tangent to AB at A and shall pass through C.

Construction. At A erect $AE \perp$ to AB. § 228

Draw OM, the $\perp$ bisector of AC, meeting AE in O. §§ 229, 228

From O as center, with a radius equal to OA, describe a ⊙.

Then this ⊙ is the required ⊙. Q. E. F.



Proof. The ⊙ constructed is tangent to line AB at A. § 185

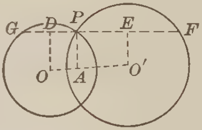
Furthermore, since OM is the $\perp$ bisector of CA, Const.

O is equally distant from A and C.

$\therefore$ the ⊙ passes through C.

Hence the ⊙ constructed is tangent to AB at A and passes through C. Q. E. D.

20. *To draw through one of the points of intersection of two circles a secant so that the two chords that are formed shall be in the given ratio* $m:n$.



Given P, a point of intersection of the intersecting ⓢ whose centers are O' and O; also the ratio $m:n$.

Required to draw through P a secant so that the two chords formed shall be in the ratio $m:n$.

Construction. Draw the line of centers $O'O$, and divide $O'O$ at A so that $O'A : OA = m : n$. § 306

Draw PA.

Through P draw line $GF \perp$ to PA, having its extremities in the given ⓢ. § 228

Then GF is the required secant. Q. E. F.

Proof. Let fall from O' and O the ⊥s $O'E$ and OD, cutting GF in E and D respectively.

Then $O'E$ and OD are each $\parallel$ to AP. § 95

$\therefore O'A : OA = EP : DP$. § 275

But $O'A : OA = m : n$. Const.

$\therefore EP : DP = m : n$. Ax. 8

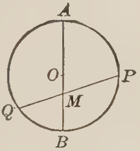
But $EP = \frac{1}{2} FP$, and $DP = \frac{1}{2} GP$. § 174

$\therefore \frac{1}{2} FP : \frac{1}{2} GP = m : n$. Ax. 9

Or $FP : GP = m : n$. Q. E. D.

21. *In a circle of* 3-*in. radius chords are drawn through a point* 1 *in. from the center. What is the product of the segments of these chords?*

Given M, a point 1 in. from the center O of the $\odot APB$ whose radius is 3 in. Let AB be a diameter of the given $\odot$ drawn through M.



Required to find the product of the segments of any one chord drawn through M in the $\odot APB$.

Solution. The product of the segments of any chord drawn through M in the $\odot APB$ is constant. § 299

Therefore $BM \times AM$ is always the product of the two segments of any one chord drawn through M in the $\odot APB$.

Now $BM = BO - OM.$ Ax. 11

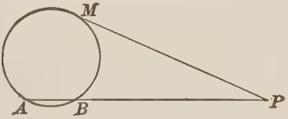
$\therefore BM = 3 - 1$, or 2. Ax. 9

Again, $AM = AO + OM.$ Ax. 11

$\therefore AM = 3 + 1$, or 4. Ax. 9

Hence $BM \times AM = 2 \times 4$, or 8, by Ax. 3. Q.E.F.

22. *The chord AB is 3 in. long, and it is produced through B to the point P so that PB is equal to 12 in. Find the tangent from P.*



Given PBA a secant, and PM a tangent, to the $\odot$; $AB = 3$ in., and $PB = 12$ in.

Required to find PM.

Solution. $\overline{PM}^2 = PB \times PA.$ § 302

Now $PA = PB + AB.$ Ax. 11

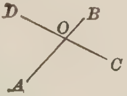
$\therefore PA = 12 + 3$, or 15. Ax. 9

Then $\overline{PM}^2 = 12 \times 15$, or 180. Ax. 9

$\therefore PM = \sqrt{180}$, or $6\sqrt{5}$. Ax. 5

Therefore $PM = 6\sqrt{5}$ in. Q.E.F.

23. *Two lines AB and CD intersect at O. How would you ascertain, by measuring OA, OB, OC, and OD, whether or not the four points A, B, C, and D lie on the same circle?*



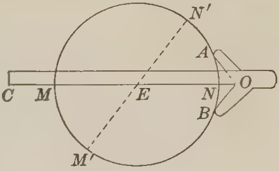
Given the lines AB and CD intersecting at O.

Required to determine, by measurement, whether A, B, C, and D lie on the same $\odot$.

Solution. If A, B, C, and D all lie on the same $\odot$, then $OA \times OB = OC \times OD.$ § 299

Hence, after measuring OA, OB, OC, and OD, if these measures satisfy the proportion, the four points all lie on the same $\odot$. Q.E.F.

24. *This figure represents an instrument for finding the centers of circular plates or sections of shafts. OC is a ruler that bisects the angle AOB, and AO and OB are equal. Show that, if A and B rest on the circle, OC passes through the center, and that by drawing two lines the center can be found.*



Given any ⊙.

Required to determine, by means of the instrument shown in the figure, the center of the given ⊙.

Construction. Place the ruler in position and trace a line MN along the edge OC of the ruler.

Then move the position of the ruler so that the arms A and B again touch the ⊙, and trace another line $M'N'$ along the edge OC of the ruler.

Then E, the point of intersection of the lines MN and $M'N'$, is the center of the given ⊙. Q.E.F.

Proof. OC bisects the $\angle AOB$. Given

∴ OC is the locus of points equidistant from AO and BO. § 152

But the center of the given ⊙ is equidistant from A and B. § 162

Therefore the center of the given ⊙ lies in OC; that is, OC passes through the center of the ⊙.

Hence MN, the line traced along the edge OC of the ruler, passes through the center of the ⊙.

∴ $M'N'$ also passes through the center of the ⊙.

∴ E, the intersection of the lines MN and $M'N'$, is the center of the given ⊙. Q.E.D.

25. *If three circles are tangent externally each to the other two, the tangents at their points of contact pass through the center of the circle inscribed in the triangle formed by joining the centers of the three given circles.*

Given the three ⓢ whose centers are A, B, and C tangent externally each to the other two; the ⊙ HGK inscribed in the triangle inclosed by the lines AB, AC, and BC; and the tangents FG, DH, and EK drawn through the points of contact of the ⓢ G, H, and K.

To prove that FG, DH, and EK pass through the center of the ⊙ GHK.

Proof. The lines FG, DH, and EK are either concurrent lines or they are not concurrent lines.

Suppose that FG, DH, and EK are *not* concurrent lines (Fig. 1).

Let FG and EK intersect in E, FG and DH in F, and DH and EK in D.

Then the lines DE, EF, and FD inclose the $\triangle EFD$.

Let FG be denoted by p, DH by m, and EK by n.

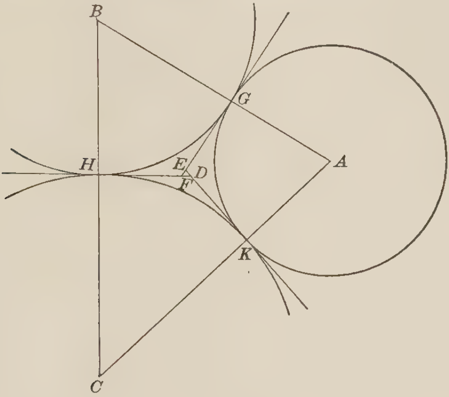


FIG. 1

Let the sides of $\triangle EDF$, namely ED, EF, and FD, be denoted by a, b, and c respectively.

Now $FG = FH$,

$EG = EK$,

and $DH = DK$. § 192

$\therefore p = m - c$, (1)

$p - b = n$, (2)

and $m = n - a$. (3)

From (1) and (2), $m - n = b + c$. Ax. 2

From (3), $m - n = - a$. Ax. 2

$\therefore -a = b + c$. Ax. 8

$\therefore a + b + c = 0$. Ax. 1

$\therefore$ the $\triangle EFD$ cannot exist; that is, points E, F, and D are one and the same point.

$\therefore$ the lines FG, DH, and EK are concurrent lines. § 154

Let O (Fig. 2) be the common point of intersection of the lines FG, DH, and EK (Fig. 1).

Draw GH, GK, and HK.

Draw OR, OS, and OT, the $\perp$ bisectors of GH, GK, and HK respectively.

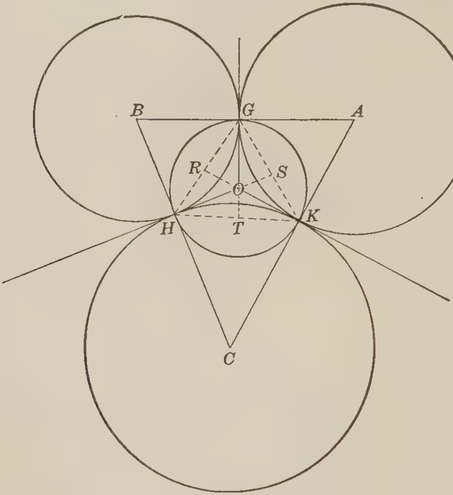


FIG. 2

Now $OG = OK$. § 192

$\therefore$ point O lies in OS. § 150

Again, $OG = OH$. § 192

$\therefore$ point O lies in OR. § 150

Also $OH = OK$. § 192

$\therefore$ point O lies in OT. § 150

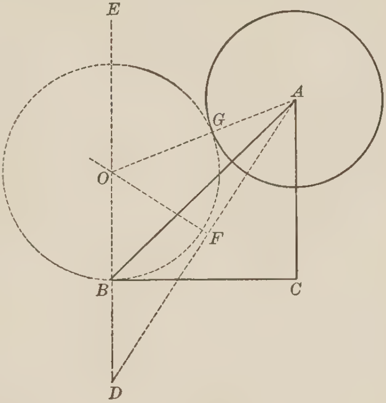
Therefore O, the common point of intersection of OG, OH, and OK, is the common point of intersection of lines OR, OS, and OT.

But the common point of intersection of OR, OS, and OT is the center of $\odot\, GHK$ inscribed in the $\triangle\, ABC$.

$\therefore$ O, the common point of intersection of the lines OG, OH, and OK, is the center of the $\odot\, GHK$.

Therefore FG, DH, and EK (Fig. 1) pass through the center of the $\odot\, GHK$. Q. E. D.

26. *In the isosceles triangle ABC, C is a right angle and AC is 4 in. With A as center and a radius 2 in. a circle is described. Required to describe another circle tangent to the first and also tangent to BC at the point B.*



Given the isosceles $\triangle ABC$, right-angled at C, with $AC = 4$ in.; also a ⊙ described about A as a center with a radius of 2 in.

Required to construct a ⊙ tangent to the given ⊙ and tangent also to BC at the point B.

Construction. Through B draw $DE \perp$ to BC. § 228

Take $BD = 2$ in.

Draw DA.

Draw FO, the $\perp$ bisector of DA. §§ 229, 228

Let FO cut DE in O.

With O as center and OB as radius draw a ⊙.

The ⊙ with center O is the required ⊙. Q.E.F.

Proof. Draw the line of centers, OA.

Let OA cut the given ⊙ in G.

Since O lies in the $\perp$ bisector of DA, Const.

$\therefore DO = AO.$ § 150

Now $DB = 2$ in., Const.

and $AG = 2$ in. Given

$\therefore DB = AG.$

$\therefore DO - DB = AO - AG,$ Ax. 2

or $OB = OG.$

That is, OG is a radius of the ⊙ with center O.

$\therefore$ point G lies on the ⊙ with center O.

$\therefore$ the two ⓢ are tangent at point G.

Since OB is $\perp$ to BC, Const.

$\therefore$ the ⊙ with center O is tangent to BC at the point B. § 184

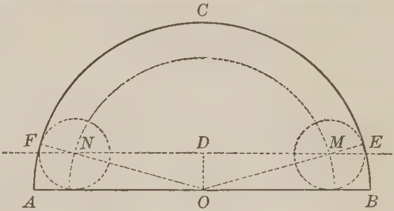
$\therefore$ the ⊙ with center O satisfies the conditions. Q.E.D.

Discussion. If $BD = 2$ in. is taken on the other side of BC, the solution will be a circle internally tangent to the given circle.

27. *Find the center of a circle of $\frac{1}{2}$ in. radius, so drawn in a semicircle of radius 2 in. as to be tangent to the semicircle itself and to its diameter.*

Given ACB, a semicircle whose radius is 2 in.

Required to find the center of a ⊙ of $\frac{1}{2}$ in. radius, so drawn in the semicircle ACB as to be tangent to the semicircle ACB and the diameter AB.



Construction. Through O, the center of the semicircle, draw a ⊥ to AB, taking $OD = \frac{1}{2}$ in. § 228

Through D draw a ∥ to AB. § 233

With center O and a radius of $1\frac{1}{2}$ in. describe an arc cutting the ∥ in M and N.

With M and N as centers and a radius of $\frac{1}{2}$ in. describe two Ⓢ.

Then the Ⓢ with centers M and N are the required Ⓢ. Q.E.F.

Proof. Since M and N are at a distance of $\frac{1}{2}$ in. from AB, § 128

∴ the Ⓢ with centers M and N are tangent to AB. § 184

Draw the lines of centers, OM and ON, cutting the small Ⓢ in E and F.

Then $OE = OM + ME$. Ax. 11

∴ $OE = 1\frac{1}{2} + \frac{1}{2} = 2$. Ax. 9

∴ OE is a radius of the semicircle.

Similarly OF is a radius of the semicircle.

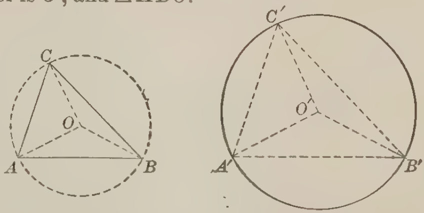
∴ the small Ⓢ are tangent internally to the semicircle.

∴ the Ⓢ with centers M and N satisfy the conditions. Q.E.D.

28. *To inscribe in a given circle a triangle similar to a given triangle.*

Given the ⊙ whose center is O', and △ ABC.

Required to inscribe in the given ⊙ a △ similar to △ ABC.



Construction. Circumscribe the ⊙ ABC about △ ABC. § 240

Draw the radii OA, OB, and OC.

Then draw any radius $O'A'$ in the given ⊙, and draw the radius $O'B'$, making with $O'A'$ an ∠ equal to ∠ AOB. § 232

Next draw $O'C'$ so that it makes with $O'A'$ an $\angle$ equal to $\angle COA$. § 232

Draw $A'B'$, $A'C'$, and $B'C'$.

Then $\triangle A'B'C'$ is the required $\triangle$. Q. E. F.

Proof. $\triangle$s AOB and $A'O'B'$ are similar. § 288

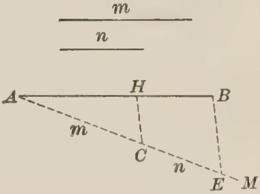
Likewise $\triangle$s AOC and $A'O'C'$ are similar,

and $\triangle$s BOC and $B'O'C'$ are similar. § 288

$\therefore$ $\triangle$s ABC and $A'B'C'$ are similar. § 293

Therefore $\triangle A'B'C'$ is the required $\triangle$. Q. E. D.

29. *To draw two straight line-segments, having given their sum and their ratio.*



Given AB, the sum of two straight line-segments whose ratio is $m : n$.

Required to find the two straight line-segments.

Construction. Draw AM so that $\angle MAB$ is any convenient $\angle$.

On AM take $AC = m$, and $CE = n$.

Join points B and E by line BE.

Through C draw $CH \parallel$ to EB, meeting AB in H. § 233

Then AH and HB are the two required straight line-segments. Q. E. F.

Proof. CH is $\parallel$ to EB. Const.

$\therefore AH : HB = AC : CE$. § 273

But $AC : CE = m : n$. Const.

$\therefore AH : HB = m : n$, by Ax. 8. Q. E. D.

BOOK IV

Exercise 50. Page 196

1. *A square and a rectangle have equal perimeters, 144 yd., and the length of the rectangle is five times the breadth. Compare the areas of the square and rectangle.*

Since the perimeter of the square is 144 yd., a side of the square is 144 yd. ÷ 4, or 36 yd.

Since the perimeter of the rectangle is 144 yd. and the length is 5 times the breadth,

if x = the number of yards in the breadth,
then $5x$ = the number of yards in the length,
and $12x$ = the number of yards in the perimeter.

$$\therefore\ 12x = 144.$$
$$\therefore\ x = 12.$$
$$\therefore\ 5x = 60.$$
$$\therefore\ \frac{\text{square}}{\text{rectangle}} = \frac{\overset{3}{\cancel{36}} \times \overset{3}{\cancel{36}}}{\underset{5}{\cancel{60}} \times \cancel{12}} = \frac{9}{5}.\quad \textit{Ans.}$$

2. *On a certain map the linear scale is 1 in. to 10 mi. How many acres are represented by a square $\frac{3}{4}$ in. on a side?*

Since 1 in. represents 10 mi., $\frac{3}{4}$ in. represents $\frac{3}{4}$ of 10 mi., or $7\frac{1}{2}$ mi.

∴ a square $\frac{3}{4}$ in. on a side represents

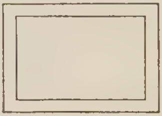
$$7\tfrac{1}{2} \times 7\tfrac{1}{2}\text{ sq. mi.} = \frac{15}{\cancel{2}} \times \frac{15}{\cancel{2}} \times \overset{160}{\overset{\cancel{320}}{\cancel{640}}}\text{ A.} = 36{,}000\text{ A.}\quad \textit{Ans.}$$

3. *Find the ratio of a lot 90 ft. long by 60 ft. wide to a field 40 rd. long by 20 rd. wide.*

$$\frac{\text{lot}}{\text{field}} = \frac{90 \times 60}{40 \times 16\frac{1}{2} \times 20 \times 16\frac{1}{2}} = \frac{90 \times 60}{40 \times \frac{33}{2} \times 20 \times \frac{33}{2}} \qquad \S\ 318$$

$$= \frac{\overset{3}{\overset{\cancel{9}}{\cancel{90}}} \times \overset{\cancel{3}}{\cancel{60}} \times \cancel{2} \times \cancel{2}}{\underset{\cancel{10}}{\underset{\cancel{20}}{\cancel{40}}} \times \cancel{20} \times \underset{11}{\cancel{33}} \times \underset{11}{\cancel{33}}} = \frac{3}{121}.\quad \textit{Ans.}$$

4. *Find the area of a gravel walk 3 ft. 6 in. wide, which surrounds a rectangular plot of grass 40 ft. long and 25 ft. wide. Make a drawing to scale before beginning to compute.*



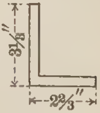
This drawing is made to the scale of 1 in. to 16 ft.

The area of the plot of grass is 40 × 25 sq. ft., or 1000 sq. ft. § 320

The area of the plot of grass and the gravel walk together is 47 × 32 sq. ft., or 1504 sq. ft. § 320

Therefore the area of the gravel walk = 1504 sq. ft. – 1000 sq. ft.
= 504 sq. ft. *Ans.*

5. *Find the number of square inches in this cross section of an* L *beam, the thickness being* $\frac{1}{3}$ *in.*



The area of the upright part of the beam
= $3\frac{1}{3} \times \frac{1}{3}$ sq. in. = $1\frac{1}{9}$ sq. in. § 320

The area of the remainder of the beam
= $2\frac{1}{3} \times \frac{1}{3}$ sq. in. = $\frac{7}{9}$ sq. in. § 320

Therefore the area of the whole beam
= $1\frac{1}{9}$ sq. in. + $\frac{7}{9}$ sq. in. = $1\frac{8}{9}$ sq. in. *Ans.*

6. *What is the perimeter of a square field that contains exactly an acre?*

1 A. = 160 sq. rd. = 160 × 272.25 sq. ft. = 43,560 sq. ft.
∴ one side of the field = $\sqrt{43{,}560}$ ft. = 208.71 ft.
∴ the perimeter of the field = 4 × 208.71 ft. = 834.84 ft. *Ans.*

7. *A machine for planing iron plates planes a space* $\frac{1}{2}$ *in. wide and 18 ft. long in 1 min. How long will it take to plane a plate 22 ft. 6 in long and 4 ft. 6 in. wide, allowing 51 min. for adjusting the machine?*

$$\tfrac{1}{2}\text{ in.} = \tfrac{1}{2}\text{ of }\tfrac{1}{12}\text{ ft.} = \tfrac{1}{24}\text{ ft.}$$

The number of minutes required

$$= \frac{22\frac{1}{2} \times 4\frac{1}{2}}{18 \times \frac{1}{24}}\ (\S\ 318) = \frac{45 \times \cancel{9} \times \overset{\overset{\overset{3}{\cancel{6}}}{\cancel{12}}}{\cancel{24}}}{\cancel{2} \times \cancel{2} \times \underset{\cancel{2}}{\cancel{18}}} = 135.$$

Therefore the total number of minutes required
= 135 min. + 51 min. = 186 min. = 3 hr. 6 min. *Ans.*

8. *How many tiles, each 8 in. square, will it take to cover a floor 24 ft. 8 in. long by 16 ft. wide?*

$$\therefore \frac{\text{area of floor}}{\text{area of tile}} = \frac{24\frac{2}{3} \times 16}{\frac{2}{3} \times \frac{2}{3}} \qquad \S\ 318$$

$$= \frac{74 \times \overset{\overset{4}{\cancel{8}}}{\cancel{16}} \times \cancel{3} \times 3}{\cancel{3} \times \cancel{2} \times \cancel{2}} = 888.$$

Therefore the number of tiles required is 888.

9. *A rectangle having an area of 48 sq. in. is three times as long as wide. What are the dimensions?*

Let $x =$ the number of inches in the width.

Then $3x =$ the number of inches in the length,

and $3x^2 =$ the number of square inches in the area. § 320

$$\therefore 3x^2 = 48.$$

$$\therefore x^2 = 16.$$

$$\therefore x = \pm 4.$$

$$\therefore 3x = \pm 12.$$

Therefore the dimensions are 12 in. and 4 in.

10. *The length of a rectangle is four times the width. If the perimeter is 60 ft., what is the area?*

Let $x =$ the number of feet in the width.

Then $4x =$ the number of feet in the length,

$10x =$ the number of feet in the perimeter,

and $4x^2 =$ the number of square feet in the area.

$$\therefore 10x = 60.$$

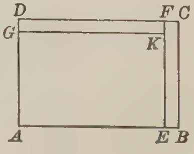
$$\therefore x = 6.$$

$$\therefore 4x^2 = 144.$$

Therefore the area is 144 sq. ft.

11. *From two adjacent sides of a rectangular field 60 rd. long and 40 rd wide a road is cut 4 rd. wide. How many acres are cut off for the road?*

If *ABCD* represents the field, the two rectangles *EBCF* and *GKFD* represent the road.



Area of *EBCF* = 40 × 4 sq. rd. § 320

= 160 sq. rd.

Area of *GKFD* = 56 × 4 sq. rd. § 320

= 224 sq. rd.

Area of road = 160 sq. rd. + 224 sq. rd.

= 384 sq. rd. = $\frac{384}{160}$ A. = 2.4 A.

12. *From one end of a rectangular sheet of iron 10 in. long a square piece is cut off, leaving 25 sq. in. in the rest of the sheet. How wide is the sheet?*

Let x = the number of inches in the width of the sheet.
Then $10x$ = the number of square inches in the area,
and x^2 = the number of square inches cut off.

$$\therefore\ 10x - x^2 = 25.$$
$$x^2 - 10x + 25 = 0.$$
$$x - 5 = 0.$$
$$\therefore\ x = 5.$$

Therefore the width of the sheet is 5 in.

Exercise 51. Page 200

1. *Find the area of the parallelogram whose base is 2.25 in. and whose altitude is* $1\frac{1}{3}$ *in.*

Area = $2.25 \times 1\frac{1}{3}$ sq. in. § 322
= $2\frac{1}{4} \times 1\frac{1}{3}$ sq. in.
= $\frac{\overset{3}{\cancel{9}}}{\cancel{4}} \times \frac{\cancel{4}}{\cancel{3}}$ sq. in. = 3 sq. in.

2. *Find the area of the parallelogram whose base is 3.44 in. and whose altitude is* $1\frac{1}{2}$ *in.*

Area = $3.44 \times 1\frac{1}{2}$ sq. in. § 322
= $(3.44 + 1.72)$ sq. in.
= 5.16 sq. in.

3. *Find the area of the parallelogram whose base is 2.7 ft. and whose altitude is 1.2 ft.*

Area = 2.7×1.2 sq. ft. § 322
= 3.24 sq. ft.

4. *Find the area of the parallelogram whose base is 5.6 ft. and whose altitude is 2.3 ft.*

Area = 5.6×2.3 sq. ft. § 322
= 12.88 sq. ft.

5. *Find the area of the parallelogram whose base is 2 ft. 3 in. and whose altitude is 7 in.*

2 ft. 3 in. = 27 in.
Area = 27×7 sq. in. § 322
= 189 sq. in.

6. *Find the area of the parallelogram whose base is 3 ft. 6 in. and whose altitude is 2 ft.*

3 ft. 6 in. $= 3\frac{1}{2}$ ft.
Area $= 3\frac{1}{2} \times 2$ sq. ft. § 322
$= 7$ sq. ft.

7. *Find the area of the triangle whose base is 1.4 in. and whose altitude is $1\frac{1}{2}$ in.*

Area $= \frac{1}{2} \times 1.4 \times 1\frac{1}{2}$ sq. in. § 325
$= 1.05$ sq. in.

8. *Find the area of the triangle whose base is 2.5 in. and whose altitude is 0.8 in.*

Area $= \frac{1}{2} \times 2.5 \times 0.8$ sq. in. § 325
$= 1$ sq. in.

9. *Find the area of the triangle whose base is $6\frac{1}{2}$ ft. and whose altitude is 3 ft.*

Area $= \frac{1}{2} \times 6\frac{1}{2} \times 3$ sq. ft. § 325
$= 9\frac{3}{4}$ sq. ft.

10. *Find the area of the triangle whose base is 5.4 ft. and whose altitude is 1.2 ft.*

Area $= \frac{1}{2} \times 5.4 \times 1.2$ sq. ft. § 325
$= 3.24$ sq. ft.

11. *Find the area of the triangle whose base is 1 ft. 6 in. and whose altitude is 8 in.*

1 ft. 6 in. $= 1\frac{1}{2}$ ft.; 8 in. $= \frac{2}{3}$ ft.
Area $= \frac{1}{2} \times 1\frac{1}{2} \times \frac{2}{3}$ sq. ft. § 325
$= \frac{1}{2} \times \frac{3}{2} \times \frac{2}{3}$ sq. ft.
$= \frac{1}{2}$ sq. ft.

12. *Find the area of the triangle whose base is 3 ft. 8 in. and whose altitude is 3 ft.*

3 ft. 8 in. $= 3\frac{2}{3}$ ft.
Area $= \frac{1}{2} \times 3\frac{2}{3} \times 3$ sq. ft. § 325
$= \frac{1}{2} \times \frac{11}{3} \times 3$ sq. ft.
$= \frac{11}{2}$ sq. ft. $= 5\frac{1}{2}$ sq. ft.

13. *Find the area of the trapezoid whose bases are 2 ft. and 1 ft. and whose altitude is 6 in.*

6 in. $= \frac{1}{2}$ ft.
Area $= \frac{1}{2} \times \frac{1}{2}(2 + 1)$ sq. ft. § 329
$= \frac{1}{2} \times \frac{1}{2} \times 3$ sq. ft.
$= \frac{3}{4}$ sq. ft.

14. *Find the area of the trapezoid whose bases are* $2\frac{1}{2}$ *ft. and* $1\frac{1}{4}$ *ft. and whose altitude is* 9 *in.*

$$\begin{aligned}9\text{ in.} &= \tfrac{3}{4}\text{ ft.}\\ \text{Area} &= \tfrac{1}{2}\times\tfrac{3}{4}\left(2\tfrac{1}{2}+1\tfrac{1}{4}\right)\text{ sq. ft.} && \S\ 329\\ &= \tfrac{1}{2}\times\tfrac{3}{4}\times 3\tfrac{3}{4}\text{ sq. ft.}\\ &= \tfrac{1}{2}\times\tfrac{3}{4}\times\tfrac{15}{4}\text{ sq. ft.}\\ &= \tfrac{45}{32}=1\tfrac{13}{32}\text{ sq. ft.}\end{aligned}$$

15. *Find the area of the trapezoid whose bases are* 3 *ft.* 7 *in. and* 2 *ft. and whose altitude is* 14 *in.*

$$\begin{aligned}3\text{ ft. }7\text{ in.} &= 3\tfrac{7}{12}\text{ ft.};\ 14\text{ in.}=1\tfrac{1}{6}\text{ ft.}\\ \text{Area} &= \tfrac{1}{2}\times 1\tfrac{1}{6}\left(3\tfrac{7}{12}+2\right)\text{ sq. ft.} && \S\ 329\\ &= \tfrac{1}{2}\times\tfrac{7}{6}\times\tfrac{67}{12}\text{ sq. ft.}\\ &= \tfrac{469}{144}\text{ sq. ft.}\\ &= 469\text{ sq. in.}\end{aligned}$$

16. *Find the area of the trapezoid whose bases are* 5 *ft.* 6 *in. and* 3 *ft. and whose altitude is* 2 *ft.*

$$\begin{aligned}5\text{ ft. }6\text{ in.} &= 5\tfrac{1}{2}\text{ ft.}\\ \text{Area} &= \tfrac{1}{2}\times 2\left(5\tfrac{1}{2}+3\right)\text{ sq. ft.} && \S\ 329\\ &= \tfrac{1}{2}\times 2\times\tfrac{17}{2}\text{ sq. ft.}\\ &= \tfrac{17}{2}\text{ sq. ft.}=8\tfrac{1}{2}\text{ sq. ft.}\end{aligned}$$

17. *Find the altitude of the parallelogram whose area is* 10 *sq. in. and whose base is* 5 *in.*

$$\begin{aligned}\text{Area} &= ab. && \S\ 322\\ \therefore\ a &= \frac{\text{area}}{b}\\ &= \tfrac{10}{5}\text{ in.}=2\text{ in.}\end{aligned}$$

18. *Find the altitude of the parallelogram whose area is* 6 *sq. in. and whose base is* 6 *in.*

$$\begin{aligned}\text{Area} &= ab. && \S\ 322\\ \therefore\ a &= \frac{\text{area}}{b}\\ &= \tfrac{6}{6}\text{ in.}=1\text{ in.}\end{aligned}$$

19. *Find the altitude of the parallelogram whose area is* 28 *sq. ft. and whose base is* 7 *ft.*

$$\begin{aligned}\text{Area} &= ab. && \S\ 322\\ \therefore\ a &= \frac{\text{area}}{b}\\ &= \tfrac{28}{7}\text{ ft.}=4\text{ ft.}\end{aligned}$$

20. *Find the altitude of the parallelogram whose area is 27 sq. ft. and whose base is 6 ft.*

$$\text{Area} = ab. \qquad \S\ 322$$

$$\therefore\ a = \frac{\text{area}}{b}$$

$$= \tfrac{27}{6}\text{ ft.} = 4\tfrac{1}{2}\text{ ft.}$$

21. *Find the altitude of the parallelogram whose area is 30 sq. ft. and whose base is 12 ft.*

$$\text{Area} = ab. \qquad \S\ 322$$

$$\therefore\ a = \frac{\text{area}}{b}$$

$$= \tfrac{30}{12}\text{ ft.} = 2\tfrac{1}{2}\text{ ft.}$$

22. *Find the altitude of the parallelogram whose area is 80 sq. in. and whose base is 16 in.*

$$\text{Area} = ab. \qquad \S\ 322$$

$$\therefore\ a = \frac{\text{area}}{b}$$

$$= \tfrac{80}{16}\text{ in.} = 5\text{ in.}$$

23. *Find the altitude of the triangle whose area is 49 sq. in. and whose base is 14 in.*

$$\text{Area} = \frac{ab}{2}. \qquad \S\ 325$$

$$\therefore\ a = \frac{2 \times \text{area}}{b}$$

$$= \frac{\cancel{2} \times \overset{7}{\cancel{49}}}{\underset{\cancel{7}}{\cancel{14}}}\text{ in.} = 7\text{ in.}$$

24. *Find the altitude of the triangle whose area is 48 sq. in. and whose base is 12 in.*

$$\text{Area} = \frac{ab}{2}. \qquad \S\ 325$$

$$\therefore\ a = \frac{2 \times \text{area}}{b}$$

$$= \frac{2 \times \overset{4}{\cancel{48}}}{\cancel{12}}\text{ in.} = 8\text{ in.}$$

25. *Find the altitude of the triangle whose area is 50 sq. ft. and whose base is 10 ft.*

$$\text{Area} = \frac{ab}{2}. \qquad \S\ 325$$

$$\therefore\ a = \frac{2 \times \text{area}}{b}$$

$$= \frac{2 \times \overset{5}{\cancel{50}}}{\cancel{10}}\text{ ft.} = 10\text{ ft.}$$

26. *Find the altitude of the triangle whose area is 160 sq. ft. and whose base is 20 ft.*

$$\text{Area} = \frac{ab}{2}. \qquad \S\ 325$$

$$\therefore\ a = \frac{2 \times \text{area}}{b}$$

$$= \frac{2 \times \overset{8}{\cancel{160}}}{\cancel{20}}\text{ ft.} = 16\text{ ft.}$$

27. *Find the altitude of the triangle whose area is 110 sq. yd. and whose base is 10 yd.*

$$\text{Area} = \frac{ab}{2}. \qquad \S\ 325$$

$$\therefore\ a = \frac{2 \times \text{area}}{b}$$

$$= \frac{2 \times \overset{11}{\cancel{110}}}{\cancel{10}}\text{ yd.} = 22\text{ yd.}$$

28. *Find the altitude of the triangle whose area is* 176 *sq. yd. and whose base is* 32 *yd.*

$$\text{Area} = \frac{ab}{2}. \qquad \S\ 325$$

$$\therefore\ a = \frac{2 \times \text{area}}{b}$$

$$= \frac{\cancel{2} \times \overset{11}{\cancel{176}}}{\underset{\cancel{2}}{\cancel{32}}} \text{ yd.} = 11 \text{ yd.}$$

29. *Find the altitude of the trapezoid whose area is* 33 *sq. in. and whose bases are* 5 *in. and* 6 *in.*

$$\text{Area} = \tfrac{1}{2}\, a\,(b + b'). \qquad \S\ 329$$

$$\therefore\ \frac{1}{2} a = \frac{\text{area}}{b + b'},$$

and $$a = \frac{2 \times \text{area}}{b + b'}$$

$$= \frac{2 \times 33}{5 + 6} \text{ in.}$$

$$= \frac{2 \times \overset{3}{\cancel{33}}}{\cancel{11}} \text{ in.}$$

$$= 6 \text{ in.}$$

30. *Find the altitude of the trapezoid whose area is* 15 *sq. in. and whose bases are* 4 *in. and* 6 *in.*

$$\text{Area} = \tfrac{1}{2}\, a\,(b + b'). \qquad \S\ 329$$

$$\therefore\ \frac{1}{2} a = \frac{\text{area}}{b + b'},$$

and $$a = \frac{2 \times \text{area}}{b + b'} \text{ in.}$$

$$= \frac{2 \times 15}{4 + 6} \text{ in.}$$

$$= \frac{\cancel{2} \times \overset{3}{\cancel{15}}}{\underset{\cancel{5}}{\cancel{10}}} \text{ in.}$$

$$= 3 \text{ in.}$$

31. *Find the altitude of the trapezoid whose area is* 13 *sq. ft. and whose bases are* 9 *ft. and* 5 *ft.*

$$\text{Area} = \tfrac{1}{2}\, a\,(b + b'). \qquad \S\ 329$$

$$\therefore\ \frac{1}{2} a = \frac{\text{area}}{b + b'},$$

and $$a = \frac{2 \times \text{area}}{b + b'}$$

$$= \frac{2 \times 13}{9 + 5} \text{ ft.}$$

$$= \frac{\cancel{2} \times 13}{\underset{7}{\cancel{14}}} \text{ ft.}$$

$$= \tfrac{13}{7} \text{ ft.} = 1\tfrac{6}{7} \text{ ft.}$$

32. *Find the altitude of the trapezoid whose area is* 70 *sq. yd. and whose bases are* 9 *yd. and* 11 *yd.*

$$\text{Area} = \tfrac{1}{2}\, a\,(b + b'). \qquad \S\ 329$$

$$\therefore\ \frac{1}{2} a = \frac{\text{area}}{b + b'},$$

and $$a = \frac{2 \times \text{area}}{b + b'}$$

$$= \frac{2 \times 70}{9 + 11} \text{ yd.}$$

$$= \frac{\cancel{2} \times \overset{7}{\cancel{70}}}{\underset{\cancel{2}}{\cancel{20}}} \text{ yd.}$$

$$= 7 \text{ yd.}$$

Exercise 52. Page 206

1. *Given the sides of a right triangle, 30 ft. and 40 ft., find the hypotenuse.*

$$\begin{aligned} c^2 &= a^2 + b^2. && \S\ 337 \\ \therefore\ c &= \sqrt{a^2 + b^2} && \text{Ax. } 5 \\ &= \sqrt{30^2 + 40^2} \text{ ft.} \\ &= \sqrt{900 + 1600} \text{ ft.} \\ &= \sqrt{2500} \text{ ft.} \\ &= 50 \text{ ft.} \end{aligned}$$

2. *Given the sides of a right triangle, 45 ft. and 60 ft., find the hypotenuse.*

$$\begin{aligned} c^2 &= a^2 + b^2. && \S\ 337 \\ \therefore\ c &= \sqrt{a^2 + b^2} && \text{Ax. } 5 \\ &= \sqrt{45^2 + 60^2} \text{ ft.} \\ &= \sqrt{2025 + 3600} \text{ ft.} \\ &= \sqrt{5625} \text{ ft.} \\ &= 75 \text{ ft.} \end{aligned}$$

3. *Given the sides of a right triangle, 20 ft. and 30 ft., find the hypotenuse to two decimal places.*

$$\begin{aligned} c^2 &= a^2 + b^2. && \S\ 337 \\ \therefore\ c &= \sqrt{a^2 + b^2} && \text{Ax. } 5 \\ &= \sqrt{20^2 + 30^2} \text{ ft.} \\ &= \sqrt{400 + 900} \text{ ft.} \\ &= \sqrt{1300} \text{ ft.} \\ &= 36.06 \text{ ft.} \end{aligned}$$

4. *Given the sides of a right triangle, 1.5 in. and 2.5 in., find the hypotenuse to two decimal places.*

$$\begin{aligned} c^2 &= a^2 + b^2. && \S\ 337 \\ \therefore\ c &= \sqrt{a^2 + b^2} && \text{Ax. } 5 \\ &= \sqrt{1.5^2 + 2.5^2} \text{ in.} \\ &= \sqrt{2.25 + 6.25} \text{ in.} \\ &= \sqrt{8.5} \text{ in.} \\ &= 2.92 \text{ in.} \end{aligned}$$

5. *Given the sides of a right triangle, 2 ft. 6 in. and 3 ft., find the hypotenuse to two decimal places.*

$$\begin{aligned} c^2 &= a^2 + b^2. && \S\ 337 \\ \therefore\ c &= \sqrt{a^2 + b^2} && \text{Ax. } 5 \\ &= \sqrt{2.5^2 + 3^2} \text{ ft.} \\ &= \sqrt{6.25 + 9} \text{ ft.} \\ &= \sqrt{15.25} \text{ ft.} \\ &= 3.91 \text{ ft.} \end{aligned}$$

6. *Given the sides of a right triangle, 3 ft. 8 in. and 2 ft., find the hypotenuse to two decimal places.*

$$\begin{aligned} c^2 &= a^2 + b^2. && \S\ 337 \\ \therefore\ c &= \sqrt{a^2 + b^2} && \text{Ax. } 5 \\ &= \sqrt{(3\tfrac{2}{3})^2 + 2^2} \text{ ft.} \\ &= \sqrt{13.4444 + 4} \text{ ft.} \\ &= \sqrt{17.4444} \text{ ft.} \\ &= 4.18 \text{ ft.} \end{aligned}$$

7. *Given the hypotenuse and one side of a right triangle, 50 ft. and 40 ft. respectively, find the other side.*

$$\begin{aligned} a^2 &= c^2 - b^2. && \S\ 338 \\ \therefore\ a &= \sqrt{c^2 - b^2} && \text{Ax. } 5 \\ &= \sqrt{50^2 - 40^2} \text{ ft.} \\ &= \sqrt{2500 - 1600} \text{ ft.} \\ &= \sqrt{900} \text{ ft.} \\ &= 30 \text{ ft.} \end{aligned}$$

8. *Given the hypotenuse and one side of a right triangle, 35 ft. and 21 ft. respectively, find the other side.*

$$\begin{aligned} a^2 &= c^2 - b^2. && \S\ 338 \\ \therefore\ a^2 &= \sqrt{c^2 - b^2} && \text{Ax. } 5 \\ &= \sqrt{35^2 - 21^2} \text{ ft.} \\ &= \sqrt{1225 - 441} \text{ ft.} \\ &= \sqrt{784} \text{ ft.} \\ &= 28 \text{ ft.} \end{aligned}$$

9. *Given the hypotenuse and one side of a right triangle, 10 ft. and 6 ft. respectively, find the other side.*

$$a^2 = c^2 - b^2. \qquad \text{§ 338}$$
$$\therefore\ a = \sqrt{c^2 - b^2} \qquad \text{Ax. 5}$$
$$= \sqrt{10^2 - 6^2}\ \text{ft.}$$
$$= \sqrt{100 - 36}\ \text{ft.}$$
$$= \sqrt{64}\ \text{ft.}$$
$$= 8\ \text{ft.}$$

10. *Given the hypotenuse and one side of a right triangle, 1.2 in. and 0.8 in. respectively, find the other side to two decimal places.*

$$a^2 = c^2 - b^2. \qquad \text{§ 338}$$
$$\therefore\ a = \sqrt{c^2 - b^2} \qquad \text{Ax. 5}$$
$$= \sqrt{1.2^2 - 0.8^2}\ \text{in.}$$
$$= \sqrt{1.44 - 0.64}\ \text{in.}$$
$$= \sqrt{.80}\ \text{in.}$$
$$= 0.89\ \text{in.}$$

11. *Given the hypotenuse and one side of a right triangle, 3 ft. 4 in. and 2 ft. respectively, find the other side to two decimal places.*

$$a^2 = c^2 - b^2. \qquad \text{§ 338}$$
$$\therefore\ a = \sqrt{c^2 - b^2} \qquad \text{Ax. 5}$$
$$= \sqrt{(3\tfrac{1}{3})^2 - 2^2}\ \text{ft.}$$
$$= \sqrt{11\tfrac{1}{9} - 4}\ \text{ft.}$$
$$= \sqrt{7.1111}\ \text{ft.}$$
$$= 2.67\ \text{ft.}$$

12. *Given the hypotenuse and one side of a right triangle, 6 ft. 2 in. and 5 ft. respectively, find the other side to two decimal places.*

$$a^2 = c^2 - b^2. \qquad \text{§ 338}$$
$$\therefore\ a = \sqrt{c^2 - b^2} \qquad \text{Ax. 5}$$
$$= \sqrt{(6\tfrac{1}{6})^2 - 5^2}\ \text{ft.}$$
$$= \sqrt{\tfrac{1369}{36} - 25}\ \text{ft.}$$
$$= \sqrt{\tfrac{1369}{36} - \tfrac{900}{30}}\ \text{ft.}$$
$$= \sqrt{\tfrac{469}{36}}\ \text{ft.}$$
$$= \frac{21.656}{6} = 3.61\ \text{ft.}$$

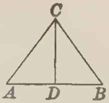
13. *A ladder 38 ft. 6 in. long is placed against a wall, with its foot 23.1 ft. from the base of the wall. How high does it reach on the wall?*

$$a^2 = c^2 - b^2. \qquad \text{§ 338}$$
$$\therefore\ a = \sqrt{c^2 - b^2}$$
$$= \sqrt{38.5^2 - 23.1^2}\ \text{ft.}$$
$$= \sqrt{61.6 \times 15.4}\ \text{ft.}$$
$$= \sqrt{4 \times 15.4 \times 15.4}\ \text{ft.}$$
$$= 2 \times 15.4\ \text{ft.}$$
$$= 30.8\ \text{ft.}$$

14. *Find the altitude of an equilateral triangle with side s.*

Given the equilateral $\triangle ABC$ with side s.

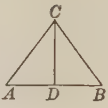
The $\perp CD$ bisects AB. § 85



$$h^2 = s^2 - (\tfrac{1}{2}s)^2. \qquad \text{§ 338}$$
$$\therefore\ h = \sqrt{s^2 - (\tfrac{1}{2}s)^2} \qquad \text{Ax. 5}$$
$$= \sqrt{s^2 - \tfrac{1}{4}s^2} = \sqrt{\tfrac{3}{4}s^2}$$
$$= \tfrac{1}{2}s\sqrt{3}.$$

15. *Find the side of an equilateral triangle with altitude h.*

Given the equilateral $\triangle ABC$ with altitude h.



$$s^2 - (\tfrac{1}{2}s)^2 = h^2. \qquad \S\ 338$$

$$s^2 - \tfrac{1}{4}s^2 = h^2.$$

$$\tfrac{3}{4}s^2 = h^2.$$

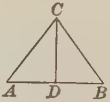
$$s^2 = \frac{4h^2}{3}.$$

$$s = \sqrt{\frac{4h^2}{3}} = 2h\sqrt{\frac{1}{3}} = 2h\sqrt{\frac{3}{9}} = \frac{2}{3}h\sqrt{3}.$$

16. *The area of an equilateral triangle with side s is $\frac{1}{4}s^2\sqrt{3}$.*

Given the equilateral $\triangle ABC$ with side s.

To prove that the area of $\triangle ABC = \frac{1}{4}s^2\sqrt{3}$.



Proof. Let the altitude of the $\triangle$ be h.

Then $$h^2 = s^2 - (\tfrac{1}{2}s)^2. \qquad \S\ 338$$

$$\therefore\ h = \sqrt{s^2 - (\tfrac{1}{2}s)^2} \qquad \text{Ax. 5}$$

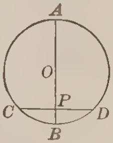
$$= \sqrt{s^2 - \tfrac{1}{4}s^2} = \sqrt{\tfrac{3}{4}s^2} = \tfrac{1}{2}s\sqrt{3}.$$

$$\text{Area} = \tfrac{1}{2}s \times \tfrac{1}{2}s\sqrt{3} \qquad \S\ 325$$

$$= \tfrac{1}{4}s^2\sqrt{3}.$$

Q. E. D.

17. *Find the length of the longest chord and of the shortest chord that can be drawn through a point 1 ft. from the center of a circle whose radius is 20 in.*



The longest chord through the given point P is the diameter AB through P. § 181

$\therefore$ the longest chord is 2 × 20 in., or 40 in.

The shortest chord through the given point P is the $\perp CD$ to the diameter AB. Ex. 1, p. 104

Now $$PB = OB - OP = 20 \text{ in.} - 12 \text{ in.} = 8 \text{ in.},$$

and $$AP = AO + OP = 20 \text{ in.} + 12 \text{ in.} = 32 \text{ in.}$$

Again, $$PC = PD, \qquad \S\ 174$$

and $$PC \times PD = AP \times PB. \qquad \S\ 299$$

$$\therefore\ \overline{PC}^2 = 32 \times 8. \qquad \text{Ax. 9}$$

$$\therefore\ PC = \sqrt{32 \times 8} \text{ in.} = \sqrt{256} \text{ in.} = 16 \text{ in.}$$

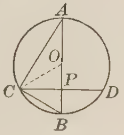
Then $$CD = 2\,PC = 32 \text{ in.}$$

Therefore the longest chord is 40 in. and the shortest is 32 in.

18. *The radius of a circle is 5 in. Through a point 3 in. from the center a diameter is drawn, and also a chord perpendicular to the diameter. Find the length of this chord, and the distance (to two decimal places) from one end of the chord to the ends of the diameter.*

Given the diameter AB through the point P and the chord $CD \perp$ to AB at P. $OA = 5$ in., $OP = 3$ in.

Required to find the length of CD, AC, and BC.



Solution. Draw OC.

Then $\overline{CP}^2 = \overline{OC}^2 - \overline{OP}^2$. § 337

$\therefore \overline{CP}^2 = 5^2 - 3^2$. Ax. 9

$CP = \sqrt{5^2 - 3^2}$ in. $= \sqrt{25 - 9}$ in.

$= \sqrt{16}$ in. $= 4$ in.

$CD = 2 \times CP$ § 174

$= 2 \times 4$ in. $= 8$ in.

Again, $AP = 5$ in. $+ 3$ in. $= 8$ in.

$\overline{AC}^2 = \overline{AP}^2 + \overline{CP}^2$. § 337

$\therefore AC = \sqrt{\overline{AP}^2 + \overline{CP}^2} = \sqrt{8^2 + 4^2}$ in. $= \sqrt{64 + 16}$ in.

$= \sqrt{80}$ in. $= 8.94$ in.

$\overline{BC}^2 = \overline{CP}^2 + \overline{PB}^2$. § 337

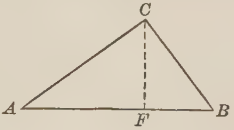
$\therefore BC = \sqrt{\overline{CP}^2 + \overline{PB}^2} = \sqrt{4^2 + 2^2}$ in. $= \sqrt{16 + 4}$ in.

$= \sqrt{20}$ in. $= 4.47$ in.

19. *In this figure the angle C is a right angle. From the relations $\overline{AC}^2 = AB \times AF$ (§ 294) and $\overline{CB}^2 = AB \times BF$, show that $\overline{AC}^2 + \overline{CB}^2 = \overline{AB}^2$.*

Given the rt. △ ABC, with the altitude CF.

To prove that $\overline{AC}^2 + \overline{CB}^2 = \overline{AB}^2$.



Proof. $\overline{AC}^2 = AB \times AF$,

and $\overline{CB}^2 = AB \times BF$. § 294

$\therefore \overline{AC}^2 + \overline{CB}^2 = AB \times AF + AB \times BF$ Ax. 1

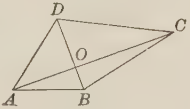
$= AB(AF + BF) = AB \times AB$. Ax. 9

$\therefore \overline{AC}^2 + \overline{CB}^2 = \overline{AB}^2$. Q. E. D.

20. *If the diagonals of a quadrilateral intersect at right angles, the sum of the squares on one pair of opposite sides is equivalent to the sum of the squares on the other pair.*

Given the quadrilateral $ABCD$, with the diagonals AC and BD intersecting at rt. ∠s at O.

To prove that $\overline{AB}^2 + \overline{CD}^2 = \overline{BC}^2 + \overline{DA}^2$.



Proof. $\overline{AB}^2 = \overline{AO}^2 + \overline{OB}^2$,

and $\overline{CD}^2 = \overline{OC}^2 + \overline{DO}^2$. § 337

$\therefore \overline{AB}^2 + \overline{CD}^2 = \overline{AO}^2 + \overline{OB}^2 + \overline{OC}^2 + \overline{DO}^2$. Ax. 1

Also $\overline{BC}^2 = \overline{OB}^2 + \overline{OC}^2$,

and $\overline{DA}^2 = \overline{AO}^2 + \overline{DO}^2$. § 337

$\therefore \overline{BC}^2 + \overline{DA}^2 = \overline{AO}^2 + \overline{OB}^2 + \overline{OC}^2 + \overline{DO}^2$. Ax. 1

$\therefore \overline{AB}^2 + \overline{CD}^2 = \overline{BC}^2 + \overline{DA}^2$, by Ax. 8. Q. E. D.

Exercise 53. Page 209

1. *Find the length, to two decimal places, of the diagonal of the square whose side is 7 in.*

$d^2 = s^2 + s^2.$ § 337

$\therefore\ d = \sqrt{s^2 + s^2}$ Ax. 5

$= \sqrt{7^2 + 7^2}$ in.

$= \sqrt{98}$ in.

$= 9.8995$ in.

9.90 in. *Ans.*

2. *Find the length, to two decimal places, of the diagonal of the square whose side is 10 in.*

$d^2 = s^2 + s^2.$ § 337

$\therefore\ d = \sqrt{s^2 + s^2}$ Ax. 5

$= \sqrt{10^2 + 10^2}$ in.

$= \sqrt{200}$ in.

$= 14.142$ in.

14.14 in. *Ans.*

3. *Find the length, to two decimal places, of the diagonal of the square whose side is 9.2 in.*

$d^2 = s^2 + s^2.$ § 337

$\therefore\ d = \sqrt{s^2 + s^2}$ Ax. 5

$= \sqrt{9.2^2 + 9.2^2}$ in.

$= \sqrt{169.28}$ in.

$= 13.010$ in.

13.01 in. *Ans.*

4. *Find the length, to two decimal places, of the diagonal of the square whose side is 1 ft. 6 in.*

1 ft. 6 in. = 18 in.

$d^2 = s^2 + s^2.$ § 337

$\therefore\ d = \sqrt{s^2 + s^2}$ Ax. 5

$= \sqrt{18^2 + 18^2}$ in.

$= \sqrt{648}$ in.

$= 25.455$ in.

25.46 in. *Ans.*

5. *Find the length, to two decimal places, of the diagonal of the square whose side is 2 ft. 3 in.*

2 ft. 3 in. = 27 in.

$d^2 = s^2 + s^2.$ § 337

$\therefore\ d = \sqrt{s^2 + s^2}$ Ax. 5

$= \sqrt{27^2 + 27^2}$ in.

$= \sqrt{1458}$ in.

$= 38.183$ in.

38.18 in. *Ans.*

6. *Find the length, to two decimal places, of the side of the square whose diagonal is 4 in.*

$d^2 = s^2 + s^2.$ § 337

$\therefore\ 2\,s^2 = d^2.$

$s^2 = \frac{d^2}{2}.$

$\therefore\ s = \sqrt{\frac{d^2}{2}} = \frac{1}{2}d\sqrt{2}.$

$\therefore\ s = \frac{1}{2} \times 4\sqrt{2}$ in.

$= 2\sqrt{2}$ in.

$= 2 \times 1.4142$ in.

$= 2.8284$ in.

2.83 in. *Ans.*

7. *Find the length, to two decimal places, of the side of the square whose diagonal is 8 in.*

$d^2 = s^2 + s^2.$ § 337

$\therefore\ 2\,s^2 = d^2.$

$s^2 = \frac{d^2}{2}.$

$\therefore\ s = \sqrt{\frac{d^2}{2}} = \frac{1}{2}d\sqrt{2}.$

$\therefore\ s = \frac{1}{2} \times 8\sqrt{2}$ in.

$= 4\sqrt{2}$ in.

$= 4 \times 1.4142$ in.

$= 5.6568$ in.

5.66 in. *Ans.*

8. *Find the length, to two decimal places, of the side of the square whose diagonal is 5 ft.*

$$d^2 = s^2 + s^2. \qquad \S\ 337$$

$$\therefore\ 2s^2 = d^2.$$

$$s^2 = \frac{d^2}{2}.$$

$$\therefore\ s = \sqrt{\frac{d^2}{2}} = \frac{1}{2}d\sqrt{2}.$$

$$\therefore\ s = \tfrac{1}{2} \times 5\sqrt{2} \text{ ft.}$$
$$= 2.5 \times 1.4142 \text{ ft.}$$
$$= 3.5355 \text{ ft.}$$

3.54 ft. *Ans.*

9. *Find the length, to two decimal places, of the side of the square whose diagonal is $\sqrt{5}$ in.*

$$d^2 = s^2 + s^2. \qquad \S\ 337$$

$$\therefore\ 2s^2 = d^2.$$

$$s^2 = \frac{d^2}{2}.$$

$$s = \sqrt{\frac{d^2}{2}}$$
$$= \sqrt{\tfrac{5}{2}} \text{ in.}$$
$$= \sqrt{2.5} \text{ in.}$$
$$= 1.581 \text{ in.}$$

1.58 in. *Ans.*

10. *Find the length, to two decimal places, of the side of the square whose diagonal is 2 ft. 6 in.*

2 ft. 6 in. = 30 in.

$$d^2 = s^2 + s^2. \qquad \S\ 337$$

$$\therefore\ 2s^2 = d^2.$$

$$s^2 = \frac{d^2}{2}.$$

$$\therefore\ s = \sqrt{\frac{d^2}{2}} = \frac{1}{2}d\sqrt{2}.$$

$$\therefore\ s = \tfrac{1}{2} \times 30\sqrt{2} \text{ in.}$$
$$= 15 \times 1.4142 \text{ in.}$$
$$= 21.213 \text{ in.}$$

21.21 in. *Ans.*

11. *The minute hand and hour hand of a clock are 6 in. and $4\frac{1}{2}$ in. long respectively. How far apart are the ends of the hands at 9 o'clock?*

At 9 o'clock the hour hand OH points at 9 and the minute hand OM points at 12, and the hands are therefore perpendicular to each other.

Given $OM = 6$ in. and $OH = 4\frac{1}{2}$ in. and $\angle O =$ a rt. $\angle$.

Required to find HM.

Solution.

$$\overline{HM}^2 = \overline{OM}^2 + \overline{OH}^2. \qquad \S\ 337$$

$$\therefore\ HM = \sqrt{\overline{OM}^2 + \overline{OH}^2} \qquad \text{Ax. } 5$$
$$= \sqrt{6^2 + (4\tfrac{1}{2})^2} \text{ in.} \qquad \text{Ax. } 9$$
$$= \sqrt{36 + 20.25} \text{ in.}$$
$$= \sqrt{56.25} \text{ in.}$$
$$= 7.5 \text{ in.}$$

12. *A rectangle whose base is* 9 *and diagonal* 15 *has the same area as a square whose side is* x. *Find the value of* x.

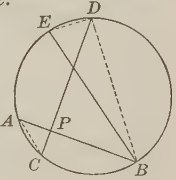
$$\begin{aligned} a^2 &= d^2 - b^2. && \text{§ 338}\\ \therefore\ a &= \sqrt{d^2 - b^2} && \text{Ax. 5}\\ &= \sqrt{15^2 - 9^2}\\ &= \sqrt{225 - 81}\\ &= \sqrt{144}\\ &= 12.\\ \text{Area of rectangle} &= 12 \times 9 = 108. && \text{§ 320}\\ \text{Area of square} &= x^2. && \text{§ 320}\\ \therefore\ x^2 &= 108.\\ \therefore\ x &= \sqrt{108} = 10.392. \end{aligned}$$

13. *A ring is screwed into a ceiling in a room* 10 *ft. high. Two rings are screwed into the floor at points* 5 *ft. and* 12 *ft. from a point directly beneath the one in the ceiling. Wires are stretched from the ceiling ring to each floor ring. How long are the wires? (Answer to two decimal places.)*

$$\begin{aligned} c^2 &= a^2 + b^2. && && \text{§ 337}\\ \therefore\ c &= \sqrt{a^2 + b^2}. && && \text{Ax. 5}\\ \therefore\ c &= \sqrt{10^2 + 5^2}\text{ ft. and} & c &= \sqrt{10^2 + 12^2}\text{ ft.}\\ &= \sqrt{100 + 25}\text{ ft.} & &= \sqrt{100 + 144}\text{ ft.}\\ &= \sqrt{125}\text{ ft.} & &= \sqrt{244}\text{ ft.}\\ &= 11.180\text{ ft.} & &= 15.623\text{ ft.} \end{aligned}$$

Therefore the wires are 11.18 ft. and 15.62 ft.

14. *The sum of the squares on the segments of two perpendicular chords is equivalent to the square on the diameter of the circle.*



Given the ⊥ chords AB and CD, intersecting at P, and the diameter BE.

To prove that $\overline{AP}^2 + \overline{PB}^2 + \overline{CP}^2 + \overline{PD}^2 = \overline{BE}^2$.

Proof. Draw AC, ED, and BD.

The $\angle BDE$ is a rt. $\angle$. § 215

$$\begin{aligned} \therefore\ \overline{BD}^2 + \overline{ED}^2 &= \overline{BE}^2. && \text{§ 337}\\ \overline{BD}^2 &= \overline{PB}^2 + \overline{PD}^2,\\ \text{and}\quad \overline{AC}^2 &= \overline{AP}^2 + \overline{CP}^2. && \text{§ 337}\\ \therefore\ \overline{BD}^2 + \overline{AC}^2 &= \overline{PB}^2 + \overline{PD}^2 + \overline{AP}^2 + \overline{CP}^2. && \text{Ax. 1} \end{aligned}$$

Now since the $\angle P$ is a rt. $\angle$, Given

$$\text{arc } AC + \text{arc } DB = 180°. \qquad \text{§ 219}$$

Also $\text{arc } ED + \text{arc } DB = 180°$. Given

$\therefore \text{arc } AC + \text{arc } DB = \text{arc } ED + \text{arc } DB$. Ax. 1

$\therefore \text{arc } AC = \text{arc } ED$. Ax. 2

$\therefore \text{chord } AC = \text{chord } ED$. § 170

$\therefore \overline{AC}^2 = \overline{ED}^2$. Ax. 5

Substitute $\overline{ED}^2$ for its equal, $\overline{AC}^2$,

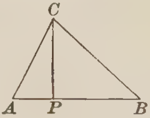
$\overline{BD}^2 + \overline{ED}^2 = \overline{PB}^2 + \overline{PD}^2 + \overline{AP}^2 + \overline{CP}^2$. Ax. 9

Substitute $\overline{BE}^2$ for its equal, $\overline{BD}^2 + \overline{ED}^2$,

$\overline{BE}^2 = \overline{PB}^2 + \overline{PD}^2 + \overline{AP}^2 + \overline{CP}^2$. Ax. 9

That is, $\overline{AP}^2 + \overline{PB}^2 + \overline{CP}^2 + \overline{PD}^2 = \overline{BE}^2$. Q.E.D.

15. *The difference of the squares on two sides of a triangle is equivalent to the difference of the squares on the segments of the third side, made by the perpendicular on the third side from the opposite vertex.*



Given the $\triangle ABC$, with $CP \perp$ to AB and BC greater than CA.

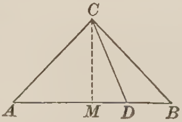
To prove that $\overline{BC}^2 - \overline{CA}^2 = \overline{PB}^2 - \overline{AP}^2$.

Proof. $\overline{BC}^2 = \overline{CP}^2 + \overline{PB}^2$,

and $\overline{CA}^2 = \overline{CP}^2 + \overline{AP}^2$. § 337

$\therefore \overline{BC}^2 - \overline{CA}^2 = \overline{PB}^2 - \overline{AP}^2$, by Ax. 2. Q.E.D.

16. *In an isosceles triangle the square on one of the equal sides is equivalent to the square on any line drawn from the vertex to the base, increased by the product of the segments of the base.*



Given the isosceles $\triangle ABC$, with AC equal to BC and CD any line drawn from C to AB.

To prove that $\overline{BC}^2 = \overline{CD}^2 + AD \times DB$.

Proof. Draw $CM \perp$ to AB.

Since CD is any line drawn from C to AB, $\angle s$ CDA and BDC are in general supplementary unequal $\angle s$.

Suppose $\angle BDC$ is obtuse.

Then $\overline{BC}^2 = \overline{CD}^2 + \overline{DB}^2 + 2\,DB \times MD$. § 342

That is, $\overline{BC}^2 = \overline{CD}^2 + DB \times (DB + 2\,MD)$.

Now $AM = MB$. § 85

That is, $AM = DB + MD$. Ax. 11

$\therefore AM + MD = DB + 2\,MD$. Ax. 1

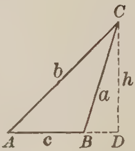
That is, $AD = DB + 2\,MD$. Ax. 11

Substitute $DB + 2\,MD$ for its equal, AD,

$\overline{BC}^2 = \overline{CD}^2 + AD \times DB$, by Ax. 9. Q.E.D.

Exercise 54. Page 211

1. *To compute the area of a triangle in terms of its sides.*



Given the $\triangle ABC$, with sides a, b, and c opposite the angles A, B, and C respectively.

To compute the area of $\triangle ABC$ in terms of a, b, and c.

Solution. At least one of the angles A or B is acute.

Suppose $\angle A$ is acute.

Draw $CD \perp$ to AB.

In the $\triangle ADC$, $h^2 = b^2 - \overline{AD}^2$. § 338

In the $\triangle ABC$, $a^2 = b^2 + c^2 - 2c \times AD$. § 341

$$\therefore AD = \frac{b^2 + c^2 - a^2}{2c}.$$

Substitute this value for AD,

$$h^2 = b^2 - \frac{(b^2 + c^2 - a^2)^2}{4c^2} \qquad \text{Ax. 9}$$

$$= \frac{4b^2c^2 - (b^2 + c^2 - a^2)^2}{4c^2}$$

$$= \frac{(2bc + b^2 + c^2 - a^2)(2bc - b^2 - c^2 + a^2)}{4c^2}$$

$$= \frac{[(b + c)^2 - a^2][a^2 - (b - c)^2]}{4c^2}$$

$$= \frac{(a + b + c)(b + c - a)(a + b - c)(a - b + c)}{4c^2}$$

Let $a + b + c = 2s$.

Then $b + c - a = a + b + c - 2a = 2s - 2a = 2(s - a)$,

$a + b - c = a + b + c - 2c = 2s - 2c = 2(s - c)$,

and $a - b + c = a + b + c - 2b = 2s - 2b = 2(s - b)$.

$$\therefore h^2 = \frac{2s \times 2(s - a) \times 2(s - b) \times 2(s - c)}{4c^2}$$

$$= \frac{4s(s - a)(s - b)(s - c)}{c^2}.$$

$$\therefore h = \frac{2}{c}\sqrt{s(s - a)(s - b)(s - c)}. \qquad \text{Ax. 5}$$

Area of $\triangle ABC = \frac{1}{2}ch$ § 325

$$= \frac{1}{2}c \times \frac{2}{c}\sqrt{s(s - a)(s - b)(s - c)}$$

$$= \sqrt{s(s - a)(s - b)(s - c)}.$$ Q.E.F.

2. *Find the area, to two decimal places, of the triangle whose sides are* 4, 5, 6.

$$2s = 4 + 5 + 6 = 15.$$
$$\therefore\ s = \tfrac{1}{2} \text{ of } 15 = 7.5.$$
$$\text{Area} = \sqrt{7.5 \times 3.5 \times 2.5 \times 1.5} \qquad \text{Ex. 1}$$
$$= \sqrt{98.4375}$$
$$= 9.921. \qquad 9.92.\ \textit{Ans.}$$

3. *Find the area, to two decimal places, of the triangle whose sides are* 5, 6, 7.

$$2s = 5 + 6 + 7 = 18.$$
$$\therefore\ s = \tfrac{1}{2} \text{ of } 18 = 9.$$
$$\text{Area} = \sqrt{9 \times 4 \times 3 \times 2} \qquad \text{Ex. 1}$$
$$= \sqrt{216}$$
$$= 14.696. \qquad 14.70.\ \textit{Ans.}$$

4. *Find the area of the triangle whose sides are* 6, 8, 10.

$$2s = 6 + 8 + 10 = 24.$$
$$\therefore\ s = \tfrac{1}{2} \text{ of } 24 = 12.$$
$$\text{Area} = \sqrt{12 \times 6 \times 4 \times 2} \qquad \text{Ex. 1}$$
$$= \sqrt{24 \times 24}$$
$$= 24. \qquad 24.\ \textit{Ans.}$$

5. *Find the area, to two decimal places, of the triangle whose sides are* 6, 8, 9.

$$2s = 6 + 8 + 9 = 23.$$
$$\therefore\ s = \tfrac{1}{2} \text{ of } 23 = 11.5.$$
$$\text{Area} = \sqrt{11.5 \times 5.5 \times 3.5 \times 2.5} \qquad \text{Ex. 1}$$
$$= \sqrt{553.4375}$$
$$= 23.525. \qquad 23.53.\ \textit{Ans.}$$

6. *Find the area, to two decimal places, of the triangle whose sides are* 7, 8, 11.

$$2s = 7 + 8 + 11 = 26.$$
$$\therefore\ s = \tfrac{1}{2} \text{ of } 26 = 13.$$
$$\text{Area} = \sqrt{13 \times 6 \times 5 \times 2} \qquad \text{Ex. 1}$$
$$= \sqrt{780}$$
$$= 27.928. \qquad 27.93.\ \textit{Ans.}$$

7. *Find the area, to two decimal places, of the triangle whose sides are* 9, 10, 11.

$$2s = 9 + 10 + 11 = 30.$$
$$\therefore\ s = \tfrac{1}{2} \text{ of } 30 = 15.$$
$$\text{Area} = \sqrt{15 \times 6 \times 5 \times 4} \qquad \text{Ex. 1}$$
$$= \sqrt{1800}$$
$$= 42.426. \qquad 42.43.\ \textit{Ans.}$$

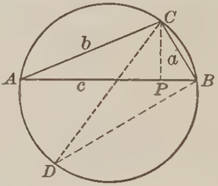
8. *Find the area, to two decimal places, of the triangle whose sides are* 1.2, 3, 2.1.

$$2s = 1.2 + 3 + 2.1 = 6.3.$$
$$\therefore s = \tfrac{1}{2} \text{ of } 6.3 = 3.15.$$
$$\text{Area} = \sqrt{3.15 \times 1.95 \times 0.15 \times 1.05} \qquad \text{Ex. 1}$$
$$= \sqrt{0.96744375}$$
$$= 0.983. \qquad 0.98. \; \textit{Ans.}$$

9. *Find the area, to two decimal places, of the triangle whose sides are* 11, 12, 13.

$$2s = 11 + 12 + 13 = 36.$$
$$\therefore s = \tfrac{1}{2} \text{ of } 36 = 18.$$
$$\text{Area} = \sqrt{18 \times 7 \times 6 \times 5} \qquad \text{Ex. 1}$$
$$= \sqrt{3780}$$
$$= 61.481. \qquad 61.48. \; \textit{Ans.}$$

10. *To compute the radius of the circle circumscribed about a triangle in terms of the sides of the triangle.*



Given the ⊙ circumscribed about the △ ABC, the sides of which are a, b, and c.

To compute r in terms of a, b, c.

Solution. Draw $CP \perp$ to AB, draw the diameter CD, and draw BD.

Then $BC \times CA = CD \times CP$, § 305

or $ab = 2rh$.

But $$h = \frac{2}{c}\sqrt{s(s-a)(s-b)(s-c)}. \qquad \text{Ex. 1}$$

$$\therefore ab = \frac{4r}{c}\sqrt{s(s-a)(s-b)(s-c)}.$$

$$\therefore r = \frac{abc}{4\sqrt{s(s-a)(s-b)(s-c)}}. \qquad \text{Q.E.F.}$$

11. *Compute the radius of the circle circumscribed about the triangle whose sides are* 3, 4, 5.

$$2s = 3 + 4 + 5 = 12.$$
$$\therefore s = \tfrac{1}{2} \text{ of } 12 = 6.$$
$$r = \frac{abc}{4\sqrt{s(s-a)(s-b)(s-c)}} \qquad \text{Ex. 10}$$
$$= \frac{3 \times 4 \times 5}{4\sqrt{6 \times 3 \times 2 \times 1}}$$
$$= \frac{3 \times 4 \times 5}{4 \times 6} = \frac{5}{2} = 2.5. \qquad 2.5. \; \textit{Ans.}$$

12. *Compute the radius of the circle circumscribed about the triangle whose sides are* 27, 36, 45.

$$2s = 27 + 36 + 45 = 108.$$

$$\therefore\ s = \tfrac{1}{2} \text{ of } 108 = 54.$$

$$r = \frac{abc}{4\sqrt{s(s-a)(s-b)(s-c)}} \qquad \text{Ex. 10}$$

$$= \frac{27 \times 36 \times 45}{4\sqrt{54 \times 27 \times 18 \times 9}}$$

$$= \frac{27 \times 36 \times 45}{4\sqrt{27 \times 27 \times 18 \times 18}}$$

$$= \frac{\cancel{27} \times \overset{2}{\cancel{36}} \times 45}{\underset{2}{\cancel{4}} \times \cancel{27} \times \cancel{18}}$$

$$= \tfrac{45}{2} = 22.5. \qquad 22.5.\ \textit{Ans.}$$

13. *Compute the radius, to two decimal places, of the circle circumscribed about the triangle whose sides are* 7, 9, 11.

$$2s = 7 + 9 + 11 = 27.$$

$$\therefore\ s = \tfrac{1}{2} \text{ of } 27 = 13\tfrac{1}{2}.$$

$$r = \frac{abc}{4\sqrt{s(s-a)(s-b)(s-c)}} \qquad \text{Ex. 10}$$

$$= \frac{7 \times 9 \times 11}{4\sqrt{13\frac{1}{2} \times 6\frac{1}{2} \times 4\frac{1}{2} \times 2\frac{1}{2}}}$$

$$= \frac{7 \times 9 \times 11}{4\sqrt{\frac{27}{2} \times \frac{13}{2} \times \frac{9}{2} \times \frac{5}{2}}}$$

$$= \frac{7 \times 9 \times 11}{\sqrt{27 \times 13 \times 9 \times 5}}$$

$$= \frac{7 \times 9 \times 11}{9\sqrt{3 \times 13 \times 5}}$$

$$= \frac{77}{\sqrt{3 \times 13 \times 5}}$$

$$= \frac{77}{3 \times 13 \times 5}\sqrt{3 \times 13 \times 5}$$

$$= \tfrac{77}{195}\sqrt{195}$$

$$= \tfrac{77}{195} \times 13.9642$$

$$= 5.514. \qquad 5.51.\ \textit{Ans.}$$

14. *Compute the radius, to two decimal places, of the circle circumscribed about the triangle whose sides are* 10, 11, 12.

$$2s = 10 + 11 + 12 = 33.$$

$$\therefore\ s = \tfrac{1}{2} \text{ of } 33 = 16\tfrac{1}{2}.$$

$$r = \frac{abc}{4\sqrt{s(s-a)(s-b)(s-c)}} \qquad \text{Ex. 10}$$

$$= \frac{10 \times 11 \times 12}{4\sqrt{16\frac{1}{2} \times 6\frac{1}{2} \times 5\frac{1}{2} \times 4\frac{1}{2}}}$$

$$= \frac{10 \times 11 \times 12}{4\sqrt{\frac{33}{2} \times \frac{13}{2} \times \frac{11}{2} \times \frac{9}{2}}}$$

$$= \frac{10 \times 11 \times 12}{\sqrt{33 \times 13 \times 11 \times 9}}$$

$$= \frac{10 \times 11 \times 12}{\sqrt{11^2 \times 3^2 \times 3 \times 13}}$$

$$= \frac{10 \times 11 \times 12}{11 \times 3\sqrt{39}}$$

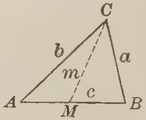
$$= \frac{40}{\sqrt{39}} = \frac{40}{39}\sqrt{39} = \frac{40}{39} \times 6.245 = 6.405.$$

6.41. *Ans.*

15. *To compute the medians of a triangle in terms of its sides.*

Given the $\triangle ABC$, with m the median on the side AB.

To compute m in terms of the sides.



Solution. Let a, b, and c be the sides opposite the $\angle$s A, B, and C respectively.

$$a^2 + b^2 = 2\left(\frac{c}{2}\right)^2 + 2m^2. \qquad \S\ 343$$

$$a^2 + b^2 = \frac{c^2}{2} + 2m^2.$$

$$2a^2 + 2b^2 = c^2 + 4m^2.$$

$$\therefore\ 4m^2 = 2a^2 + 2b^2 - c^2.$$

$$\therefore\ m^2 = \frac{2(a^2 + b^2) - c^2}{4}.$$

$$\therefore\ m = \tfrac{1}{2}\sqrt{2(a^2 + b^2) - c^2}. \qquad \text{Q.E.F.}$$

Similarly $$m' = \tfrac{1}{2}\sqrt{2(b^2 + c^2) - a^2},$$

and $$m'' = \tfrac{1}{2}\sqrt{2(a^2 + c^2) - b^2}.$$

16. *Compute the three medians, to two decimal places, of the triangle whose sides are* 3, 4, 5.

Here $a=3$, $b=4$, $c=5$.

$$\begin{array}{lll}
\therefore\ m=\frac{1}{2}\sqrt{2(a^2+b^2)-c^2} & m'=\frac{1}{2}\sqrt{2(b^2+c^2)-a^2} & m''=\frac{1}{2}\sqrt{2(a^2+c^2)-b^2} \\
 & & \text{Ex. 15} \\
=\frac{1}{2}\sqrt{2(3^2+4^2)-5^2} & =\frac{1}{2}\sqrt{2(4^2+5^2)-3^2} & =\frac{1}{2}\sqrt{2(3^2+5^2)-4^2} \\
=\frac{1}{2}\sqrt{2\times 25-25} & =\frac{1}{2}\sqrt{2\times 41-9} & =\frac{1}{2}\sqrt{2\times 34-16} \\
=\frac{1}{2}\sqrt{50-25} & =\frac{1}{2}\sqrt{82-9} & =\frac{1}{2}\sqrt{68-16} \\
=\frac{1}{2}\sqrt{25} & =\frac{1}{2}\sqrt{73} & =\frac{1}{2}\sqrt{52} \\
=\frac{1}{2}\times 5 & =\frac{1}{2}\times 8.544 & =\frac{1}{2}\times 7.211 \\
=2.5. & =4.272. & =3.606.
\end{array}$$

Therefore the three medians are 2.5, 4.27, and 3.61.

17. *Compute the three medians, to two decimal places, of the triangle whose sides are* 6, 8, 10.

Here $a=6$, $b=8$, $c=10$.

$$\begin{array}{lll}
\therefore\ m=\frac{1}{2}\sqrt{2(a^2+b^2)-c^2} & m'=\frac{1}{2}\sqrt{2(b^2+c^2)-a^2} & m''=\frac{1}{2}\sqrt{2(a^2+c^2)-b^2} \\
 & & \text{Ex. 15} \\
=\frac{1}{2}\sqrt{2(6^2+8^2)-10^2} & =\frac{1}{2}\sqrt{2(8^2+10^2)-6^2} & =\frac{1}{2}\sqrt{2(6^2+10^2)-8^2} \\
=\frac{1}{2}\sqrt{2\times 100-100} & =\frac{1}{2}\sqrt{2\times 164-36} & =\frac{1}{2}\sqrt{2\times 136-64^2} \\
=\frac{1}{2}\sqrt{200-100} & =\frac{1}{2}\sqrt{328-36} & =\frac{1}{2}\sqrt{272-64} \\
=\frac{1}{2}\sqrt{100} & =\frac{1}{2}\sqrt{292} & =\frac{1}{2}\sqrt{208} \\
=\frac{1}{2}\times 10 & =\frac{1}{2}\times 17.088 & =\frac{1}{2}\times 14.422 \\
=5. & =8.544. & =7.211.
\end{array}$$

Therefore the three medians are 5, 8.54, and 7.21.

18. *Compute the three medians, to two decimal places, of the triangle whose sides are* 6, 7, 8.

Here $a=6$, $b=7$, $c=8$.

$$\begin{array}{lll}
.\ m=\frac{1}{2}\sqrt{2(a^2+b^2)-c^2} & m'=\frac{1}{2}\sqrt{2(b^2+c^2)-a^2} & m''=\frac{1}{2}\sqrt{2(a^2+c^2)-b^2} \\
 & & \text{Ex. 15} \\
=\frac{1}{2}\sqrt{2(6^2+7^2)-8^2} & =\frac{1}{2}\sqrt{2(7^2+8^2)-6^2} & =\frac{1}{2}\sqrt{2(6^2+8^2)-7^2} \\
=\frac{1}{2}\sqrt{2\times 85-64} & =\frac{1}{2}\sqrt{2\times 113-36} & =\frac{1}{2}\sqrt{2\times 100-49} \\
=\frac{1}{2}\sqrt{170-64} & =\frac{1}{2}\sqrt{226-36} & =\frac{1}{2}\sqrt{200-49} \\
=\frac{1}{2}\sqrt{106} & =\frac{1}{2}\sqrt{190} & =\frac{1}{2}\sqrt{151} \\
=\frac{1}{2}\times 10.295 & =\frac{1}{2}\times 13.784 & =\frac{1}{2}\times 12.288 \\
=5.148. & =6.892. & =6.144.
\end{array}$$

Therefore the three medians are 5.15, 6.89, and 6.14.

19. *Compute the three medians, to two decimal places, of the triangle whose sides are* 7, 9, 11.

Here $a = 7$, $b = 9$, $c = 11$.

$$\begin{array}{lll}
\therefore\ m = \tfrac{1}{2}\sqrt{2(a^2+b^2)-c^2} & m' = \tfrac{1}{2}\sqrt{2(b^2+c^2)-a^2} & m'' = \tfrac{1}{2}\sqrt{2(a^2+c^2)-b^2} \\
 & & \text{Ex. 15} \\
\quad = \tfrac{1}{2}\sqrt{2(7^2+9^2)-11^2} & = \tfrac{1}{2}\sqrt{2(9^2+11^2)-7^2} & = \tfrac{1}{2}\sqrt{2(7^2+11^2)-9^2} \\
\quad = \tfrac{1}{2}\sqrt{2\times 130-121} & = \tfrac{1}{2}\sqrt{2\times 202-49} & = \tfrac{1}{2}\sqrt{2\times 170-81} \\
\quad = \tfrac{1}{2}\sqrt{260-121} & = \tfrac{1}{2}\sqrt{404-49} & = \tfrac{1}{2}\sqrt{340-81} \\
\quad = \tfrac{1}{2}\sqrt{139} & = \tfrac{1}{2}\sqrt{355} & = \tfrac{1}{2}\sqrt{259} \\
\quad = \tfrac{1}{2}\times 11.789 & = \tfrac{1}{2}\times 18.841 & = \tfrac{1}{2}\times 16.093 \\
\quad = 5.894. & = 9.421. & = 8.047.
\end{array}$$

Therefore the medians are 5.89, 9.42, and 8.05.

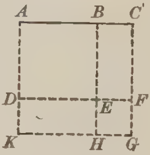
20. *If the sides of a triangle are* 7, 9, *and* 11, *is the angle opposite the side* 11 *right, acute, or obtuse?*

Now $11^2 < 7^2 + 9^2$,

for $121 < 49 + 81$.

Therefore the ∠ opposite the side 11 is acute. § 341

21. *The square constructed upon the sum of two lines is equivalent to the sum of the squares constructed upon these two lines, increased by twice the rectangle of these lines.*



Given the lines AB and BC and their sum AC.

To prove that $\overline{AC}^2 = \overline{AB}^2 + \overline{BC}^2 + 2\,AB \times BC$.

Proof. Construct the square $AKGC$ upon AC, and the square $ADEB$ upon AB.

Produce BE and DE to meet KG and CG at H and F respectively.

Now AK, BH, and CG are ∥ lines,

and AC, DF, and KG are ∥ lines. § 95

∴ the sides of the square $EHGF$ are each equal to BC, and the sides of the rectangles $DKHE$ and $BEFC$ are equal to AB and BC. § 134

∴ square $AKGC$ = square $ADEB$ + square $EHGF$ + rectangle $DKHE$ + rectangle $BEFC$.

Now square $AKGC = \overline{AC}^2$,

square $ADEB = \overline{AB}^2$,

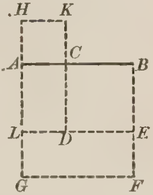
square $EHGF = \overline{BC}^2$,

rectangle $DKHE = AB \times BC$,

and rectangle $BEFC = AB \times BC$. § 320

∴ $\overline{AC}^2 = \overline{AB}^2 + \overline{BC}^2 + 2\,AB \times BC$, by Ax. 9. Q. E. D.

22. *The square constructed upon the difference of two lines is equivalent to the sum of the squares constructed upon these two lines, diminished by twice their rectangle.*



Given the lines AB and AC, and BC their difference.

To prove that $\overline{BC}^2 = \overline{AB}^2 + \overline{AC}^2 - 2\,AB \times AC$.

Proof. Construct the square $AGFB$ upon AB, the square $ACKH$ upon AC, and the square $CDEB$ upon BC.

Produce ED to meet AG in L.

The dimensions of the rectangles $LGFE$ and $HLDK$ are AB and AC.

Now the square $CDEB$ is the difference between the whole figure and the sum of these rectangles.

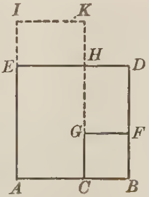
That is, square $CDEB$ = square $AGFB$ + square $ACKH$ − rectangle $LGFE$ − rectangle $HLDK$.

Now square $CDEB = \overline{BC}^2$,
square $AGFB = \overline{AB}^2$,
square $ACKH = \overline{AC}^2$,
rectangle $LGFE = AB \times AC$,
and rectangle $HLDK = AB \times AC$. § 320

$\therefore \overline{BC}^2 = \overline{AB}^2 + \overline{AC}^2 - 2\,AB \times AC$, by Ax. 9. Q. E. D.

23. *The difference between the squares constructed upon two lines is equivalent to the rectangle of the sum and difference of these lines.*

Given the square $ABDE$ constructed upon AB, and the square $CBFG$ constructed upon BC.



To prove that $\overline{AB}^2 - \overline{BC}^2 = (AB + BC)(AB - BC)$.

Proof. Produce CG to meet DE in H.

The difference between the square $ABDE$ and the square $CBFG$ is the polygon $ACGFDE$, which is composed of the rectangles $ACHE$ and $GFDH$.

Produce AE to I and CH to K, making EI and HK each equal to BC, and draw IK.

Then rectangle $EHKI$ = rectangle $GFDH$. § 323

∴ the difference between the squares $ABDE$ and $CBFG$ is equivalent to the rectangle $ACKI$ which has for dimensions AI, equal to $AB + BC$, and AC, equal to $AB - BC$.

Now square $ABDE = \overline{AB}^2$,
square $CBFG = \overline{BC}^2$,
and rectangle $ACKI = (AB + BC)(AB - BC)$. § 320

$\therefore \overline{AB}^2 - \overline{BC}^2 = (AB + BC)(AB - BC)$, by Ax. 9. Q. E. D.

Exercise 55. Page 223

1. *The sides of a triangle are* 0.7 *in.*, 0.6 *in.*, *and* 0.7 *in. respectively. Is the largest angle acute, right, or obtuse?*

The largest ∠s are the base ∠s opposite the equal sides. § 113
The largest ∠s cannot be right or obtuse. § 109
Therefore the largest ∠ is acute.

2. *The sides of a triangle are* 5.1 *in.*, 6.8 *in.*, *and* 8.5 *in. respectively. Is the largest angle acute, right, or obtuse?*

The largest ∠ is the ∠ opposite the side 8.5 in. § 113
Since $8.5^2 = 5.1^2 + 6.8^2$
the △ is a rt. △, § 337
and the largest ∠ is a rt. ∠. § 109

3. *Find the area of an isosceles triangle whose perimeter is* 14 *in. and base* 4 *in.* (*one decimal place*).

Since the perimeter is 14 in. and base 4 in., one of the equal sides $= \frac{1}{2}$ of (14 in. – 4 in.) = 5 in.

$\therefore a^2 = 5^2 - 2^2.$ § 338
$\therefore a = \sqrt{5^2 - 2^2}$ in. $= \sqrt{25 - 4}$ in. $= \sqrt{21}$ in. Ax. 5
Area of $\triangle = \frac{1}{2} ab$ § 325
$= \frac{1}{2}\sqrt{21} \times 4$ sq. in.
$= 2\sqrt{21}$ sq. in.
$= 2 \times 4.58$ sq. in.
$= 9.16$ sq. in. 9.2 sq. in. *Ans.*

4. *Find the area of an equilateral triangle whose perimeter is* 18 *in.* (*one decimal place*).

Here one side $= \frac{1}{3}$ of 18 in. = 6 in.

$a^2 = 6^2 - 3^2.$ § 338
$\therefore a = \sqrt{6^2 - 3^2}$ in. $= \sqrt{36 - 9}$ in. $= \sqrt{27}$ in. Ax. 5
Area of $\triangle = \frac{1}{2} ab$ § 325
$= \frac{1}{2}\sqrt{27} \times 6$ sq. in.
$= 3\sqrt{27}$ sq. in.
$= 3 \times 5.20$ sq. in.
$= 15.60$ sq. in. 15.6 sq. in. *Ans.*

5. *Find the area of a right triangle, the hypotenuse being* 1.7 *in. and one of the other sides being* 0.8 *in.*

The square on the third side $= 1.7^2 - 0.8^2$. § 338

$$\begin{aligned}\therefore \text{ the third side} &= \sqrt{1.7^2 - 0.8^2}\text{ in.}\\ &= \sqrt{2.89 - 0.64}\text{ in.}\\ &= \sqrt{2.25}\text{ in.}\\ &= 1.5\text{ in.}\end{aligned}$$

$$\begin{aligned}\text{Area of } \triangle &= \tfrac{1}{2}\,ab = \tfrac{1}{2} \times 0.8 \times 1.5\text{ sq. in.}\\ &= 0.6\text{ sq. in.}\end{aligned}$$ § 325

6. *Find the ratio of the altitudes of two triangles of equal area, the base of one being* 1.5 *in. and that of the other* 4.5 *in.*

Now $$\frac{\text{area of 1st }\triangle}{\text{area of 2d }\triangle} = 1 = \frac{a \times 1.5}{a' \times 4.5}.$$ § 327

$$\therefore\ \frac{a}{a'} = \frac{4.5}{1.5} = \frac{3}{1} = 3.$$

7. *The bases of a trapezoid are* 34 *in. and* 30 *in., and the altitude is* 2 *in. Find the side of a square having the same area.*

$$\begin{aligned}\text{Area of trapezoid} &= \tfrac{1}{2}\,a\,(b + b')\\ &= \tfrac{1}{2} \times 2\,(34 + 30)\text{ sq. in.}\\ &= 1 \times 64\text{ sq. in.}\\ &= 64\text{ sq. in.}\end{aligned}$$ § 329

$$\therefore \text{ one side of equivalent square} = \sqrt{64}\text{ in.} = 8\text{ in.}$$

8. *What is the area of the isosceles right triangle in which the hypotenuse is* $\sqrt{2}$.

Here $$a = b.$$

$$\therefore\ h^2 = a^2 + a^2.$$ § 337

$$\begin{aligned}(\sqrt{2})^2 &= 2\,a^2.\\ 2\,a^2 &= 2.\\ a^2 &= 1.\\ \therefore\ a &= 1.\end{aligned}$$

$$\begin{aligned}\text{Area of } \triangle &= \tfrac{1}{2}\,ab\\ &= \tfrac{1}{2} \times 1 \times 1\\ &= 0.5.\end{aligned}$$ § 325

9. *What is the area of the isosceles right triangle in which the hypotenuse is* $7\sqrt{2}$.

Here

$$a = b.$$

$$\therefore h^2 = a^2 + a^2. \qquad \text{§ 337}$$

$$(7\sqrt{2})^2 = 2a^2.$$

$$2a^2 = 98.$$

$$a^2 = 49.$$

$$\therefore a = 7.$$

$$\text{Area of } \triangle = \tfrac{1}{2}ab \qquad \text{§ 325}$$

$$= \tfrac{1}{2} \times 7 \times 7$$

$$= 24.5.$$

10. *If the side of an equilateral triangle is* $2\sqrt{3}$, *what is the altitude of the triangle? the area of the triangle?*

$$\text{Square on altitude} = (2\sqrt{3})^2 - (\sqrt{3})^2. \qquad \text{§ 338}$$

$$\therefore \text{altitude} = \sqrt{(2\sqrt{3})^2 - (\sqrt{3})^2} = \sqrt{12 - 3} = \sqrt{9} = 3.$$

$$\text{Area of } \triangle = \tfrac{1}{2}ab \qquad \text{§ 325}$$

$$= \tfrac{1}{2} \times 3 \times 2\sqrt{3} = 3\sqrt{3}.$$

11. *If the side of an equilateral triangle is* 1 *ft., what is the area of the triangle?*

$$\text{Square on altitude} = 1^2 - (\tfrac{1}{2})^2. \qquad \text{§ 338}$$

$$\therefore \text{altitude} = \sqrt{1 - \tfrac{1}{4}} \text{ ft.} = \sqrt{\tfrac{3}{4}} \text{ ft.} = \tfrac{1}{2}\sqrt{3} \text{ ft.}$$

$$\text{Area of } \triangle = \tfrac{1}{2}ab \qquad \text{§ 325}$$

$$= \tfrac{1}{2} \times \tfrac{1}{2}\sqrt{3} \times 1 \text{ sq. ft.} = \tfrac{1}{4}\sqrt{3} \text{ sq. ft.}$$

12. *If the area of an equilateral triangle is* 43.3 *sq. in., what is the base of the triangle?* (*Take* $\sqrt{3} = 1.732$.)

$$\text{Square on altitude} = s^2 - (\tfrac{1}{2}s)^2. \qquad \text{§ 338}$$

$$\therefore \text{altitude} = \sqrt{s^2 - \left(\frac{1}{2}s\right)^2} = \sqrt{\frac{3s^2}{4}} = \frac{1}{2}s\sqrt{3}.$$

$$\text{Area of } \triangle = \tfrac{1}{2}ab. \qquad \text{§ 325}$$

$$\therefore 43.3 = \tfrac{1}{2} \times \tfrac{1}{2}s\sqrt{3} \times s = \tfrac{1}{4}s^2\sqrt{3}.$$

$$\therefore 43.3 = \tfrac{1}{4}s^2 \times 1.732.$$

$$43.3 = 0.433\,s^2.$$

$$\therefore s = 10.$$

13. *The sides of a triangle are* 2.8 *in.,* 3.5 *in., and* 2.1 *in. respectively. Draw the figure carefully and see what kind of a triangle it is. Verify this conclusion by applying a geometric test, and find the area of the triangle.*

Draw a line AB equal to 2.8 in.

With A as a center and a radius equal to 3.5 in. describe an arc, and with B as a center and a radius equal to 2.1 in. describe an arc intersecting the first arc at C.

Then ABC is the $\triangle$ required.

The $\triangle ABC$ is evidently a rt. $\triangle$,

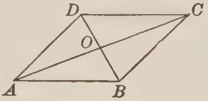
for $3.5^2 = 2.8^2 + 2.1^2$. § 337

Area of $\triangle ABC = \frac{1}{2} \times 2.8 \times 2.1$ sq. in. § 325

$= 2.94$ sq. in.

Exercise 56. Page 224

1. *The area of a rhombus is equal to half the product of its diagonals.*



Given the rhombus $ABCD$, with its diagonals AC and BD intersecting at O.

To prove that area of $ABCD = \frac{1}{2}(AC \times BD)$.

Proof. In the $\triangle$s AOD and COD,

$AD = CD$, § 119

$OD = OD$, Iden.

and $AO = CO$. § 131

$\therefore$ the $\triangle AOD$ is congruent to $\triangle COD$. § 80

$\therefore \angle DOA = \angle COD$. § 67

$\therefore \angle DOA = 1$ rt. $\angle$. § 26

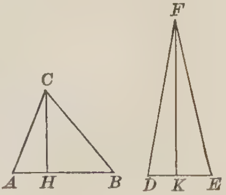
Now area of $\triangle ABD = \frac{1}{2} BD \times AO$,

and area of $\triangle BCD = \frac{1}{2} BD \times CO$. § 325

$\therefore \triangle ABD + \triangle BCD = \frac{1}{2} BD \times (AO + CO)$. Ax. 1

That is, area of $ABCD = \frac{1}{2} BD \times AC$, by Ax. 9. Q. E. D.

2. *Two triangles are equivalent if the base of the first is equal to half the altitude of the second, and the altitude of the first is equal to twice the base of the second.*



Given the $\triangle$s ABC and DEF with the altitudes CH and FK; $AB = \frac{1}{2} FK$, and $CH = 2\,DE$.

To prove that $\triangle$s ABC and DEF are equivalent.

Proof. Area of $\triangle ABC = \frac{1}{2} AB \times CH$,

and area of $\triangle DEF = \frac{1}{2} DE \times FK$. § 325

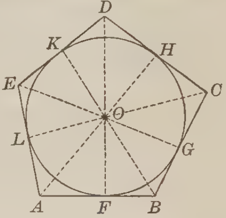
But $AB = \frac{1}{2} FK$ and $CH = 2\,DE$. Given

$\therefore$ area of $\triangle ABC = \frac{1}{2} \times \frac{1}{2} FK \times 2\,DE$ Ax. 9

$= \frac{1}{2} DE \times FK$.

$\therefore$ area of $\triangle ABC$ = area of $\triangle DEF$, by Ax. 8. Q. E. D.

3. *The area of a circumscribed polygon is equal to half the product of its perimeter by the radius of the inscribed circle.*



Given the polygon $ABCDE$, with perimeter p, circumscribed about the ⊙ with radius OF.

To prove that area of $ABCDE = \frac{1}{2}p \times OF$.

Proof. Draw the radii OF, OG, OH, OK, OL to the points of contact, and draw OA, OB, OC, OD, OE.

Now $OF = OG = OH = OK = OL$. § 162

Area of $\triangle ABO = \frac{1}{2} AB \times OF$. § 325

Likewise the area of each of the ⊿ BCO, CDO, DEO, and EAO is equal to half the product of its base by the radius of the inscribed ⊙.

But the sum of these five triangles is the polygon, and the sum of the bases is the perimeter of the polygon. Ax. 11.

∴ area of $ABCDE = \frac{1}{2}p \times OF$, by Ax. 9. Q.E.D.

4. *Two parallelograms are equivalent if their altitudes are reciprocally proportional to their bases.*

Given the ▱ P, with base b and altitude a, and the ▱ P', with base b' and altitude a', such that $a : a' = b' : b$.

To prove that $P = P'$.

Proof. $P = ab$,

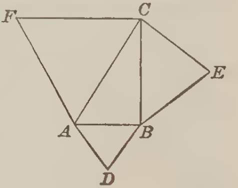
and $P' = a'b'$. § 322

Since $a : a' = b' : b$, Given

$ab = a'b'$. § 261

∴ $P = P'$, by Ax. 9. Q.E.D.

5. *If equilateral triangles are constructed on the sides of a right triangle, the triangle on the hypotenuse is equivalent to the sum of the triangles on the other two sides.*



Given the rt. $\triangle ABC$, with ACF the equilateral △ constructed upon the hypotenuse, and ADB and BEC the equilateral ⊿ constructed upon the other two sides.

To prove that $\triangle ACF = \triangle ADB + \triangle BEC$.

Proof. Now ⊿ ACF, ADB, and BEC are similar. § 285

$$\therefore \frac{\triangle ADB}{\triangle ACF} = \frac{\overline{AB}^2}{\overline{AC}^2},$$

and
$$\frac{\triangle BEC}{\triangle ACF} = \frac{\overline{BC}^2}{\overline{AC}^2}. \quad \text{§ 333}$$

$\therefore \dfrac{\triangle ADB + \triangle BEC}{\triangle ACF} = \dfrac{\overline{AB}^2 + \overline{BC}^2}{\overline{AC}^2}$. Ax. 1

But $\overline{AB}^2 + \overline{BC}^2 = \overline{AC}^2$. § 337

$\therefore \triangle ACF = \triangle ADB + \triangle BEC$, by § 263. Q.E.D.

6. *If similar polygons are constructed on the sides of a right triangle as corresponding sides, the polygon on the hypotenuse is equivalent to the sum of the polygons on the other two sides.*

Given the rt. $\triangle ABC$, with P a polygon constructed upon the hypotenuse AC, and P' and P'' polygons similar to P and constructed upon AB and BC respectively.

To prove that $P = P' + P''$.

Proof. $\dfrac{P'}{P} = \dfrac{\overline{AB}^2}{\overline{AC}^2}$,

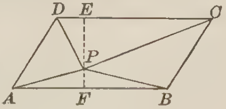
and $\dfrac{P''}{P} = \dfrac{\overline{BC}^2}{\overline{AC}^2}$. § 333

$\therefore \dfrac{P' + P''}{P} = \dfrac{\overline{AB}^2 + \overline{BC}^2}{\overline{AC}^2}$. Ax. 1

But $\overline{AB}^2 + \overline{BC}^2 = \overline{AC}^2$. § 337

$\therefore P = P' + P''$, by § 263. Q.E.D.

7. *If lines are drawn from any point within a parallelogram to the four vertices, the sum of either pair of triangles with parallel bases is equivalent to the sum of the other pair.*



Given the $\square ABCD$, with PA, PB, PC, and PD lines drawn to the four vertices from any point within the $\square$.

To prove that $\triangle ABP + \triangle CDP = \triangle BCP + \triangle DAP$.

Proof. Through P draw $EF \perp$ to AB.

$\therefore EF$ is $\perp$ to DC also. § 97

Then area of $\triangle ABP = \frac{1}{2}(AB \times PF)$,

and area of $\triangle CDP = \frac{1}{2}(DC \times PE)$. § 325

$\therefore \triangle ABP + \triangle CDP = \frac{1}{2}(AB \times PF) + \frac{1}{2}(DC \times PE)$. Ax. 1

But $AB = DC$, § 125

and $PF + PE = EF$. Ax. 11

$\therefore \triangle ABP + \triangle CDP = \frac{1}{2}(AB \times EF)$. Ax. 9

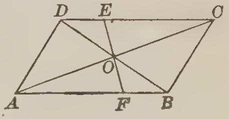
But $\square ABCD = AB \times EF$. § 322

$\therefore \triangle ABP + \triangle CDP = \frac{1}{2} \square ABCD$.

$\therefore \triangle BCP + \triangle DAP = \frac{1}{2} \square ABCD$. Ax. 2

$\therefore \triangle ABP + \triangle CDP = \triangle BCP + \triangle DAP$, by Ax. 4. Q.E.D.

8. *Every line drawn through the intersection of the diagonals of a parallelogram bisects the parallelogram.*



Given the line EF drawn through the intersection O of the diagonals AC and BD of the ▱ $ABCD$.

To prove that $AFED = CEFB$.

Proof. In the $\triangle$s AFO and CEO,

$AO = CO$, § 131

$\angle FAO = \angle ECO$, § 100

and $\angle AOF = \angle COE$. § 60

$\therefore \triangle AFO$ is congruent to $\triangle CEO$. § 72

Similarly $\triangle DOE$ is congruent to $\triangle BOF$.

In the $\triangle$s AOD and COB,

$AO = CO$,

and $DO = BO$, § 131

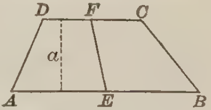
and $\angle DOA = \angle BOC$. § 60

$\therefore \triangle AOD$ is congruent to $\triangle COB$. § 68

$\therefore \triangle AFO + \triangle DOE + \triangle AOD = \triangle CEO + \triangle BOF + \triangle COB$. Ax. 1

That is, $AFED = CEFB$, by Ax. 9. Q. E. D.

9. *The line that bisects the bases of a trapezoid divides the trapezoid into two equivalent parts.*



Given the trapezoid $ABCD$ and the line EF drawn through the mid-points of the bases AB and CD.

To prove that $AEFD$ and $EBCF$ are equivalent.

Proof. Let a represent the altitude to the given trapezoid.

Then area of trapezoid $AEFD = \frac{1}{2} a (AE + DF)$,

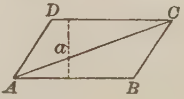
and area of trapezoid $EBCF = \frac{1}{2} a (EB + FC)$. § 329

But $AE = EB$, and $DF = FC$. Given

$\therefore AE + DF = EB + FC$. Ax. 1

$\therefore$ trapezoids $AEFD$ and $EBCF$ are equivalent, by Ax. 8. Q. E. D.

10. *If a quadrilateral with two sides parallel is bisected by either diagonal, the quadrilateral is a parallelogram.*



Given the quadrilateral $ABCD$, with the ∥ sides AB and DC, and with the diagonal AC bisecting the quadrilateral.

To prove that $ABCD$ is a ▱.

Proof. The given quadrilateral is a trapezoid. § 118

Let the line a represent the altitude of the trapezoid $ABCD$.

Now area of trapezoid $ABCD = \frac{1}{2} a (AB + DC)$, § 329

area of $\triangle ABC = \frac{1}{2} a \times AB$,

and area of $\triangle ACD = \frac{1}{2} a \times DC$. § 325

But area of $\triangle ABC =$ area of $\triangle ACD$. Given

$\therefore \frac{1}{2} a \times AB = \frac{1}{2} a \times DC$. Ax. 8

$\therefore AB = DC$. Ax. 4

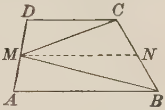
Also AB is $\parallel$ to DC. Given

$\therefore ABCD$ is a ▱, by § 130. Q. E. D.

11. *The triangle formed by two lines drawn from the mid-point of either of the nonparallel sides of a trapezoid to the opposite vertices is equivalent to half the trapezoid.*

Given the trapezoid $ABCD$ and the lines MB and MC drawn from M, the mid-point of AD, to the opposite vertices B and C.

To prove that $\triangle MBC = \frac{1}{2}$ trapezoid $ABCD$.



Proof. Let a represent the altitude of the given trapezoid.

From M draw MN to N, the mid-point of BC.

Then MN is $\parallel$ to AB and DC. § 137

Now area of $\triangle MBN = \frac{1}{2} \times \frac{1}{2} a \times MN$,

and area of $\triangle ABM = \frac{1}{2} \times \frac{1}{2} a \times AB$. § 325

Also area of $\triangle MNC = \frac{1}{2} \times \frac{1}{2} a \times MN$,

and area of $\triangle MCD = \frac{1}{2} \times \frac{1}{2} a \times DC$. § 325

But $MN = \frac{1}{2}(AB + DC)$. § 137

$\therefore \triangle MBN + \triangle MNC = \triangle ABM + \triangle MCD$. Ax. 1

That is, $\triangle MBC = \frac{1}{2}$ trapezoid $ABCD$. Q. E. D.

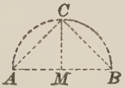
Exercise 57. Page 225

1. *Given a square, to construct a square of half its area.*

Given the square S.

Required to construct a square equivalent to $\frac{1}{2} S$.

Construction. Draw a line AB equal to a side of S.

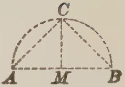


On AB as a diameter describe a semicircle.

At M, the mid-point of AB, erect a $\perp$ cutting the ⊙ at C. Draw CA.

Construct the square S' having a side equal to AC.

Then S' is the square required. Q. E. F.



Proof. Draw BC.

$\angle ACB = 1$ rt. $\angle$. § 215

$\therefore \overline{AC}^2 + \overline{BC}^2 = \overline{AB}^2$. § 337

But $AC = BC$. § 150

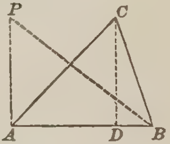
$\therefore \overline{AC}^2 = \overline{BC}^2$. Ax. 5

$\therefore 2\,\overline{AC}^2 = \overline{AB}^2$. Ax. 9

$\therefore \overline{AC}^2 = \frac{1}{2}\overline{AB}^2$. Ax. 4

$\therefore S' = \frac{1}{2}S$. Q.E.D.

2. *To construct a right triangle equivalent to a given oblique triangle.*



Given the oblique $\triangle ABC$.

Required to construct a right $\triangle$ equivalent to $\triangle ABC$.

Construction. Through the vertex C draw CD, the altitude of the $\triangle ABC$.

At A erect the $\perp$ AP to AB and equal to CD, and draw PB.

Then the rt. $\triangle ABP$ is the rt. $\triangle$ required. Q.E.F.

Proof. Rt. $\triangle ABP = \triangle ABC$, by § 326. Q.E.D.

3. *To construct a triangle equivalent to the sum of two given triangles.*

Given the $\triangle$s ABC and DEF, with bases b and b' and altitudes a and a' respectively.

Required to construct a $\triangle$ equivalent to the sum of $\triangle$s ABC and DEF.

Construction. Find m, the mean proportional between b and $\frac{1}{2}a$, and m', the mean proportional between b' and $\frac{1}{2}a'$. § 309

Find c so that $c^2 = m^2 + m'^2$. § 344

Then any $\triangle$, as T, having base $2c$ and altitude c is the $\triangle$ required. Q.E.F.

Proof. Now area of $\triangle ABC = \frac{1}{2}ab$,

and area of $\triangle DEF = \frac{1}{2}a'b'$. § 325

Also area of $T = \frac{1}{2}(c \times 2c)$ § 325

$= c^2$.

But $c^2 = m^2 + m'^2$, Const.

and $m^2 = \frac{1}{2}ab$, and $m'^2 = \frac{1}{2}a'b'$. Const.

$\therefore c^2 = \frac{1}{2}ab + \frac{1}{2}a'b'$. Ax. 9

$\therefore T = \triangle ABC + \triangle DEF$, by Ax. 8. Q.E.D.

4. *To construct a triangle equivalent to a given triangle, and having one side equal to a given line.*

Given the $\triangle ABC$ and the line l.

Required to construct a $\triangle$ equivalent to $\triangle ABC$, and having one side equal to l.

Construction. Draw a, the altitude of the $\triangle ABC$, and denote AB by b.

Find a', the fourth proportional to l, b, and a. § 307

Then any $\triangle$, as T, having base l and altitude a' is the $\triangle$ required. Q.E.F.

Proof. Now area of $\triangle ABC = \frac{1}{2}ab$,

and area of $\triangle T = \frac{1}{2}a'l$. § 325

But $l : b = a : a'$. Const.

$\therefore a'l = ab$. § 261

$\therefore \frac{1}{2}a'l = \frac{1}{2}ab$. Ax. 4

$\therefore T = \triangle ABC$, by Ax. 8. Q.E.D.

5. *To construct a rectangle equivalent to a given parallelogram, and having its altitude equal to a given line.*

Given the $\square$ $ABCD$ and the line l.

Required to construct a rectangle equivalent to $ABCD$ and having altitude l.

Construction. Draw a, the altitude of the $\square ABCD$, and denote AB by b.

Find b', the fourth proportional to l, a, and b. § 307

Construct the rectangle R with base b' and altitude l.

Then R is the rectangle required. Q.E.F.

Proof. Now area of $\square ABCD = ab$, § 322

and area of rectangle $R = lb'$. § 320

But $l : a = b : b'$. Const.

$\therefore lb' = ab$. § 261

$\therefore$ rectangle $R = \square ABCD$, by Ax. 8. Q.E.D.

6. *To construct a right triangle equivalent to a given triangle, and having one of the sides of the right angle equal to a given line.*

Given the $\triangle ABC$ and the line l.

Required to construct a rt. $\triangle$ equivalent to the $\triangle ABC$, and having one of the sides of the rt. $\angle$ equal to l.

Construction. Draw a, the altitude of the $\triangle ABC$, and denote AB by b.

Find a', the fourth proportional to l, b, and a. § 307

At either extremity of l erect a $\perp$ to l and equal to a'.

Connect the extremity of this $\perp$ with the other extremity of l, thus forming the rt. $\triangle T$.

Then T is the rt. $\triangle$ required. Q. E. F.

Proof. The $\triangle T$ is a rt. $\triangle$. Const.

Now area of $\triangle ABC = \frac{1}{2} ab$,

and area of $\triangle T = \frac{1}{2} a'l$. § 325

But $l : b = a : a'$. Const.

$\therefore\ a'l = ab$. § 261

$\therefore\ \frac{1}{2} a'l = \frac{1}{2} ab$. Ax. 4

$\therefore$ rt. $\triangle T = \triangle ABC$, by Ax. 8. Q. E. D.

7. *To construct a right triangle equivalent to a given triangle, and having its hypotenuse equal to a given line.*

Given the $\triangle ABC$ and the line l.

Required to construct a rt. $\triangle$ equivalent to $\triangle ABC$ and having the hypotenuse equal to l.

Construction. Draw a, the altitude of the $\triangle ABC$, and denote AB by b.

Find a', the fourth proportional to l, b, and a. § 307

On l as a diameter describe a semicircle.

Draw a line $\parallel$ to l and at a distance a' from it, cutting the semicircle in the points C' and C''. §§ 228, 233

Connect either C' or C'' with the extremities of l, thus forming the $\triangle T$.

Then T is the rt. $\triangle$ required. Q. E. F.

Proof. The $\angle C' = 1$ rt. $\angle$,

and also $\angle C'' = 1$ rt. $\angle$. § 215

$\therefore\ \triangle T$ is a rt. $\triangle$. § 63

Now area of $\triangle ABC = \frac{1}{2} ab$,

and area of $\triangle T = \frac{1}{2} a'l$. § 325

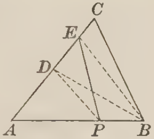
But $l : b = a : a'$. Const.

$\therefore\ a'l = ab$. § 261

$\therefore\ \frac{1}{2} a'l = \frac{1}{2} ab$. Ax. 4

$\therefore$ rt. $\triangle T = \triangle ABC$, by Ax. 8. Q. E. D.

8. *To divide a given triangle into two equivalent parts by a line through a given point P in the base.*



Given the $\triangle ABC$, and P a point in the base AB.

Required to draw through P, a line that shall divide the $\triangle ABC$ into two equivalent parts.

Construction. Draw BD to the mid-point of AC, and draw PD.

Draw $BE \parallel$ to PD, cutting AC in E.

Draw PE.

Then PE is the line required. Q. E. F.

Proof. The ⟁ DAB and CDB have equal bases AD and DC, and the same altitude, and are therefore equivalent. § 326

$\therefore \triangle DAB = \frac{1}{2} \triangle ABC$. § 327

In the ⟁ APE and DAB,

the $\triangle ADP$ is common,

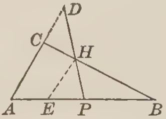
and $\triangle DPE = \triangle DPB$, § 326

for they have the same base DP and equal altitudes, the $\perp$ between the $\parallel$ lines DP and EB.

$\therefore \triangle APE = \triangle DAB$.

$\therefore \triangle APE = \frac{1}{2} \triangle ABC$, by Ax. 9. Q. E. D.

9. *To draw from a given point P in the base AB of a triangle ABC, a line to AC produced, so that it may be bisected by BC.*



Given the $\triangle ABC$, and P a point in AB.

Required to draw a line from P to AC produced so that it may be bisected by BC.

Construction. Bisect AP in E. § 229

Draw $EH \parallel$ to AC, cutting BC in H. § 233

Draw PH, and produce PH to meet AC produced in D.

Then PD is the line required. Q. E. F.

Proof. Since $PE = EA$, Const.

and EH is $\parallel$ to AD, Const.

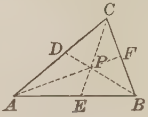
$\therefore PH = HD$. § 135

That is, PD is bisected by BC. Q. E. D.

10. *To find a point within a given triangle such that the lines from this point to the vertices shall divide the triangle into three equivalent triangles.*

Given the $\triangle ABC$.

Required to find a point within the $\triangle ABC$ such that the lines from this point to the vertices shall divide the $\triangle$ into three equivalent $\triangle$.



Construction. Draw the medians AF and BD, intersecting at the point P.

Then P is the point required. Q. E. F.

Proof. The third median CE passes through the point P. Ex. 5, p. 78

$$\triangle AEC = \tfrac{1}{2}\triangle ABC,$$

and $$\triangle ABD = \tfrac{1}{2}\triangle ABC. \qquad \S\ 327$$

$$\therefore \triangle AEC = \triangle ABD. \qquad \text{Ax. } 8$$

Subtract from each $\triangle$ the common quadrilateral $AEPD$.

Then $$\triangle EBP = \triangle DPC. \qquad \text{Ax. } 2$$

Also $$\triangle AEP = \triangle EBP,$$

and $$\triangle APD = \triangle DPC. \qquad \S\ 326$$

$$\therefore \triangle AEP = \triangle APD. \qquad \text{Ax. } 8$$

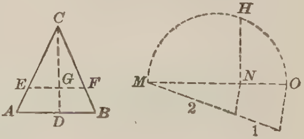
$$\therefore \triangle AEP + \triangle EBP = \triangle APD + \triangle DPC. \qquad \text{Ax. } 1$$

That is, $$\triangle ABP = \triangle APC.$$

Similarly $$\triangle ABP = \triangle PBC.$$

$\therefore \triangle ABP = \triangle APC = \triangle PBC$, by Ax. 8. Q. E. D.

11. *To divide a given triangle into two equivalent parts by a line parallel to one of the sides.*



Given the $\triangle ABC$.

Required to divide the $\triangle ABC$ into equivalent parts by a line $\parallel$ to AB.

Construction. Draw CD, the altitude of the $\triangle ABC$.

Draw MN equal to CD.

Find NH so that $\overline{MN}^2 : \overline{NH}^2 = 2:1$. § 355

On CD take CG equal to NH, and through G draw $EF \parallel$ to AB.

Then EF is the line required. Q. E. F.

Proof. Since EF is $\parallel$ to AB, Const.

the $\triangle ABC$ and EFC are mutually equiangular.

$\therefore$ the $\triangle ABC$ and EFC are similar. § 285

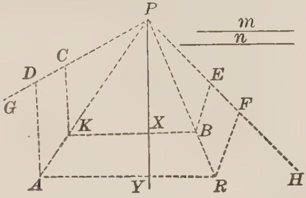
$$\therefore \triangle ABC : \triangle EFC = \overline{CD}^2 : \overline{CG}^2. \qquad \S\ 335$$

$$\therefore \triangle ABC : \triangle EFC = \overline{MN}^2 : \overline{NH}^2. \qquad \text{Ax. } 9$$

But $$\overline{MN}^2 : \overline{NH}^2 = 2 : 1. \qquad \text{Const.}$$

$\therefore \triangle ABC : \triangle EFC = 2 : 1$, by Ax. 9. Q. E. D.

12. *Through a given point to draw a line so that the segments intercepted between the point and perpendiculars drawn to the line from two other given points may have a given ratio.*



Given the points P, A, and B not in the same straight line, and the lines m and n.

Required to draw through P a line so that the segments intercepted between P and the ⊥s from B and A to it have the given ratio $m : n$.

Construction. Draw PA and PB.

Then draw PG, making any convenient ∠ with PA; and PH, making any convenient ∠ with PB.

Suppose $PA > PB$.

On PG lay off $PD = n$, and $PC = m$.

Then draw DA.

Also draw CK ∥ to DA, meeting PA in K. § 233

On PH lay off $PE = m$, and $PF = n$.

Then draw EB.

Also draw FR ∥ to EB, meeting PB or PB produced in R. § 233

Finally, draw KB and AR.

From P let fall the ⊥ PY to AR, cutting KB and AR in X and Y respectively. § 227

Then PY is the line required. Q.E.F.

Proof. $$\frac{PK}{PA} = \frac{PC}{PD}.$$ § 273

But $$\frac{PC}{PD} = \frac{m}{n}.$$ Const.

$$\therefore \frac{PK}{PA} = \frac{m}{n}.$$ Ax. 8

Likewise $$\frac{PB}{PR} = \frac{m}{n}.$$

$$\therefore \frac{PK}{PA} = \frac{PB}{PR}.$$ Ax. 8

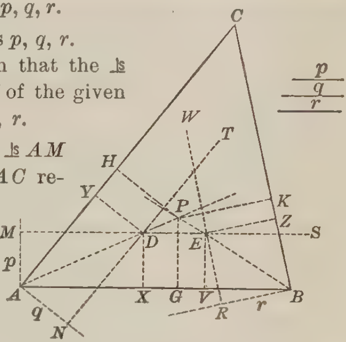
∴ KB is ∥ to AR. § 276

∴ KB is ⊥ to PY. § 97

Furthermore $$\frac{PX}{PY} = \frac{PK}{PA}.$$ § 273

$$\therefore \frac{PX}{PY} = \frac{m}{n},$$ by Ax. 8. Q.E.D.

13. *To find a point such that the perpendiculars from it to the sides of a given triangle shall be in the ratio p, q, r.*



Given the $\triangle ABC$ and the lines p, q, r.

Required to find a point such that the $\perp$s from it to the sides AB, AC, BC of the given $\triangle ABC$ shall be in the ratio p, q, r.

Construction. At A erect the $\perp$s AM and AN to the sides AB and AC respectively, making $AM = p$, and $AN = q$. § 228

At B erect the $\perp$ BR to the side BC, making $BR = r$. § 228

Through M, N, and R draw the lines MS, NT, and $RW \parallel$ to AB, AC, and BC respectively. § 233

Let MS and NT intersect at D, and MS and RW at E.

Draw AD and BE.

Produce AD and BE to intersect at P.

Then P is the point required. Q.E.F.

Proof. Draw PG, PH, and $PK \perp$ to AB, AC, and BC respectively.

Also draw DX and $DY \perp$ to AB and AC respectively, and EV and $EZ \perp$ to AB and BC respectively. § 227

The rt. $\triangle$s AGP and AXD are similar. § 286

$$\therefore \frac{PG}{DX} = \frac{PA}{DA}. \qquad \text{§ 282}$$

Also the rt. $\triangle$s AHP and AYD are similar. § 286

$$\therefore \frac{PH}{DY} = \frac{PA}{DA}. \qquad \text{§ 282}$$

$$\therefore \frac{PG}{DX} = \frac{PH}{DY}. \qquad \text{Ax. 8}$$

Similarly $$\frac{PG}{EV} = \frac{PK}{EZ}. \qquad \text{Ax. 8}$$

Since $$DX = EV, \qquad \text{§ 128}$$

$$\therefore \frac{PG}{DX} = \frac{PK}{EZ} = \frac{PH}{DY}. \qquad \text{Ax. 8}$$

But $DX = p$,

$DY = q$,

and $EZ = r$. § 128

$$\therefore \frac{PG}{p} = \frac{PH}{q} = \frac{PK}{r},$$ by Ax. 9. Q.E.D.

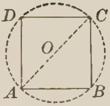
BOOK V

Exercise 59. Page 232

1. *Find the radius of the square whose side is 5 in.*

Given the square $ABCD$ with $AB = 5$ in.

Required to find OA.



Solution.

$$OA = \tfrac{1}{2} AC. \qquad \text{§ 131}$$
$$\overline{AC}^2 = \overline{AB}^2 + \overline{BC}^2. \qquad \text{§ 337}$$
$$\therefore\ AC = \sqrt{\overline{AB}^2 + \overline{BC}^2} \qquad \text{Ax. 5}$$
$$= \sqrt{5^2 + 5^2}\text{ in.}$$
$$= \sqrt{50}\text{ in.}$$
$$= 7.071\text{ in.}$$
$$\therefore\ OA = \tfrac{1}{2}\text{ of } 7.071\text{ in.} = 3.54\text{ in.}$$

2. *Find the side of the square whose radius is 7 in.*

Given in the square $ABCD$ the radius $OA = 7$ in.

Required to find AB.

Solution.

$$\overline{AC}^2 = \overline{AB}^2 + \overline{BC}^2. \qquad \text{§ 337}$$
$$\therefore\ \overline{AC}^2 = 2\,\overline{AB}^2.$$
$$AB = \sqrt{\tfrac{1}{2}\,\overline{AC}^2}.$$

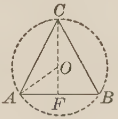
Since

$$AC = 2\,OA, \qquad \text{§ 131}$$
$$\therefore\ AB = \sqrt{2\,\overline{OA}^2}$$
$$= \sqrt{2 \times 7^2} = \sqrt{98}$$
$$= 9.899\text{ in.}$$

3. *Find the radius of the equilateral triangle whose side is 2 in.*

Given the equilateral $\triangle ABC$ with $AB = 2$ in.

Required to find the radius OA.



Solution.

$$OA = \tfrac{2}{3}\,CF. \qquad \text{Ex. 5, p. 78}$$
$$\overline{CF}^2 = \overline{BC}^2 - \overline{FB}^2. \qquad \text{§ 337}$$
$$\therefore\ CF = \sqrt{\overline{BC}^2 - \overline{FB}^2} \qquad \text{Ax. 5}$$
$$= \sqrt{2^2 - 1^2}\text{ in.} = \sqrt{3}\text{ in.} = 1.732\text{ in.}$$
$$\therefore\ OA = \tfrac{2}{3}\text{ of } 1.732\text{ in.} = 1.154\text{ in.}$$

4. *Find the side of the equilateral triangle whose radius is 3 in.*

Given in the equilateral $\triangle ABC$ the radius equal to 3 in.
Required to find the side of the $\triangle$.

Solution. $CF = \frac{3}{2} AO$ Ex. 5, p. 78

$= \frac{3}{2}$ of 3 in. $= 4.5$ in.

$\overline{BC}^2 = \overline{FB}^2 + \overline{CF}^2.$ § 337

$FB = \frac{1}{2} BC.$

$\therefore \overline{BC}^2 = \frac{1}{4} \overline{BC}^2 + \overline{CF}^2.$ Ax. 9

$\frac{3}{4} \overline{BC}^2 = \overline{CF}^2.$

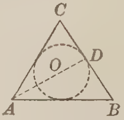
$BC = \frac{2}{3} CF\sqrt{3}$

$= \frac{2}{3}\sqrt{3} \times 4.5 \text{ in.} = 3\sqrt{3} \text{ in.} = 3 \times 1.732 \text{ in.}$

$= 5.196$ in.

5. *Find the apothem of the equilateral triangle whose side is $\sqrt{3}$ in.*

Given the equilateral $\triangle ABC$ with $AB = \sqrt{3}$.
Required to find OD, the apothem.



Solution. $\overline{AD}^2 = \overline{AB}^2 - \overline{BD}^2.$ § 338

$\therefore AD = \sqrt{\overline{AB}^2 - \overline{BD}^2}$ Ax. 5

$= \sqrt{(\sqrt{3})^2 - (\frac{1}{2}\sqrt{3})^2} \text{ in.}$

$= \sqrt{3 - 0.75} \text{ in.}$

$= \sqrt{2.25} \text{ in.}$

$= 1.5$ in.

$OD = \frac{1}{3} AD$ Ex. 5, p. 78

$= \frac{1}{3}$ of 1.5 in. $= 0.5$ in.

6. *Find the side of the equilateral triangle whose apothem is $2\sqrt{3}$ in.*

Given in the equilateral $\triangle ABC$ the apothem equal to $2\sqrt{3}$ in.
Required to find the side of the $\triangle$.

Solution. $AD = 3\, OD.$ Ex. 5, p. 78

$OD = 2\sqrt{3}$ in. Given

$\therefore AD = 3 \times 2\sqrt{3} \text{ in.} = 6\sqrt{3} \text{ in.}$

$\overline{AD}^2 = \overline{AB}^2 - \overline{BD}^2.$ § 338

Since $BD = \frac{1}{2} AB,$

$\overline{AD}^2 = \overline{AB}^2 - \frac{1}{4}\overline{AB}^2 = \frac{3}{4}\overline{AB}^2.$

$(6\sqrt{3})^2 = \frac{3}{4}\overline{AB}^2.$

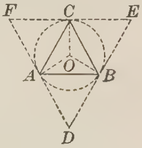
$\therefore \overline{AB}^2 = \frac{4}{3}$ of $(6\sqrt{3})^2 = \frac{4}{3}$ of $108 = 144.$

$\therefore AB = \sqrt{144} \text{ in.} = 12 \text{ in.}$

7. *How many degrees are there in the angle at the center of an equiangular triangle? of a regular hexagon?*

$$A = \tfrac{1}{3} \text{ of } 360^\circ = 120^\circ.$$
$$A = \tfrac{1}{6} \text{ of } 360^\circ = 60^\circ.$$

8. *Given an equilateral triangle inscribed in a circle, to circumscribe an equilateral triangle about the circle.*



Given the equilateral $\triangle ABC$ inscribed in the ⊙.

Required to circumscribe an equilateral △ about the ⊙.

Construction. Draw the radii OA, OB, and OC.

Through A, B, and C draw lines $\perp$ to OA, OB, and OC respectively, and intersecting at the points D, E, and F.

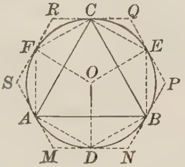
Then $\triangle DEF$ is the △ required. Q.E.F.

Proof. Chord AB = chord BC = chord CA. Given

$\therefore$ arc AB = arc BC = arc CA. § 172

$\therefore$ $\triangle DEF$ is the equilateral circumscribed △, by § 367. Q.E.D.

9. *Given an equilateral triangle inscribed in a circle, to inscribe a regular hexagon in the circle, and to circumscribe a regular hexagon about the circle.*



Given the equilateral $\triangle ABC$ inscribed in the ⊙ with center O.

Required to inscribe a regular hexagon in the ⊙ and to circumscribe a regular hexagon about the ⊙.

Construction. Draw the radii OD, OE, OF $\perp$ to the chords AB, BC, CA respectively.

Draw the lines AD, DB, BE, EC, CF, and FA.

Draw the radii OA, OB, and OC.

Through A, D, B, E, C, and F draw lines $\perp$ to OA, OD, OB, OE, OC, and OF respectively, and intersecting in the points M, N, P, Q, R, S.

Then $ADBECF$ is the regular inscribed hexagon required, and $MNPQRS$ is the regular circumscribed hexagon required. Q.E.F.

Proof. Chord AB = chord BC = chord CA. Given

$\therefore$ arc AB = arc BC = arc CA. § 172

D, E, and F are the mid-points of the arcs AB, BC, and CA respectively. § 174

$\therefore$ $ADBECF$ is a regular inscribed hexagon. § 370

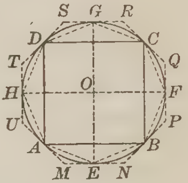
Again, MN, NP, PQ, QR, RS, and SA are tangent to the ⊙. § 184

$\therefore$ $MNPQRS$ is a regular circumscribed hexagon, by § 371. Q.E.D.

10. *Given a square inscribed in a circle, to inscribe a regular octagon in the circle, and to circumscribe a regular octagon about the circle.*

Given the square $ABCD$ inscribed in the ⊙ with center O.

Required to inscribe a regular octagon in the ⊙ and circumscribe a regular octagon about the ⊙.



Construction. Draw the radii OE, OF, OG, and $OH \perp$ to the chords AB, BC, CD, and DA respectively.

Draw the lines AE, EB, BF, FC, CG, GD, DH, HA.

Draw the radii OA, OB, OC, and OD.

Through A, E, B, F, C, G, D, H draw lines ⊥ to OA, OE, OB, OF, OC, OG, OD, OH respectively, and intersecting in the points M, N, P, Q, R, S, T, U.

Then $AEBFCGDH$ is the regular inscribed octagon required and $MNPQRSTU$ is the regular circumscribed octagon required. Q. E. F.

Proof. Chord AB = chord BC = chord CD = chord DA. § 65

∴ arc AB = arc BC = arc CD = arc DA. § 172

∴ E, F, G, H are the mid-points of the arcs AB, BC, CD, DA respectively. § 174

∴ $AEBFCGDH$ is a regular inscribed octagon. § 370

Again, MN, NP, PQ, QR, RS, ST, TU, UM are tangent to the ⊙. § 184

∴ $MNPQRSTU$ is a regular circumscribed octagon, by § 371. Q. E. D.

11. *How many degrees are there in the angle at the center of a regular octagon? in each angle of a regular octagon? in the sum of these two angles?*

Each ∠ at the center $= \frac{1}{8}$ of $360° = 45°$.

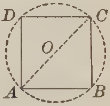
Each ∠ of the regular octagon $= 180° - 45° = 135°$. § 364

The sum of an ∠ at the center and an ∠ of the regular octagon = 180°, by § 364.

12. *What is the area of the square inscribed in a circle of radius 2 in.?*

Given a square inscribed in the ⊙ of radius 2 in.

To find the area of the square.



Solution. $\overline{AB}^2 + \overline{BC}^2 = \overline{AC}^2$. § 337

But $AB = BC$, § 65

and $AC = 2\,AO$. § 131

$\therefore 2\,\overline{AB}^2 = 4\,\overline{AO}^2$.

$\therefore \overline{AB}^2 = 2\,(2)^2 = 8$ sq. in. Ax. 9

That is, area of square $= \overline{AB}^2 = 8$ sq. in. Q. E. F.

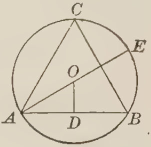
13. *The apothem of an equilateral triangle is equal to half the radius.*

Given the equilateral $\triangle ABC$, OC its radius, and OP its apothem.

To prove that $OP = \frac{1}{2} OC$.

Proof. The proof is the same as the proof for Problem 5, Exercise 16.

14. *Prove that the apothem of an equilateral triangle is equal to one fourth the diameter of the circumscribed circle. From this show how an equilateral triangle may be inscribed in a circle.*



Given OD, the apothem of the equilateral $\triangle ABC$, about which the $\odot ABC$ is circumscribed; also AOE, the diameter of the circumscribed $\odot$.

To prove that $OD = \frac{1}{4}$ of the diameter AOE.

Proof. $OD = \frac{1}{2} AO$. Ex. 13, p. 232

But $AO = \frac{1}{2}$ of the diameter AOE.

$\therefore OD = \frac{1}{4}$ of the diameter AOE, by Ax. 9. Q. E. D.

$\therefore$ to inscribe an equilateral $\triangle$ in a given $\odot$, proceed as follows:

Bisect the radius OA. § 229

With O as a center and $\frac{1}{2} OA$ as a radius describe a $\odot$.

Draw a chord AB of the given $\odot ABC$ tangent to the inner $\odot$ at any point D. § 246

Then draw AC and BC tangent to the inner $\odot$, meeting at C. § 246

Then the $\triangle ABC$ will be an inscribed equilateral $\triangle$ of the $\odot ABC$.

Exercise 60. Page 235

1. *Find the ratio of the perimeters and the ratio of the areas of two regular hexagons, their sides being 2 in. and 4 in. respectively.*

$p : p' = 2 : 4 = 1 : 2.$ § 375

$a : a' = 2^2 : 4^2 = 1^2 : 2^2 = 1 : 4.$ § 374

2. *Find the ratio of the perimeters and the ratio of the areas of two regular octagons, their sides having the ratio 2 : 6.*

$p : p' = 2 : 6 = 1 : 3.$ § 375

$a : a' = 2^2 : 6^2 = 1^2 : 3^2 = 1 : 9.$ § 374

3. *Find the ratio of the perimeters of two squares whose areas are 121 sq. in. and $30\frac{1}{4}$ sq. in. respectively.*

Here $a : a' = 121 : 30\frac{1}{4} = 4 : 1.$

Now $a : a' = s^2 : s'^2.$ § 374

But $s^2 : s'^2 = p^2 : p'^2.$

$\therefore p^2 : p'^2 = a : a' = 4 : 1.$

$\therefore p : p' = \sqrt{4} : \sqrt{1} = 2 : 1.$

4. *Find the ratio of the perimeters and the ratio of the areas of two equilateral triangles whose altitudes are 3 in. and 12 in. respectively.*

$$p : p' = s : s' = h : h' = 3 : 12 = 1 : 4.$$
$$a : a' = s^2 : s'^2 = p^2 : p'^2 = 1^2 : 4^2 = 1 : 16.$$

5. *The area of one equiangular triangle is 9 times that of another. Required the ratio of their altitudes.*

$$s^2 : s'^2 = a : a'.$$
$$\therefore\ s : s' = \sqrt{a} : \sqrt{a'} = \sqrt{9} : \sqrt{1} = 3 : 1.$$
$$\therefore\ h : h' = s : s' = 3 : 1.$$

6. *The area of the cross section of a steel beam 1 in. thick is 12 sq. in. What is the area of the cross section of a beam of the same proportions and $1\frac{1}{8}$ in. thick?*

$$a : a' = 1^2 : 1\tfrac{1}{8}^2.$$
$$12 : a' = 1 : \tfrac{81}{64}.$$
$$\therefore\ a' = 12 \times \tfrac{81}{64} = \tfrac{243}{16} = 15\tfrac{3}{16}.$$ $15\frac{3}{16}$ sq. in. *Ans.*

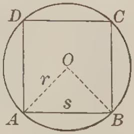
7. *Squares are inscribed in two circles of radii 2 in. and 6 in. respectively. Find the ratio of the areas of the squares, and also the ratio of the perimeters.*

$$a : a' = r^2 : r'^2 = 2^2 : 6^2 = 1^2 : 3^2 = 1 : 9. \qquad \S\ 376$$
$$p : p' = r : r' = 2 : 6 = 1 : 3. \qquad \S\ 375$$

8. *Squares are inscribed in two circles of radii 2 in. and 8 in. respectively, and on the sides of these squares equilateral triangles are constructed. What is the ratio of the areas of these triangles?*

$$s : s' = r : r' = 2 : 8 = 1 : 4.$$
$$a : a' = 1^2 : 4^2 = 1 : 16.$$

9. *A round log a foot in diameter is sawed so as to have the cross section the largest square possible. What is the area of this square? What would be the area of the cross section of the square beam cut from a log of half this diameter?*



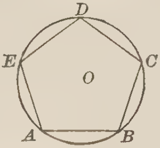
In the rt. $\triangle ABO$,

$$AO = 6 \text{ in.},\ BO = 6 \text{ in., and } AB = s.$$
$$\overline{AB}^2 = \overline{AO}^2 + \overline{BO}^2 \qquad \S\ 337$$
$$= 6^2 + 6^2 = 36 + 36 = 72.$$
$\therefore$ area of square $= 72$ sq. in.
$$a' : a = s'^2 : s^2 = r'^2 : r^2.$$
$$a' : 72 = 1^2 : 2^2 = 1 : 4.$$

$$4a' = 72.$$
$$a' = 18.$$

$\therefore$ area of second square $= 18$ sq. in.

10. *Every equiangular polygon inscribed in a circle is regular if it has an odd number of sides.*



Given the equiangular polygon $ABCDE$ of an odd number of sides inscribed in a $\odot$.

To prove that $ABCDE$ is regular.

Proof. $\angle A = \angle B$. Given

$\therefore$ arc $BCDE =$ arc $CDEA$. § 214

Subtracting each of these equal arcs from the circumference,

arc $EAB =$ arc ABC. Ax. 2

Subtract the common arc AB.

$\therefore$ arc $EA =$ arc BC.

In like manner, arc $AB =$ arc CD,

arc $BC =$ arc DE,

arc $CD =$ arc EA.

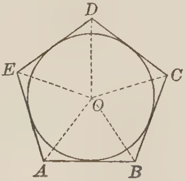
$\therefore$ arc $AB =$ arc $BC =$ arc $CD =$ arc $DE =$ arc EA. Ax. 8

$\therefore$ chord $AB =$ chord $BC =$ chord $CD =$ chord $DE =$ chord EA. § 170

Now the polygon $ABCDE$ is equiangular. Given

$\therefore$ the polygon $ABCDE$ is regular, by § 357. Q.E.D.

11. *Every equilateral polygon circumscribed about a circle is regular if it has an odd number of sides.*



Given the equilateral polygon $ABCDE$, of an odd number of sides, circumscribed about the $\odot$ whose center is O.

To prove that $ABCDE$ is regular.

Proof. Draw the lines OA, OB, OC, OD, and OE.

In the $\triangle$ ABO and BCO, $AB = BC$. Given

$\angle OBA = \angle CBO$. § 192

$OB = OB$. Iden.

$\therefore \triangle ABO$ is congruent to $\triangle BCO$. § 68

$\therefore$ $OA = OC$. § 67

Similarly $OC = OE$.

$\therefore$ $OA = OE$. Ax. 8

In like manner, $OA = OB = OC = OD = OE$.

That is, a $\odot$ may be circumscribed about $ABCDE$ with O as center.

$\therefore$ the polygon $ABCDE$ is regular, by § 365. Q.E.D.

Exercise 61. Page 241

1. *Two circles are constructed with radii* $1\frac{1}{32}$ *in. and* $4\frac{1}{8}$ *in. respectively. The circumference of the second is how many times that of the first?*

$$c : c' = r : r' \qquad \S\ 382$$
$$= 4\tfrac{1}{8}\text{ in.} : 1\tfrac{1}{32}\text{ in.} = 4 : 1 = 4.$$

2. *The circumference of one circle is* 3 *times that of another. The square on the radius of the first is how many times the square on the radius of the second?*

$$c : c' = r : r'. \qquad \S\ 382$$
$$\therefore\ r : r' = c : c' = 3 : 1.$$
$$\therefore\ r^2 : r'^2 = 3^2 : 1^2 = 9 : 1 = 9.$$

3. *The circumference of one circle is* $2\frac{1}{2}$ *times that of another. The equilateral triangle constructed on the diameter of the first has how many times the area of the equilateral triangle constructed on the diameter of the second?*

$$c : c' = r : r'. \qquad \S\ 382$$
$$\therefore\ d : d' = 2r : 2r' = c : c'.$$
$$\therefore\ d : d' = 2\tfrac{1}{2} : 1 = 5 : 2.$$
$$a : a' = d^2 : d'^2 \qquad \S\ 376$$
$$= 5^2 : 2^2 = 25 : 4 = 6\tfrac{1}{4}.$$

4. *A circle with a diameter of* 5 *in. has a circumference of* 15.708 *in. What is the circumference of a pipe that has a diameter of* 2 *in.?*

$$c : c' = r : r' \qquad \S\ 382$$
$$= 2r : 2r' = d : d'.$$
$$c : 15.708\text{ in.} = 2 : 5.$$
$$\therefore\ c = \frac{2 \times 15.708\text{ in.}}{5} = 6.283\text{ in.}$$

5. *A wheel with a circumference of* 4 *ft. has a diameter of* 1.27 *ft., expressed to two decimal places. What is the circumference of a wheel with a diameter of* $1.58\frac{3}{4}$ *ft.?*

$$c : c' = r : r' \qquad \S\ 382$$
$$= 2r : 2r' = d : d'.$$
$$\therefore\ c : 4\text{ ft.} = 1.58\tfrac{3}{4} : 1.27.$$
$$\therefore\ c = \frac{1.58\frac{3}{4} \times 4\text{ ft.}}{1.27} = 5\text{ ft.}$$

6. *A regular hexagon is 2 in. on a side. Find its apothem and its area to two decimal places.*

$$\text{Apothem} = \sqrt{2^2 - 1^2}\text{ in.} = \sqrt{3}\text{ in.} = 1.732\text{ in.}$$
$$\text{Perimeter} = 6 \times 2\text{ in.} = 12\text{ in.}$$
$$\text{Area} = \tfrac{1}{2} \times 12 \times 1.732\text{ sq. in.} = 10.392\text{ sq. in.}$$

1.73 in., 10.39 sq. in. *Ans.*

7. *An equilateral triangle is 2 in. on a side. Find its apothem and its area to two decimal places.*

$$\text{Altitude} = \sqrt{2^2 - 1^2}\text{ in.} = \sqrt{3}\text{ in.} = 1.732\text{ in.}$$
$$\text{Apothem} = \tfrac{1}{3}\text{ of } 1.732\text{ in.} = 0.577\text{ in.}$$
$$\text{Perimeter} = 3 \times 2\text{ in.} = 6\text{ in.}$$
$$\text{Area} = \tfrac{1}{2} \times 6 \times 0.577\text{ in.} = 1.731\text{ sq. in.}$$

0.58 in., 1.73 sq. in. *Ans.*

8. *The radius of one circle is 2½ times that of another. The area of the smaller is 15.2 sq. in. What is the area of the larger?*

$$s : s' = r^2 : r'^2. \qquad \S\ 390$$
$$\therefore\ s : 15.2\text{ sq. in.} = (2\tfrac{1}{2})^2 : 1^2 = \tfrac{25}{4} : 1 = 6\tfrac{1}{4} : 1.$$
$$\therefore\ s = 6\tfrac{1}{4} \times 15.2\text{ sq. in.} = 95\text{ sq. in.}$$

9. *The radius of one circle is 3¼ times that of another. The area of the smaller is 17.75 sq. in. What is the area of the larger?*

$$s : s' = r^2 : r'^2. \qquad \S\ 390$$
$$\therefore\ s : 17.75\text{ sq. in.} = (3\tfrac{1}{4})^2 : 1^2 = \tfrac{169}{16} : 1 = 10\tfrac{9}{16} : 1.$$
$$\therefore\ s = 10\tfrac{9}{16} \times 17.75\text{ sq. in.} = 187.484\text{ sq. in.}$$

10. *The circumferences of two cylindrical steel shafts are respectively 3 in. and 1¼ in. The area of a cross section of the first is how many times that of a cross section of the second?*

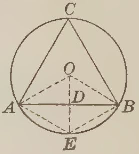
$$r : r' = c : c'. \qquad \S\ 382$$
$$s : s' = r^2 : r'^2. \qquad \S\ 390$$
$$\therefore\ s : s' = c^2 : c'^2$$
$$= 3^2 : (1\tfrac{1}{4})^2 = 9 : 1\tfrac{9}{16} = 144 : 25 = \tfrac{144}{25} = 5.76.$$

11. *The arc of a sector of a circle 2¼ in. in diameter is 1¾ in. What is the area of the sector?*

$$r = \tfrac{1}{2}\text{ of } 2\tfrac{1}{4}\text{ in.} = 1\tfrac{1}{8}\text{ in.}$$
$$s = \tfrac{1}{2}ra \qquad \S\ 391$$
$$= \tfrac{1}{2} \times 1\tfrac{1}{8} \times 1\tfrac{3}{4}\text{ sq. in.} = \tfrac{63}{64}\text{ sq. in.}$$

Exercise 62. Page 245

1. *If r denotes the radius of a regular inscribed triangle, a the apothem, s one side, A an angle, and C the angle at the center, show that* $s = r\sqrt{3}$, $a = \frac{1}{2}r$, $A = 60°$, $C = 120°$.



Given the regular inscribed $\triangle ABC$, with radius r, apothem a, side s, $\angle A$, and $\angle C$ at the center.

To prove that
1. $s = r\sqrt{3}$;
2. $a = \frac{1}{2}r$;
3. $A = 60°$;
4. $C = 120°$.

Draw the radii OA and OB, and draw $OE \perp$ to AB, cutting AB at D and the $\odot$ at E. Draw AE and BE.

Then $OA = r$, $OD = a$, $AB = s$, $\angle BAC = A$, and $\angle AOB = C$.

Proof. 1. E is the mid-point of the arc AB. § 174

$\therefore AO = AE = BO = BE.$ § 394

$\therefore AB$ and OE bisect each other at D. § 151

$\therefore \overline{AD}^2 = \overline{AO}^2 - \overline{OD}^2.$ § 338

That is, $(\frac{1}{2}s)^2 = r^2 - (\frac{1}{2}r)^2.$

$$\therefore \tfrac{1}{4}s^2 = r^2 - \tfrac{1}{4}r^2.$$

$$s^2 = 3r^2.$$

$$s = r\sqrt{3}.$$

2. Since AB bisects OE,

$$OD = \tfrac{1}{2}OE.$$

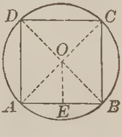
That is, $a = \frac{1}{2}r.$

3. The $\triangle ABC$ is equiangular. § 75

$$\therefore A = 60°.$$

4. $C = 120°$, by § 364. Q.E.D.

2. *If r denotes the radius of a regular inscribed quadrilateral, a the apothem, s one side, A an angle, and C the angle at the center, show that* $s = r\sqrt{2}$, $a = \frac{1}{2}r\sqrt{2}$, $A = 90°$, $C = 90°$.



Given the regular inscribed quadrilateral $ABCD$, with radius r, apothem a, side s, $\angle A$, and $\angle C$ at the center O.

To prove that
1. $s = r\sqrt{2}$;
2. $a = \frac{1}{2}r\sqrt{2}$;
3. $A = 90°$;
4. $C = 90°$.

Draw the diagonals AC and BD intersecting at O, and draw OE ⊥ to AB.

Then $OA = r$, $OE = a$, $AB = s$, $\angle BAD = A$, and $\angle AOB = C$.

Proof. 1. $AC = AB\sqrt{2}$. § 339

$\therefore AO = \frac{1}{2}AB\sqrt{2}$. Ax. 4

That is, $r = \frac{1}{2}s\sqrt{2}$.

$$\therefore s = \frac{r}{\frac{1}{2}\sqrt{2}} = r\sqrt{2}.$$

2. $OE = \frac{1}{2}AD = \frac{1}{2}AB = \frac{1}{2}s$.

Substitute for s its equal $r\sqrt{2}$.

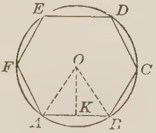
Then $a = \frac{1}{2}r\sqrt{2}$.

3. $\angle BAD$ is a rt. $\angle$. § 215

$\therefore A = 90°$.

4. $C = 90°$, by § 364. Q. E. D.

3. *If r denotes the radius of a regular inscribed hexagon, a the apothem, s one side, A an angle, and C the angle at the center, show that $s = r$, $a = \frac{1}{2}r\sqrt{3}$, $A = 120°$, $C = 60°$.*



Given the regular inscribed hexagon $ABCDEF$, with radius r, apothem a, side s, $\angle A$, and $\angle C$ at the center O.

To prove that
1. $s = r$;
2. $a = \frac{1}{2}r\sqrt{3}$;
3. $A = 120°$;
4. $C = 60°$.

Draw the radii OA and OB, and draw OK ⊥ to AB.

Then $OA = r$, $OK = a$, $AB = s$, $\angle BAF = A$, $\angle AOB = C$.

Proof. 1. $s = r$. § 394

2. $AK = \frac{1}{2}AB$. § 174

Now $\overline{OK}^2 = \overline{OA}^2 - \overline{AK}^2$. § 338

That is, $a^2 = r^2 - (\frac{1}{2}s)^2$

$= r^2 - \frac{1}{4}r^2 = \frac{3}{4}r^2$. Ax. 9

$\therefore a = \frac{1}{2}r\sqrt{3}$.

3. $\angle BAF = \frac{2(6-2)}{6}$ rt. $\angle$s. § 145

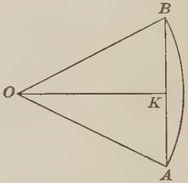
$\therefore A = \frac{4}{3}$ of a rt. $\angle = 120°$.

4. $C = 60°$, by § 364. Q. E. D.

4. *If r denotes the radius of a regular inscribed decagon, a the apothem, s one side, A an angle, and C the angle at the center, show that*

$$s = \frac{r(\sqrt{5}-1)}{2},\ a = \tfrac{1}{4} r\sqrt{10+2\sqrt{5}},\ A = 144°,\ C = 36°.$$

Given AB, one side of a regular inscribed decagon with radius r, apothem a, side s, $\angle A$, and $\angle C$ at the center O.



To prove that 1. $s = \frac{r(\sqrt{5}-1)}{2}$;

2. $a = \frac{1}{4} r\sqrt{10+2\sqrt{5}}$;

3. $A = 144°$;

4. $C = 36°$.

Draw the radii OA and OB, and draw $OK \perp$ to AB.

Then $OA = r$, $OK = a$, $AB = s$, $2\angle A = A$, and $\angle AOB = C$.

Proof. 1. $OA : AB = AB : OA - AB$. § 397

That is, $r : s = s : r - s$.

$\therefore s^2 = r^2 - sr$. § 261

$\therefore s^2 + sr = r^2$.

$\therefore 4s^2 + 4sr + r^2 = 5r^2$.

$\therefore 2s + r = r\sqrt{5}$.

$\therefore 2s = r\sqrt{5} - r$.

$$s = \frac{r(\sqrt{5}-1)}{2}.$$

2. $AK = \frac{1}{2} AB$. § 174

Now $\overline{OK}^2 = \overline{OA}^2 - \overline{AK}^2$. § 338

That is, $a^2 = r^2 - \left[\frac{r(\sqrt{5}-1)}{4}\right]^2$ Ax. 9

$$= r^2 - \frac{r^2(6-2\sqrt{5})}{16}$$

$$= \frac{16r^2 - 6r^2 + 2r^2\sqrt{5}}{16}$$

$$= \frac{r^2(10+2\sqrt{5})}{16}.$$

$\therefore a = \frac{1}{4} r\sqrt{10+2\sqrt{5}}$.

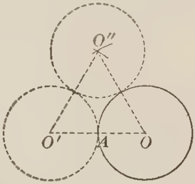
3. $A = \frac{2(10-2)}{10}$ rt. $\angle$s. § 145

$\therefore A = \frac{8}{5}$ of a rt. $\angle = 144°$.

4. $C = 36°$, by § 364. Q. E. D.

Exercise 63. Page 247

1. *A five-cent piece is placed on the table. How many five-cent pieces can be placed around it, each tangent to it and tangent to two of the others? Prove it.*



Given the $\odot$ with center O and radius r.

Required to find how many ⓢ of the same radius can be described each tangent to the given $\odot$ and also tangent to two of the other ⓢ.

Construction. Draw any radius OA and produce it to O', making $AO' = OA$.

With O' as a center and OA as a radius describe a $\odot$.

With O and O' as centers and radii equal to OO' describe arcs intersecting at O''.

With O'' as a center and a radius equal to OA describe a $\odot$.

Then each of these ⓢ is tangent to the given $\odot$ and also tangent to the other.

Proof. Draw $O'O''$, OO''.

$OO' = OO'' = O'O''$. Const.

$\therefore \triangle OO'O''$ is an equilateral $\triangle$. § 62

Since $OO' = 2\,OA$, the ⓢ O and O' are tangent to each other.

Likewise ⓢ O and O'', and ⓢ O' and O'' are tangent to each other.

Again, $\angle O = \angle O' = \angle O''$. § 74

$\therefore \angle O = \angle O' = 60°$. § 107

Since the whole angular space about $O = 360°$, § 41

six $\angle$s, each equal to $\angle O$, can be drawn about O.

Therefore six equal ⓢ can be described, each tangent to the given $\odot$ and also tangent to two of the others. Q. E. D.

2. *What is the perimeter of an equilateral triangle inscribed in a circle with radius* 1 *in.?*

Here $r = 1$ in.

Therefore $s = \sqrt{3}$ in. Ex. 1, p. 245

$\therefore p = 3s = 3\sqrt{3}$ in.

3. *What is the perimeter of an equilateral triangle circumscribed about a circle with radius* 1 *in.?*

Here $a = 1$ in.

Since $s = r\sqrt{3}$ and $a = \frac{1}{2}r$, Ex. 1, p. 245

$\therefore r = 2a = 2$ in.,

and $s = 2\sqrt{3}$ in.

$\therefore p = 3s = 3 \times 2\sqrt{3}$ in. $= 6\sqrt{3}$ in.

4. *What is the perimeter of a regular hexagon circumscribed about a circle with radius* 1 *in.?*

Here $a = 1$ in.

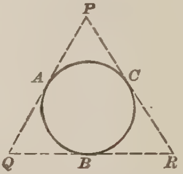
Since $s = r$ and $a = \frac{1}{2} r\sqrt{3}$, Ex. 3, p. 245

$$\therefore r = \frac{a}{\frac{1}{2}\sqrt{3}} = \frac{2}{3}a\sqrt{3} = \frac{2}{3}\sqrt{3} \text{ in.},$$

and $s = \frac{2}{3}\sqrt{3}$ in.

$$\therefore p = 6s$$
$$= 6 \times \tfrac{2}{3}\sqrt{3} \text{ in.} = 4\sqrt{3} \text{ in.}$$

5. *Required to circumscribe about a given circle a regular triangle.*



Given the ⊙ ABC.

Required to circumscribe a regular △ about the ⊙ ABC.

Construction. Find the points A, B, C, the vertices of an inscribed regular △. § 395

Through each of the points A, B, C draw a tangent. § 246

Let the three tangents intersect in the points P, Q, R.

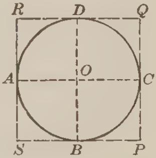
Then △ PQR is the △ required. Q. E. F.

Proof. Now arc AB = arc BC = arc CA, Const.

and PQ, QR, and RP are tangents at A, B, and C. Const.

∴ △ PQR is regular, by § 367. Q. E. D.

6. *Required to circumscribe about a given circle a regular quadrilateral.*



Given the ⊙ with center O.

Required to circumscribe a regular quadrilateral about the ⊙ with center O.

Construction. Find the points A, B, C, D, the vertices of an inscribed regular quadrilateral. § 392

Through each of the points A, B, C, D draw a tangent. § 246

Let the four tangents intersect in the points P, Q, R, S.

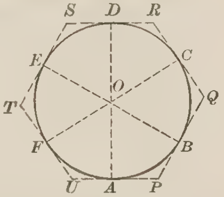
Then the square $PQRS$ is the regular circumscribed quadrilateral required. Q. E. F.

Proof. Now arc AB = arc BC = arc CD = arc DA, Const.

and PQ, QR, RS, and SP are tangent at B, C, D, and A. Const.

∴ the quadrilateral $PQRS$ is regular, by § 367. Q. E. D.

7. *Required to circumscribe about a given circle a regular hexagon.*



Given the ⊙ with center O.

Required to circumscribe a regular hexagon about the given ⊙.

Construction. Divide the ⊙ into six equal arcs at the points A, B, C, D, E, and F. § 394

Through each of the points A, B, C, D, E, F draw a tangent. § 246

Let the six tangents meet in the points P, Q, R, S, T, U.

Then $PQRSTU$ is the regular circumscribed hexagon required. Q.E.F.

Proof. Now $\text{arc } AB = \text{arc } BC = \text{arc } CD = \text{arc } DE = \text{arc } EF$
$= \text{arc } FA = 60°$, Const.

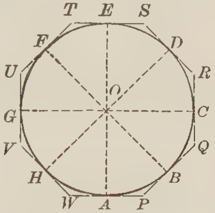
and PQ, QR, RS, ST, TU, and UP are tangents at B, C, D, E, F, and A. Const.

∴ the hexagon $PQRSTU$ is regular, by § 367. Q.E.D.

8. *Required to circumscribe about a given circle a regular octagon.*

Given the ⊙ with center O.

Required to circumscribe a regular octagon about the given ⊙.



Construction. Divide the ⊙ into eight equal arcs at the points A, B, C, D, E, F, G, H. § 393

Through these eight points draw ⊥s to the radii drawn to these points, and designate the points of intersection of these ⊥s by P, Q, R, S, T, U, V, W.

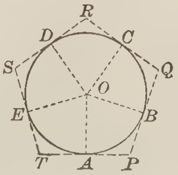
Then $PQRSTUVW$ is the regular circumscribed octagon required. Q.E.F.

Proof. The lines PQ, QR, RS, ST, TU, UV, VW, WP are tangent to the ⊙. § 184

The eight points of contact are the vertices of a regular inscribed octagon. Const.

∴ $PQRSTUVW$ is the regular circumscribed octagon required, by § 368. Q.E.D.

9. *Required to circumscribe about a given circle a regular pentagon.*



Given the ⊙ with center O.

Required to circumscribe a regular pentagon about the given ⊙.

Construction. Divide the ⊙ into five equal arcs at the points A, B, C, D, E. § 398

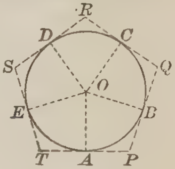
Through these five points draw ⊥s to the radii drawn to the points, and designate the points of intersection of these ⊥s by P, Q, R, S, T.

Then $PQRST$ is the regular circumscribed pentagon required. Q. E. F.

Proof. The lines PQ, QR, RS, ST, TP are tangent to the circle. § 174

The five points of contact are the vertices of a regular inscribed pentagon. Const.

∴ $PQRST$ is the regular circumscribed pentagon required, by § 368. Q. E. D.



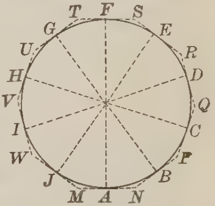
10. *Required to circumscribe about a given circle a regular decagon.*

Given the ⊙ with center O.

Required to circumscribe a regular decagon about the ⊙.

Construction. Divide the ⊙ into ten equal arcs at the points A, B, C, D, E, F, G, H, I, J. § 397

Through these ten points draw ⊥s to the radii drawn to these points, intersecting in the points M, N, P, Q, R, S, T, U, V, W.



Then $MNPQRSTUVW$ is the regular circumscribed decagon required. Q. E. F.

Proof. The lines MN, NP, PQ, QR, RS, ST, TU, UV, VW, WM, are tangent to the circle. § 174

The ten points of contact are the vertices of a regular inscribed decagon. Const.

∴ $MNPQRSTUVW$ is the regular circumscribed decagon required, by § 368. Q. E. D.

11. *Required to describe a circle whose circumference equals the sum of the circumferences of two circles of given radii.*

Let c and c' be the circumferences of two Ⓢ with radii r and r' respectively.

Required to describe a ⊙ with circumference equal to $c + c'$.

Analysis. $c = 2\pi r,$

and $c' = 2\pi r'.$ § 385

$\therefore\ c + c' = 2\pi(r + r').$ Ax. 1

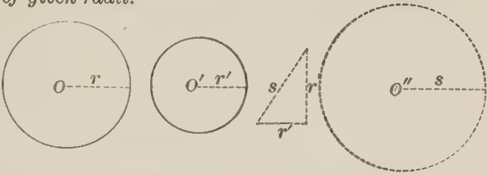
Construction. With any convenient point as center and $r + r'$ as radius describe a ⊙.

This ⊙ is the ⊙ required. Q. E. F.

Proof. The circumference of the ⊙ of radius $r + r'$ is $2\pi(r + r')$.

That is, the circumference $= 2\pi r + 2\pi r' = c + c'$, by Ax. 9. Q. E. D

12. *Required to describe a circle whose area equals the sum of the areas of two circles of given radii.*



Given the two ⓢ whose centers are O and O' respectively, with radii r and r' respectively.

Required to construct a ⊙ whose area is the sum of the areas of the two given ⓢ.

Construction. Construct a rt. △ with r and r' as the legs.

Let s denote the hypotenuse of this rt. △.

With s as a radius describe the ⊙ whose center is O''.

Then the ⊙ whose center is O'' is the ⊙ required. Q.E.F.

Proof. The sum of the areas of the ⓢ whose centers are O and O' is $\pi r^2 + \pi r'^2$, or $\pi (r^2 + r'^2)$. § 389

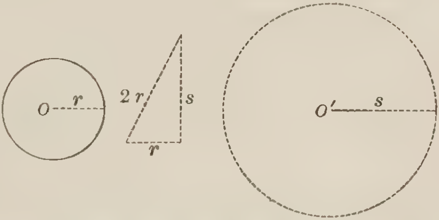
But $r^2 + r'^2 = s^2$. § 337

$\therefore$ the sum of the areas of the two given ⓢ $= \pi s^2$. Ax. 9

Now the area of the ⊙ whose center is $O'' = \pi s^2$. § 389

Therefore $O'' = O + O'$. Q.E.D.

13. *Required to describe a circle having three times the area of a given circle.*



Given the ⊙ whose center is O and radius r.

Required to construct a ⊙ whose area is three times the area of the given ⊙.

Construction. Construct a rt. △ with one leg r and hypotenuse $2r$.

Let s denote the other leg of this rt. △.

With s as a radius describe the ⊙ whose center is O'.

Then the ⊙ whose center is O' is the ⊙ required. Q.E.F.

Proof. $s^2 = (2r)^2 - r^2$ § 338

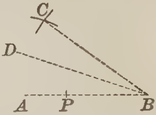
$= 3r^2$.

$\therefore \pi s^2 = 3\pi r^2$. Ax. 3

But πr^2 is the area of the ⊙ whose center is O, and πs^2 the area of the ⊙ whose center is O'. § 389

Therefore the ⊙ whose center is O' has three times the area of the given ⊙. Q.E.D.

14. *Required to construct an angle of* 18°.



Construction. Draw any line AB.

Divide AB internally at P in extreme and mean ratio, so that $AB : PB = PB : AP$. § 311

From A as a center with PB as a radius describe an arc.

From B as a center with a radius equal to AB describe an arc intersecting the former arc in C.

Draw BC.

Then draw BD, the bisector of the $\angle CBA$. § 231

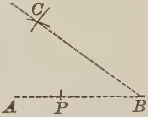
Then the $\angle DBA$ is the $\angle$ required. Q.E.F.

Proof. By § 397, it is shown that $\angle CBA$ is $\frac{1}{10}$ of 4 rt. $\angle$s.

$\therefore \angle CBA$ is an $\angle$ of 36°.

$\therefore \angle DBA$ is an $\angle$ of 18°, by Ax. 4. Q.E.D.

15. *Required to construct an angle of* 36°.



Construction. Draw any line AB.

Divide AB internally at P in extreme and mean ratio, so that $AB : PB = PB : AP$. § 311

From A as a center with PB as a radius describe an arc.

From B as a center with a radius equal to AB describe an arc intersecting the former arc in C.

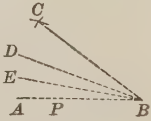
Draw BC.

Then the $\angle CBA$ is the $\angle$ required. Q.E.F.

Proof. By § 397, it is shown that $\angle CBA$ is $\frac{1}{10}$ of 4 rt. $\angle$s.

$\therefore \angle CBA$ is an $\angle$ of 36°. Q.E.D.

16. *Required to construct an angle of* 9°.



Construction. Draw any line AB.

Divide AB internally at P in extreme and mean ratio, so that $AB : PB = PB : AP$. § 311

From A as a center with PB as a radius describe an arc.

From B as a center with a radius equal to AB describe an arc intersecting the former arc in C.

Draw BC.

Then draw BD, the bisector of the $\angle CBA$, and BE, the bisector of the $\angle DBA$. § 231

Then the $\angle EBA$ is the $\angle$ required. Q.E.F.

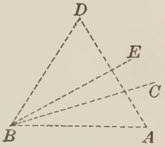
Proof. By § 397, it is shown that $\angle CBA$ is $\frac{1}{10}$ of 4 rt. $\angle$s.

$\therefore \angle CBA$ is an $\angle$ of 36°.

But $\angle EBA$ is $\frac{1}{4}$ $\angle CBA$. Const.

$\therefore \angle EBA$ is an $\angle$ of 9°, by Ax. 4. Q.E.D.

17. *Required to construct an angle of* 12°.



Construction. As in Ex. 14, construct $\angle ABC = 18°$.

On AB construct the equilateral $\triangle ABD$. § 237

Draw BE, the bisector of the $\angle ABD$ § 231

Then the $\angle CBE$ is the $\angle$ required. Q.E.F.

Proof. $\angle ABD = 60°$. § 145

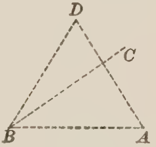
$\therefore \angle ABE = 30°$. Ax. 4

$\angle ABC = 18°$. Const.

Now $\angle CBE = \angle ABE - \angle ABC$.

$\therefore \angle CBE = 30° - 18°$, or 12°, by Ax. 9. Q.E.D.

18. *Required to construct an angle of* 24°.



Construction. As in Ex. 15, construct $\angle ABC = 36°$.

On AB construct the equilateral $\triangle ABD$. § 237

Then the $\angle CBD$ is the $\angle$ required. Q.E.F.

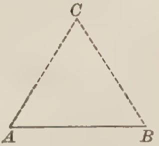
Proof. $\angle ABD = 60°$. § 145

$\angle ABC = 36°$. Const.

Now $\angle CBD = \angle ABD - \angle ABC$.

$\therefore \angle CBD = 60° - 36°$, or 24°, by Ax. 9. Q.E.D.

19. *Required to construct with a side of given length an equilateral triangle.*



Given the line AB.

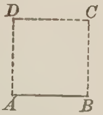
Required to describe on AB as a side an equilateral $\triangle$.

Construction. From A and B as centers with a radius equal to AB describe arcs intersecting at C. § 237

Draw AC and BC.

Then the $\triangle ABC$ is the $\triangle$ required. Q.E.F.

20. *Required to construct with a side of given length a square.*



Given the line AB.

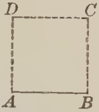
Required to describe on AB as a side a square.

Construction. At point B erect $BC \perp$ to AB, making $BC = AB$. § 228

From A and C as centers with AB as a radius describe arcs intersecting at D.

Draw AD and CD.

Then $ABCD$ is the square required. Q.E.F



Proof. The quadrilateral $ABCD$ is equilateral. Const.

Furthermore, $ABCD$ is a ▱. § 129

$\therefore \angle D$ is a rt. $\angle$. § 124

$\therefore$ ∠s A and C are rt. ∠s. § 124

$\therefore$ $ABCD$ is a square, by § 65. Q.E.D.

21. *Required to construct with a side of given length a regular hexagon.*

Given the line AB.

Required to construct a regular hexagon having AB as a side.

Construction. Describe any ⊙ with AB as a radius.

Apply AB as a chord successively six times to the ⊙ constructed.

Then join the six points of division of the ⊙, thus forming the hexagon $ABCDEF$.

Then $ABCDEF$ is the regular hexagon required. Q.E.F.

Proof. See § 394.

22. *Required to construct with a side of given length a regular octagon.*

Given the line AB.

Required to construct a regular octagon having AB as a side.

Construction. By § 393, construct any regular octagon T.

Then construct the regular octagon T' with AB as a side. § 312

Then T' is the regular octagon required. Q.E.F.

23. *Required to construct with a side of given length a regular pentagon.*

Given the line AB.

Required to construct a regular pentagon having AB as a side.

Construction. By § 398, construct any regular pentagon T.

Then construct the regular pentagon T' with AB as a side. § 312

Then T' is the regular pentagon required. Q.E.F.

24. *Required to construct with a side of given length a regular decagon.*

Given the line AB.

Required to construct a regular decagon having AB as a side.

Construction. By § 397, construct any regular decagon T.

Then construct the regular decagon T' with AB as a side. § 312

Then T' is the regular decagon required. Q.E.F

25. *Required to construct with a side of given length a regular dodecagon.*

Given the line AB.

Required to construct a regular dodecagon having AB as a side.

Construction. By § 396, construct any regular dodecagon T.

Then construct the regular dodecagon T' with AB as a side. § 312

Then T' is the regular dodecagon required. Q.E.F.

26. *Required to construct with a side of given length a regular pentadecagon.*

Given the line AB.

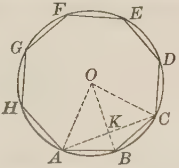
Required to construct a regular pentadecagon having AB as a side.

Construction. By § 400, construct any regular pentadecagon T.

Then construct the regular pentadecagon T' with AB as a side. § 312

Then T' is the regular pentadecagon required. Q.E.F.

27. *From a circular log 16 in. in diameter a builder wishes to cut a column with its cross section as large a regular octagon as possible. Find the length of each side.*



Given the octagon $ABCDEFGH$ inscribed in a ⊙ whose diameter is 16 in.

Required to find s.

Solution. Draw the radii OA, OB, OC.

Draw AC, one side of the inscribed square, intersecting AB at K.

Then OB is the ⊥ bisector of AC. § 151

Now $AC = r\sqrt{2}$,

and $OK = \frac{1}{2} r\sqrt{2}$. Ex. 2, p. 245

$$\begin{aligned}\therefore BK &= OB - OK\\ &= r - \tfrac{1}{2} r\sqrt{2}\\ &= \tfrac{1}{2} r(2-\sqrt{2}).\end{aligned}$$

Again, $\overline{AB}^2 = \overline{AK}^2 + \overline{BK}^2$. § 337

That is,

$$\begin{aligned}s^2 &= \left(\tfrac{1}{2} r\sqrt{2}\right)^2 + \left[\tfrac{1}{2} r(2-\sqrt{2})\right]^2\\ &= \tfrac{1}{2} r^2 + \tfrac{1}{4} r^2(6 - 4\sqrt{2})\\ &= \tfrac{1}{2} r^2 + \tfrac{3}{2} r^2 - r^2\sqrt{2}\\ &= 2r^2 - r^2\sqrt{2}\\ &= r^2(2-\sqrt{2}).\end{aligned}$$

$$\therefore s = r\sqrt{2-\sqrt{2}}.$$

Since $r = 8$ in. Given

$$\begin{aligned}\therefore s &= 8\sqrt{2-\sqrt{2}} \text{ in.}\\ &= 8 \times 0.766 \text{ in.} = 6.128 \text{ in.}\end{aligned}$$

Exercise 64. Page 250

1. *Using the value* 3.1416 *for* π, *find the circumference of the circle with radius* 3 *in.*

$c = 2\pi r$ § 385
$= 2 \times 3.1416 \times 3$ in.
$= 18.8496$ in.

2. *Using the value* 3.1416 *for* π, *find the circumference of the circle with radius* 5 *in.*

$c = 2\pi r$ § 385
$= 2 \times 3.1416 \times 5$ in.
$= 31.416$ in.

3. *Using the value* 3.1416 *for* π, *find the circumference of the circle with radius* 2.7 *in.*

$c = 2\pi r$ § 385
$= 2 \times 3.1416 \times 2.7$ in.
$= 16.96464$ in.

4. *Using the value* 3.1416 *for* π, *find the circumference of the circle with radius* 3.4 *in.*

$c = 2\pi r$ § 385
$= 2 \times 3.1416 \times 3.4$ in.
$= 21.36288$ in.

5. *Using the value* 3.1416 *for* π, *find the circumference of the circle with radius* $7\frac{1}{2}$ *in.*

$c = 2\pi r$ § 385
$= 2 \times 3.1416 \times 7\frac{1}{2}$ in.
$= 47.124$ in.

6. *Using the value* 3.1416 *for* π, *find the circumference of the circle with radius* $6\frac{3}{4}$ *in.*

$c = 2\pi r$ § 385
$= 2 \times 3.1416 \times 6\frac{3}{4}$ in.
$= 42.4116$ in.

7. *Using the value* 3.1416 *for* π, *find the circumference of the circle with radius* 2 *ft.* 8 *in.*

2 ft. 8 in. $= 2\frac{2}{3}$ ft.
$c = 2\pi r$ § 385
$= 2 \times 3.1416 \times 2\frac{2}{3}$ ft.
$= 16.7552$ ft.

8. *Using the value* 3.1416 *for* π, *find the circumference of the circle with radius* 3 *ft.* 7 *in.*

3 ft. 7 in. $= 3\frac{7}{12}$ ft.
$c = 2\pi r$ § 385
$= 2 \times 3.1416 \times 3\frac{7}{12}$ ft.
$= 22.5148$ ft.

9. *Using the value* 3.1416 *for* π, *find the circumference of the circle with diameter* 9 *in.*

$c = \pi d$ § 384
$= 3.1416 \times 9$ in.
$= 28.2744$ in.

10. *Using the value* 3.1416 *for* π, *find the circumference of the circle with diameter* 12 *in.*

$c = \pi d$ § 384
$= 3.1416 \times 12$ in.
$= 37.6992$ in.

11. *Using the value* 3.1416 *for* π, *find the circumference of the circle with diameter* 5.9 *in.*

$c = \pi d$ § 384
$= 3.1416 \times 5.9$ *in.*
$= 18.53544$ in.

12. *Using the value* 3.1416 *for* π, *find the circumference of the circle with diameter* 7.3 *in.*

$c = \pi d$ § 384
$= 3.1416 \times 7.3$ in.
$= 22.93368$ in.

13. *Using the value* 3.1416 *for* π, *find the circumference of the circle with diameter* $2\frac{1}{2}$ *ft.*

$c = \pi d$ § 384
$= 3.1416 \times 2\frac{1}{2}$ ft.
$= 7.854$ ft.

14. *Using the value* 3.1416 *for* π, *find the circumference of the circle with diameter* $3\frac{1}{8}$ *in.*

$c = \pi d$ § 384
$= 3.1416 \times 3\frac{1}{8}$ in.
$= 9.8175$ in.

15. *Using the value* 3.1416 *for* π, *find the circumference of the circle with diameter* 29 *centimeters.*

$c = \pi d$ § 384
$= 3.1416 \times 29$ centimeters.
$= 91.1064$ centimeters.

16. *Using the value* 3.1416 *for* π, *find the circumference of the circle with diameter* 47 *millimeters.*

$c = \pi d$ § 384
$= 3.1416 \times 47$ millimeters.
$= 147.6552$ millimeters.

17 *Find the radius of the circle with circumference* 7π

$c = 2\pi r.$ § 385

$$\therefore r = \frac{c}{2\pi} = \frac{7\pi}{2\pi} = \frac{7}{2} = 3.5.$$

18. *Find the radius of the circle with circumference* $3\frac{1}{3}\pi$.

$c = 2\pi r.$ § 385

$$\therefore r = \frac{c}{2\pi} = \frac{3\frac{1}{3}\pi}{2\pi} = \frac{1}{2} \text{ of } \frac{10}{3} = 1\frac{2}{3}.$$

19. *Using the value* 3.1416 *for* π, *find the radius of the circle with circumference* 15.708 *in.*

$c = 2\pi r.$ § 385

$$2r = \frac{c}{\pi} = \frac{15.708}{3.1416} \text{ in.} = 5 \text{ in.}$$

$\therefore r = \frac{1}{2}$ of 5 in. $= 2\frac{1}{2}$ in.

20. *Using the value* 3.1416 *for* π, *find the radius of the circle with circumference* 21.9912 *in.*

$c = 2\pi r.$ § 385

$$\therefore 2r = \frac{c}{\pi} = \frac{21.9912}{3.1416} \text{ in.} = 7 \text{ in.}$$

$\therefore r = \frac{1}{2}$ of 7 in. $= 3\frac{1}{2}$ in.

21. *Using the value 3.1416 for π, find the radius of the circle with circumference 18.8496 in.*

$$c = 2\pi r. \qquad \S\ 385$$

$$\therefore 2r = \frac{c}{\pi}$$

$$= \frac{18.8496}{3.1416} \text{ in.}$$

$$= 6 \text{ in.}$$

$$\therefore r = \tfrac{1}{2} \text{ of } 6 \text{ in.} = 3 \text{ in.}$$

22. *Using the value 3.1416 for π, find the radius of the circle with circumference 125.664 in.*

$$c = 2\pi r. \qquad \S\ 385$$

$$\therefore 2r = \frac{c}{\pi}$$

$$= \frac{125.664}{3.1416} \text{ in.}$$

$$= 40 \text{ in.}$$

$$\therefore r = \tfrac{1}{2} \text{ of } 40 \text{ in.} = 20 \text{ in.}$$

23. *Using the value 3.1416 for π, find the radius of the circle with circumference 345.576 ft.*

$$c = 2\pi r. \qquad \S\ 385$$

$$\therefore 2r = \frac{c}{\pi}$$

$$= \frac{345.576}{3.1416} \text{ ft.}$$

$$= 110 \text{ ft.}$$

$$\therefore r = \tfrac{1}{2} \text{ of } 110 \text{ ft.}$$

$$= 55 \text{ ft.}$$

24. *Using the value 3.1416 for π, find the radius of the circle with circumference 3487.176 in.*

$$c = 2\pi r. \qquad \S\ 385$$

$$\therefore 2r = \frac{c}{\pi}$$

$$= \frac{3487.176}{3.1416} \text{ in.}$$

$$= 1110 \text{ ft.}$$

$$\therefore r = \tfrac{1}{2} \text{ of } 1110 \text{ in.}$$

$$= 555 \text{ in.}$$

25. *Find the diameter of the circle with circumference 15π.*

$$c = \pi d. \qquad \S\ 384$$

$$\therefore d = \frac{c}{\pi}$$

$$= \frac{15\pi}{\pi}$$

$$= 15.$$

26. *Find the diameter of the circle with circumference π^2.*

$$c = \pi d. \qquad \S\ 384$$

$$\therefore d = \frac{c}{\pi}$$

$$= \frac{\pi^2}{\pi} = \pi.$$

27. *Find the diameter of the circle with circumference $2\pi r$.*

$$c = \pi d. \qquad \S\ 384$$

$$\therefore d = \frac{c}{\pi}$$

$$= \frac{2\pi r}{\pi} = 2r.$$

28. *Find the diameter of the circle with circumference $7\pi a^2$.*

$$c = \pi d. \qquad \S\ 384$$

$$\therefore d = \frac{c}{\pi}$$

$$= \frac{7\pi a^2}{\pi}$$

$$= 7a^2.$$

29. *Using the value* 3.1416 *for* π, *find the diameter of the circle with circumference* 188.496 *in.*

$$c = \pi d. \qquad \S\ 384$$
$$\therefore\ d = \frac{c}{\pi}$$
$$= \frac{188.496}{3.1416}\text{ in.}$$
$$= 60\text{ in.}$$

30. *Using the value* 3.1416 *for* π, *find the diameter of the circle with circumference* 219.912 *in.*

$$c = \pi d. \qquad \S\ 384$$
$$\therefore\ d = \frac{c}{\pi}$$
$$= \frac{219.912}{3.1416}\text{ in.}$$
$$= 70\text{ in.}$$

31. *Using the value* 3.1416 *for* π, *find the diameter of the circle with circumference* 3361.512 *in.*

$$c = \pi d. \qquad \S\ 384$$
$$\therefore\ d = \frac{c}{\pi}$$
$$= \frac{3361.512}{3.1416}\text{ in.}$$
$$= 1070\text{ in.}$$

32. *Using the value* 3.1416 *for* π, *find the diameter of the circle with circumference* 3173.016 *in.*

$$c = \pi d. \qquad \S\ 384$$
$$\therefore\ d = \frac{c}{\pi}$$
$$= \frac{3173.016}{3.1416}\text{ in.}$$
$$= 1010\text{ in.}$$

33. *Find the area of the circle with radius* $5x$.

$$\text{Area} = \pi r^2 \qquad \S\ 389$$
$$= \pi(5x)^2$$
$$= 25\pi x^2.$$

34. *Find the area of the circle with radius* 2π.

$$\text{Area} = \pi r^2 \qquad \S\ 389$$
$$= \pi(2\pi)^2$$
$$= \pi \times 4\pi^2$$
$$= 4\pi^3.$$

35. *Using the value* 3.1416 *for* π, *find the area of the circle with radius* 27 *ft.*

$$\text{Area} = \pi r^2 \qquad \S\ 389$$
$$= 3.1416 \times 27^2\text{ sq. ft.}$$
$$= 3.1416 \times 729\text{ sq. ft.}$$
$$= 2290.2264\text{ sq. ft.}$$

36. *Using the value* 3.1416 *for* π, *find the area of the circle with radius* 4.8 *ft.*

$$\text{Area} = \pi r^2 \qquad \S\ 389$$
$$= 3.1416 \times 4.8^2\text{ sq. ft.}$$
$$= 3.1416 \times 23.04\text{ sq. ft.}$$
$$= 72.382464\text{ sq. ft.}$$

37. *Using the value* 3.1416 *for* π, *find the area of the circle with radius* $3\frac{1}{8}$ *in.*

$$\text{Area} = \pi r^2 \qquad \S\ 389$$
$$= 3.1416 \times (3\tfrac{1}{8})^2\text{ sq. in.}$$
$$= 3.1416 \times 9\tfrac{49}{64}\text{ sq. in.}$$
$$= 30.6796875\text{ sq. in.}$$

38. *Using the value* 3.1416 *for* π, *find the area of the circle with radius* $4\frac{5}{8}$ *in.*

$$\text{Area} = \pi r^2 \qquad \S\ 389$$
$$= 3.1416 \times (4\tfrac{5}{8})^2\text{ sq. in.}$$
$$= 3.1416 \times 21.390625\text{ sq. in.}$$
$$= 67.2007875\text{ sq. in.}$$

39. *Using the value 3.1416 for π, find the area of the circle with radius 2 ft. 6 in.*

$$\begin{aligned} 2\text{ ft. }6\text{ in.} &= 2.5\text{ ft.} \\ \text{Area} &= \pi r^2 \qquad \S\ 389 \\ &= 3.1416 \times 2.5^2\text{ sq. ft.} \\ &= 3.1416 \times 6.25\text{ sq. ft.} \\ &= 19.635\text{ sq. ft.} \end{aligned}$$

40. *Using the value 3.1416 for π, find the area of the circle with radius* 7 ft. 9 in.

$$\begin{aligned} 7\text{ ft. }9\text{ in.} &= 7.75\text{ ft.} \\ \text{Area} &= \pi r^2 \qquad \S\ 389 \\ &= 3.1416 \times 7.75^2\text{ sq. ft.} \\ &= 3.1416 \times 60.0625\text{ sq. ft.} \\ &= 188.69235\text{ sq. ft.} \end{aligned}$$

41. *Find the area of the circle with diameter $16\,ab$.*

$$\begin{aligned} r &= \tfrac{1}{2}d = \tfrac{1}{2}\text{ of }16\,ab = 8\,ab. \\ \text{Area} &= \pi r^2 \qquad \S\ 389 \\ &= \pi(8\,ab)^2 \\ &= 64\,\pi a^2b^2. \end{aligned}$$

42. *Find the area of the circle with diameter $24\,\pi^2$.*

$$\begin{aligned} r &= \tfrac{1}{2}d = \tfrac{1}{2}\text{ of }24\,\pi^2 = 12\,\pi^2. \\ \text{Area} &= \pi r^2 \qquad \S\ 389 \\ &= \pi(12\,\pi^2)^2 \\ &= 144\,\pi^5. \end{aligned}$$

43. *Using the value 3.1416 for π, find the area of the circle with diameter 2.5 ft.*

$$\begin{aligned} r &= \tfrac{1}{2}d = \tfrac{1}{2}\text{ of }2.5\text{ ft.} = 1.25\text{ ft.} \\ \text{Area} &= \pi r^2 \qquad \S\ 389 \\ &= 3.1416 \times 1.25^2\text{ sq. ft.} \\ &= 3.1416 \times 1.5625\text{ sq. ft.} \\ &= 4.90875\text{ sq. ft.} \end{aligned}$$

44. *Using the value 3.1416 for π, find the area of the circle with diameter 7.3 in.*

$$\begin{aligned} r &= \tfrac{1}{2}d = \tfrac{1}{2}\text{ of }7.3\text{ in.} = 3.65\text{ in.} \\ \text{Area} &= \pi r^2 \qquad \S\ 389 \\ &= 3.1416 \times 3.65^2\text{ sq. in.} \\ &= 3.1416 \times 13.3225\text{ sq. in.} \\ &= 41.853966\text{ sq. in.} \end{aligned}$$

45. *Using the value* 3.1416 *for* π, *find the area of the circle with diameter* $3\frac{2}{3}$ *yd.*

$r = \frac{1}{2}d = \frac{1}{2}$ of $3\frac{2}{3}$ yd. $= \frac{11}{6}$ yd.

Area $= \pi r^2$ § 389

$= 3.1416 \times (\frac{11}{6})^2$ sq. yd.

$= 3.1416 \times \frac{121}{36}$ sq. yd.

$= 10.5593$ sq. yd.

46. *Using the value* 3.1416 *for* π, *find the area of the circle with diameter* $4\frac{5}{8}$ yd.

$r = \frac{1}{2}d = \frac{1}{2}$ of $4\frac{5}{8}$ yd. $= 2\frac{5}{16}$ yd.

Area $= \pi r^2$ § 389

$= 3.1416 \times (\frac{37}{16})^2$ sq. yd.

$= 16.8002$ sq. yd.

47. *Using the value* 3.1416 *for* π, *find the area of the circle with diameter* 3 *ft.* 2 *in.*

$r = \frac{1}{2}d = \frac{1}{2}$ of 3 ft. 2 in.

$= \frac{1}{2}$ of 38 in. $= 19$ in.

Area $= \pi r^2$ § 389

$= 3.1416 \times 19^2$ sq. in.

$= 3.1416 \times 361$ sq. in.

$= 1134.1176$ sq. in.

48. *Using the value* 3.1416 *for* π, *find the area of the circle with diameter* 4 *ft.* 1 *in.*

$r = \frac{1}{2}d = \frac{1}{2}$ of 4 ft. 1 in.

$= \frac{1}{2}$ of 49 in. $= 24.5$ in.

Area $= \pi r^2$ § 389

$= 3.1416 \times 24.5^2$ sq. in.

$= 3.1416 \times 600.25$ sq. in.

$= 1885.7454$ sq. in.

49. *Find the area of the circle with circumference* 2π.

$c = 2\pi r.$ § 385

$$\therefore\ r = \frac{c}{2\pi} = \frac{2\pi}{2\pi} = 1.$$

Area $= \frac{1}{2}rc$ § 388

$= \frac{1}{2}$ of 2π

$= \pi.$

50. *Find the area of the circle with circumference* 4π.

$c = 2\pi r.$ § 385

$$\therefore\ r = \frac{c}{2\pi} = \frac{4\pi}{2\pi} = 2.$$

Area $= \frac{1}{2}rc$ § 388

$= \frac{1}{2}$ of $2 \times 4\pi$

$= 4\pi.$

51. *Find the area of the circle with circumference* πa.

$$c = 2\pi r. \qquad \S\ 385$$

$$\therefore r = \frac{c}{2\pi} = \frac{\pi a}{2\pi} = \frac{a}{2}.$$

$$\text{Area} = \tfrac{1}{2} rc \qquad \S\ 388$$

$$= \frac{1}{2} \text{ of } \frac{a}{2} \times \pi a$$

$$= \frac{\pi a^2}{4}.$$

52. *Find the area of the circle with circumference* $14\pi a^2$.

$$c = 2\pi r. \qquad \S\ 385$$

$$\therefore r = \frac{c}{2\pi} = \frac{14\pi a^2}{2\pi} = 7a^2.$$

$$\text{Area} = \tfrac{1}{2} rc \qquad \S\ 388$$

$$= \tfrac{1}{2} \text{ of } 7a^2 \times 14\pi a^2$$

$$= 49\pi a^4.$$

53. *Using the value* 3.1416 *for* π, *find the area of the circle with circumference* 18.8496 in.

$$c = 2\pi r. \qquad \S\ 385$$

$$\therefore r = \frac{c}{2\pi} = \frac{18.8496}{2 \times 3.1416} \text{ in.} = 3 \text{ in.}$$

$$\text{Area} = \tfrac{1}{2} rc \qquad \S\ 388$$

$$= \tfrac{1}{2} \text{ of } 3 \times 18.8496 \text{ sq. in.}$$

$$= 28.2744 \text{ sq. in.}$$

54. *Using the value* 3.1416 *for* π, *find the area of the circle with circumference* 329.868 *in.*

$$c = 2\pi r. \qquad \S\ 385$$

$$\therefore r = \frac{c}{2\pi} = \frac{329.868}{2 \times 3.1416} \text{ in.} = 52.5 \text{ in.}$$

$$\text{Area} = \tfrac{1}{2} rc. \qquad \S\ 388$$

$$= \tfrac{1}{2} \text{ of } 52.5 \times 329.868 \text{ sq. in.}$$

$$= 8659.035 \text{ sq. in.}$$

55. *Using the value* 3.1416 *for* π, *find the area of the circle with circumference* 333.0096 *in.*

$$c = 2\pi r. \qquad \S\ 385$$

$$\therefore r = \frac{c}{2\pi} = \frac{333.0096}{2 \times 3.1416} \text{ in.} = 53 \text{ in.}$$

$$\text{Area} = \tfrac{1}{2} rc. \qquad \S\ 388$$

$$= \tfrac{1}{2} \text{ of } 53 \times 333.0096 \text{ sq. in.}$$

$$= 8824.7544 \text{ sq. in.}$$

56. *Using the value 3.1416 for π, find the area of the circle with circumference 364.4256 in.*

$$c = 2\pi r. \qquad \S\ 385$$

$$\therefore\ r = \frac{c}{2\pi} = \frac{364.4256}{2 \times 3.1416} \text{ in.} = 58 \text{ in.}$$

$$\begin{aligned} \text{Area} &= \tfrac{1}{2} rc \\ &= \tfrac{1}{2} \text{ of } 58 \times 364.4256 \text{ sq. in.} \\ &= 10568.3424 \text{ sq. in.} \end{aligned}$$

57. *Find the radius of the circle with area $\pi a^2 b^4$.*

$$s = \pi r^2. \qquad \S\ 389$$

$$\begin{aligned} \therefore\ r^2 &= \frac{s}{\pi}. \\ \therefore\ r &= \sqrt{\frac{s}{\pi}} \\ &= \sqrt{\frac{\pi a^2 b^4}{\pi}} \\ &= \sqrt{a^2 b^4} \\ &= ab^2. \end{aligned}$$

58. *Find the radius of the circle with area $4\pi m^4 n^6$.*

$$s = \pi r^2. \qquad \S\ 389$$

$$\begin{aligned} \therefore\ r^2 &= \frac{s}{\pi}. \\ \therefore\ r &= \sqrt{\frac{s}{\pi}} \\ &= \sqrt{\frac{4\pi m^4 n^6}{\pi}} \\ &= \sqrt{4 m^4 n^6} \\ &= 2 m^2 n^3. \end{aligned}$$

59. *Find the radius of the circle with area π.*

$$s = \pi r^2. \qquad \S\ 389$$

$$\begin{aligned} \therefore\ r^2 &= \frac{s}{\pi}. \\ \therefore\ r &= \sqrt{\frac{s}{\pi}} \\ &= \sqrt{\frac{\pi}{\pi}} \\ &= \sqrt{1} \\ &= 1. \end{aligned}$$

60. *Find the radius of the circle with area 2π.*

$$s = \pi r^2. \qquad \S\ 389$$

$$\begin{aligned} \therefore\ r^2 &= \frac{s}{\pi}. \\ \therefore\ r &= \sqrt{\frac{s}{\pi}} \\ &= \sqrt{\frac{2\pi}{\pi}} \\ &= \sqrt{2} \\ &= 1.414. \end{aligned}$$

61. *Using the value 3.1416 for π, find the radius of the circle with area* 12.5664.

$$s = \pi r^2. \qquad \S\ 389$$

$$\begin{aligned} \therefore\ r^2 &= \frac{s}{\pi}. \\ \therefore\ r &= \sqrt{\frac{s}{\pi}} \\ &= \sqrt{\frac{12.5664}{3.1416}} \\ &= \sqrt{4} \\ &= 2. \end{aligned}$$

62. *Using the value 3.1416 for π, find the radius of the circle with area* 28.2744.

$$s = \pi r^2. \qquad \S\ 389$$

$$\begin{aligned} \therefore\ r^2 &= \frac{s}{\pi}. \\ \therefore\ r &= \sqrt{\frac{s}{\pi}} \\ &= \sqrt{\frac{28.2744}{3.1416}} \\ &= \sqrt{9} \\ &= 3. \end{aligned}$$

63. *Using the value* 3.1416 *for* π, *find the radius of the circle with area* 78.54.

$$s = \pi r^2. \qquad \S\ 389$$

$$\therefore r^2 = \frac{s}{\pi}.$$

$$\therefore r = \sqrt{\frac{s}{\pi}}$$

$$= \sqrt{\frac{78.54}{3.1416}}$$

$$= \sqrt{25}$$

$$= 5.$$

64. *Using the value* 3.1416 *for* π, *find the radius of the circle with area* 113.0976.

$$s = \pi r^2. \qquad \S\ 389$$

$$\therefore r^2 = \frac{s}{\pi}.$$

$$\therefore r = \sqrt{\frac{s}{\pi}}$$

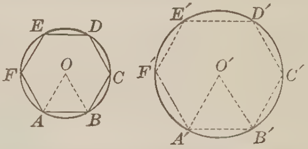
$$= \sqrt{\frac{113.0976}{3.1416}}$$

$$= \sqrt{36}$$

$$= 6.$$

Exercise 65. Page 251

1. *To inscribe in a given circle a regular polygon similar to a given regular polygon.*



Given the regular polygon $ABCDEF$ and the ⊙ with center O'.

Required to inscribe in the ⊙ a regular polygon similar to $ABCDEF$.

Construction. From O, the center of the given polygon, draw the radii OA and OB.

From O', the center of the given ⊙, draw the radii $O'A'$ and $O'B'$, making the $\angle O' = \angle O$. § 232

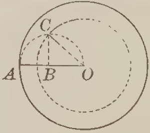
Draw $A'B'$.

Then $A'B'$ is a side of the regular polygon required. Q.E.F.

Proof. Each polygon has as many sides as the $\angle O$, or the $\angle O'$, is contained times in 4 rt. ∠s.

Therefore the polygon $ABCDEF$ is similar to the polygon $A'B'C'D'E'F'$, by § 373. Q.E.D.

2. *To divide by a concentric circle the area of a given circle into two equivalent parts.*



Given the ⊙ with center O and radius OA.

Required to divide the area of the ⊙ into two equivalent parts by a concentric ⊙.

Construction. Bisect the line OA at B. § 229

With B as center and OB as radius draw a semicircle.

At B draw the ⊥ BC intersecting the semicircle at C. § 228

Draw OC.

With O as center and OC as radius describe a ⊙.

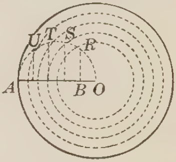
This ⊙ is the concentric ⊙ required.

Proof. $\pi\overline{OC}^2$ is the area of the inner ⊙, § 389

and $\pi\overline{OA}^2$ is the area of the given ⊙. § 389

Now
$$\overline{OC}^2 = \overline{OB}^2 + \overline{BC}^2 \qquad \text{§ 337}$$
$$= 2\,\overline{OB}^2 \qquad \text{Ax. 9}$$
$$= \tfrac{1}{2}\,\overline{OA}^2.$$
$$\therefore\ \pi\overline{OC}^2 = \tfrac{1}{2}\pi\overline{OA}^2, \text{ by Ax. 3.} \qquad \text{Q.E.D.}$$

3. *To divide by concentric circles the area of a given circle into n equivalent parts.*



Given the ⊙ with center O and radius OA.

Required to divide the area of the ⊙ into n equivalent parts by concentric Ⓢ.

Construction. Divide OA into n equal parts. § 234

At B, the first point of division from O, erect a ⊥ meeting the semicircle on OA as a diameter at R. § 228

With O as center and OR as radius describe a ⊙.

Similarly, with O as center and radii equal to the distances from O to the points of intersection of the ⊥s erected at the other points of division with the semicircle, describe Ⓢ.

These n Ⓢ are the n concentric Ⓢ required. Q.E.F.

Proof. Now
$$\overline{OR}^2 = \overline{OB}^2 + \overline{BR}^2. \qquad \text{§ 337}$$
But
$$\overline{BR}^2 = OB \times BA. \qquad \text{§ 297}$$
$$\therefore\ \overline{OR}^2 = \overline{OB}^2 + OB \times BA \qquad \text{Ax. 9}$$
$$= OB(OB + BA)$$
$$= \frac{OA}{n} \times OA \qquad \text{Ax. 9}$$
$$= \frac{1}{n}\,\overline{OA}^2.$$
$$\therefore\ \pi\overline{OR}^2 = \frac{1}{n}\pi\overline{OA}^2. \qquad \text{Ax. 3}$$

But $\pi\overline{OR}^2$ is the area of the ⊙ of which OR is the radius,

and $\pi\overline{OA}^2$ is the area of the given ⊙. § 389

In like manner, $\pi\overline{OS}^2 = \frac{2}{n}\pi\overline{OA}^2$, and so on. Q.E.D.

4. *To describe a circle whose circumference is equal to the difference of two circumferences of given radii.*

Given c and c' the circumferences of two Ⓢ whose radii are r and r' respectively, and let c be greater than c'.

Required to describe a ⊙ whose circumference is equal to $c - c'$.

Construction. With $r - r'$ as a radius describe a $\odot$.

This $\odot$ is the $\odot$ required. Q.E.F.

Proof. The circumference of the $\odot$ with radius r is $2\pi r$,

the circumference of the $\odot$ with radius r' is $2\pi r'$,

and the circumference of the $\odot$ with radius $r - r'$ is $2\pi(r - r')$. § 385

Now $2\pi r - 2\pi r' = 2\pi(r - r')$. Q.E.D.

5. *To describe a circle the ratio of whose area to that of a given circle shall be equal to the given ratio $m : n$.*

Given the $\odot$ c, with radius r.

Required a $\odot$ c' which shall be to the $\odot$ c as m is to n.

Construction. Find r'^2, the fourth proportional to n, m, and r^2. § 355

With r' as radius describe a $\odot$ c'.

Then c' is the $\odot$ required. Q.E.F.

Proof. $c = \pi r^2$,

$c' = \pi r'^2$. § 389

But $n : m = r^2 : r'^2$. Const.

$\therefore r'^2 = \dfrac{mr^2}{n}$. § 261

$\therefore c' = \dfrac{\pi m r^2}{n}$. Ax. 9

$$\therefore c' : c = \frac{\pi m r^2}{n} : \pi r^2$$

$$= \frac{m}{n} : 1$$

$$= m : n.$$ Q.E.D.

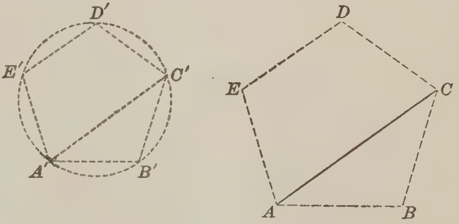
6. *To construct a regular pentagon, given one of the diagonals.*

Given the line AC.

Required to construct a regular pentagon with AC as one of the diagonals.

Construction. Inscribe a regular pentagon $A'B'C'D'E'$ in any $\odot$ $A'B'C'$. § 398

Draw $A'C'$.



On AC, corresponding to $A'C'$, construct the $\triangle ABC$ similar to the $\triangle A'B'C'$. § 312

On AC, corresponding to $A'C'$, construct the polygon $ACDE$ similar to the polygon $A'C'D'E'$. § 312

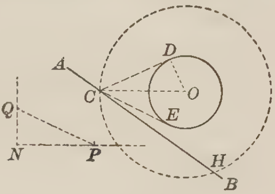
Then $ABCDE$ is the regular pentagon required. Q. E. F.

Proof. The pentagon $ABCDE$ is similar to the pentagon $A'B'C'D'E'$. § 293

Now $A'B'C'D'E'$ is a regular pentagon. Const.

$\therefore$ $ABCDE$ is a regular pentagon. Q. E. D.

7. *To draw a tangent to a given circle such that the segment intercepted between the point of contact and a given line shall have a given length.*



Given the line AB, the length m, and the ⊙ DE, with center O.

To find a point on AB from which a tangent to the ⊙ shall equal m.

Analysis. Suppose C is the required point on AB, and CD the required tangent.

Draw OC and OD.

Then $\overline{OC}^2 = \overline{OD}^2 + \overline{CD}^2$. § 337

$\therefore$ $\overline{OC}^2 = OD^2 + m^2$. Ax. 9

Construction. Draw two lines ⊥ to each other at a point N, and take NP equal to m, and NQ equal to OD. § 228

Draw QP.

With O as center and QP as radius describe a ⊙ cutting AB in C.

From C draw the tangent CD.

Then CD is the tangent required. Q. E. F.

Proof. Since OD is ⊥ to CD, § 185

$\therefore$ $\overline{CD}^2 = \overline{CO}^2 - \overline{OD}^2$. § 338

$\therefore$ $CD = \sqrt{\overline{CO}^2 - \overline{OD}^2}$ Ax. 5

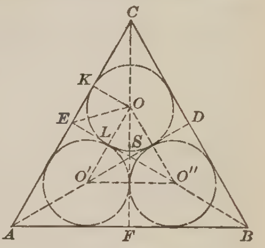
$= \sqrt{\overline{QP}^2 - \overline{NQ}^2}$ Ax. 9

$= NP = m$. Q. E. D.

Discussion. From C another tangent CE may be drawn which will fulfill the required conditions. Two other tangents from H may also be drawn.

Hence there are, in general, four tangents possible, two from each of the points in which the constructed circle cuts AB. It is evident that if the line AB is at the distance m from the center, the two points C and H coincide, and but two tangents may be drawn; and if the line AB is at a greater distance from the center than m, the constructed circle does not meet AB and there is no solution.

8. *In a given equilateral triangle to inscribe three equal circles tangent each to the other two, each circle being tangent to two sides of the triangle.*



Given the equilateral △ *ABC*.

Required to describe in the △ *ABC* three equal ⓢ tangent each to the other two, each ⊙ being tangent to two sides of the △ *ABC*.

Analysis. Each of these ⓢ is evidently inscribed in a quadrilateral formed by the sides and altitudes of the △ *ABC*, as *ECDS*.

The center of the ⊙ must lie on *CF*, as *CF* is the locus of all points equidistant from *CA* and *CB*.

As this ⊙ must also touch *BE*, its center lies on *EO*, the bisector of the ∠ *BEC*.

Hence, the center is at *O*, the intersection of *CF* and *EO*, and the radius is *OK*, the ⊥ from *O* upon *CE*, *SE*, *SD*, or *CD*.

Construction. Draw the altitudes *CF*, *BE*, *AD*. § 227

Bisect the ∠ *BEC* by *EO*, intersecting *CF* in *O*. § 231

From *O* draw the ⊥ *OK* to *CA*.

Then with *O* as center and *OK* as radius describe a ⊙.

This ⊙ is inscribed in *ECDS* and is one of the required ⓢ.

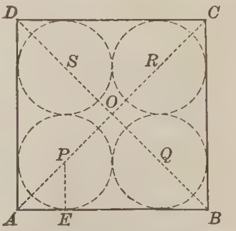
The same construction may be repeated for each of the other quadrilaterals *EAFS* and *FBDS*. Q.E.F.

Proof. *O* is equidistant from *CA*, *CB*, and *BE*. § 152

Since *O* and *O′* are equidistant from *AC*, therefore the ⊥s from *O* and *O′* upon the altitude *BE* lie in one straight line.

∴ the ⓢ *O* and *O′* are tangent to each other. Q.E.D.

9. *In a given square to inscribe four equal circles, so that each circle shall be tangent to two of the others and also tangent to two sides of the square.*



Given the square *ABCD*.

Required to inscribe in the square *ABCD* four equal ⓢ each tangent to two of the other ⓢ and also tangent to two sides of the square.

Analysis. Suppose the problem solved, and *P*, *Q*, *R*, *S* the centers of the required ⓢ.

From the equality of the ⓢ it is clear that their centers must be symmetrically arranged about *O*, the center of the square.

Hence, we see that the ∠s formed at O by OP, OQ, OR, and OS are each 90°, and, therefore, that P, Q, R, S lie in the diagonals of the square.

Construction. Draw the diagonals AC and BD, intersecting at O.

Bisect OA, OB, OC, OD at P, Q, R, S, respectively.

With P as center and radius equal to the distance from P to the sides of the square describe a ⊙. Construct similar ⓢ with Q, R, S as centers.

These ⓢ are the ⓢ required. Q.E.F.

Proof. The rt. △s AEP and ABC are similar. § 287

$\therefore PE : BC = AP : AC = 1 : 4.$ § 282

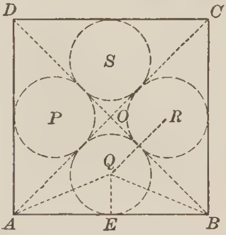
$PE = \frac{1}{4} BC.$

Now $PQ = \frac{1}{2} AB$ § 136

$= 2\,PE.$ Ax. 9

∴ each ⊙ touches two others and two sides of the square. Q.E.D.

10. *In a given square to inscribe four equal circles, so that each circle shall be tangent to two of the others and also tangent to one side and only one side of the square.*



Given the square $ABCD$.

Required to inscribe in the square $ABCD$ four equal ⓢ each tangent to two of the other ⓢ and one side of the square.

Analysis. $OA = OB = OC = OD$, § 358

and $AB = BC = CD = DA$. § 65

∴ the △s ABO, BCO, CDO, and DAO are congruent. § 80

Construction. Draw the diagonals AC and BD, intersecting at O.

Inscribe a ⊙ in each of the △s ABO, BCO, CDO, and DAO. § 244

These four ⓢ are the ⓢ required. Q.E.F.

Proof. Each ⊙ is tangent to a side of the square. Const.

The △s ABO, BCO, CDO, and DAO are congruent. § 80

∴ the inscribed ⓢ are equal.

∴ R and Q are equidistant from AC.

∴ RQ is ∥ to AC.

Again, AC is ⊥ to BD.

∴ RQ is ⊥ to BD. § 97

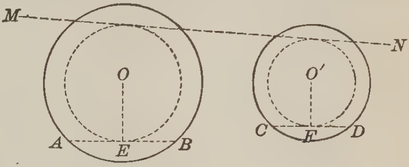
Now the two ⓢ Q and R are tangent to BD. Const.

∴ RQ goes through the point of contact. § 187

∴ the ⓢ Q and R are tangent to each other.

In like manner, each ⊙ is tangent to two others. Q.E.D

11. *To draw a common secant to two given circles exterior to each other, such that the intercepted chords shall have the given lengths a and b.*



Given the two ⓢ, exterior to each other, with centers O and O', and given the lengths a and b.

Required to draw a common secant such that the intercepted chords shall have the lengths a and b.

Construction. In the ⊙ with center O draw a chord AB equal to a, and in the ⊙ with center O' draw a chord CD equal to b.

Draw $OE \perp$ to AB, and $O'F \perp$ to CD. § 227

With center O and radius OE describe a ⊙, and with center O' and radius $O'F$ describe a ⊙.

Draw MN a common tangent to these two auxiliary ⓢ, using the method of Ex. 1, p. 145 (see p. 139 of this book).

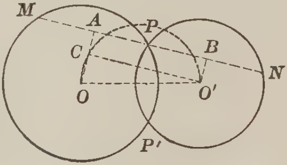
Then MN is the common secant required. Q.E.F.

Proof. Any chord of the given ⊙ with center O tangent to the concentric auxiliary ⊙ is equal to a, and any chord of the given ⊙ with center O' tangent to the concentric auxiliary ⊙ is equal to b.

Therefore the secant MN cuts a chord of length a from the given ⊙ with center O and cuts a chord of length b from the given ⊙ with center O'. Q.E.D.

Discussion. Since the auxiliary circles are exterior to each other, two common external and two common internal tangents may be drawn, each of which satisfies the conditions.

12. *To draw through a point of intersection of two given intersecting circles a common secant of a given length.*



Given two ⓢ with centers O and O' intersecting in the points P and P', and given the length m.

Required to draw through P a common secant equal to m.

Construction. Draw OO', and on OO' as a diameter describe a semicircle.

With O' as center and a radius equal to $\frac{1}{2}m$ describe an arc cutting the semicircle at C, and draw OC and $O'C$.

Through P draw the common secant $MN \parallel$ to CO'.

Then MN is the common secant required. Q.E.F.

Proof. Produce OC to meet MN at A, and draw $O'B \perp$ to MN.

OA is $\perp$ to MN. § 97

$\therefore MA = AP$,

and $PB = BN$. § 174

$\therefore AB = \frac{1}{2} MN$.

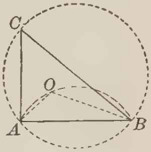
But $AB = CO' = \frac{1}{2} m$. § 127

$\therefore MN = m$, by Ax. 3. Q.E.D.

Exercise 66. Page 252

1. *Find the locus of the center of the circle inscribed in a triangle that has a given base and a given angle at the vertex.*

Given the $\triangle ABC$, with base AB and the $\angle$ at the vertex equal to the $\angle m$.



Required to find the locus of the center of the ⊙ inscribed in the $\triangle ABC$.

Construction. On AB as a chord, describe a segment of a ⊙ in which may be inscribed an $\angle$ equal to $90° + \frac{1}{2} m$. § 247

The arc of this segment is the locus required. Q.E.F.

Proof. The locus of the vertex C is the arc of the segment on AB as a chord in which the given $\angle m$ may be inscribed.

The center of the inscribed ⊙ is the intersection of the bisectors AO and BO of the $\measuredangle BAC$ and CBA.

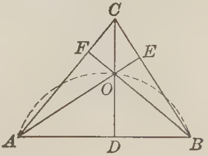
Now $\angle C = 180° - (\angle A + \angle B)$,

and $\angle O = 180° - (\frac{1}{2}\angle A + \frac{1}{2}\angle B)$ § 107

$= 90° + \frac{1}{2}\angle C$.

Therefore $\angle O$ is constant, and O lies on the arc of the segment on AB as a chord in which may be inscribed an $\angle$ equal to $90°$ + half the given $\angle m$. Q.E.D.

2. *Find the locus of the intersection of the perpendiculars from the three vertices to the opposite sides of a triangle that has a given base and a given angle at the vertex.*

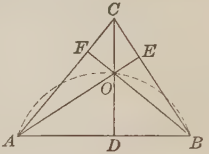


Given the $\triangle ABC$, with base AB and the $\angle$ at the vertex equal to the $\angle m$, and AE, BF, and CD drawn from the vertices $\perp$ to the opposite sides and intersecting at O.

Required to find the locus of O.

Construction. On AB as a chord describe the segment of a ⊙ in which may be inscribed an ∠ equal to the supplement of the given vertical ∠ of the △. § 247

The arc of this segment is the locus required. Q.E.F.



Proof. In the quadrilateral $OECF$, the sum of the $\measuredangle s = 4$ rt. $\measuredangle s$. § 144

But $\angle CEO = \angle OFC = 1$ rt. ∠. Given

$$\therefore \angle EOF = 180^\circ - \angle FCE.$$

But $\angle EOF = \angle AOB$. § 60

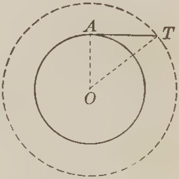
$$\therefore \angle AOB = 180^\circ - \angle FCE. \quad \text{Ax. 8}$$

That is, the $\angle AOB$ is constant and equal to the supplement of the given vertical ∠ of the △. Q.E.D.

3. *Find the locus of the extremity of a tangent to a given circle, if the length of the tangent is equal to a given line.*

Given the ⊙ with center O and AT tangent to the ⊙ at any point A and equal to m.

Required to find the locus of T.



Construction. Describe a ⊙ concentric with the given ⊙ having for radius the distance OT.

The circumference of the outer ⊙ is the locus required. Q.E.F.

Proof. Draw OA and OT.

OA is ⊥ to AT. § 185

$$\therefore \overline{OT}^2 = \overline{AT}^2 + \overline{OA}^2. \quad \text{§ 337}$$

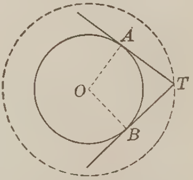
$$\therefore OT = \sqrt{m^2 + \overline{OA}^2}.$$

That is, the length OT is constant and equal to $\sqrt{m^2 + \overline{OA}^2}$. Q.E.D.

4. *Find the locus of a point from which tangents drawn to a given circle form a given angle.*

Given the ⊙ with center O and the $\angle m$.

Required to find the locus of a point from which tangents drawn to the ⊙ form the $\angle m$.



Construction. Draw two radii OA and OB so that $\angle BOA = 180^\circ - \angle m$. § 232

Through B and A draw two tangents intersecting at T. § 228

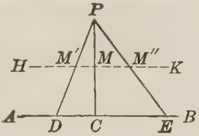
Describe a ⊙ with center O and a radius equal to OT.

Then this ⊙ is the locus required. Q.E.F

Proof. The sum of the ∠s of the quadrilateral $AOBT = 4$ rt. ∠s. § 144
But ∠s A and B are rt. ∠s. § 185
∴ $\angle O$ is the supplement of $\angle T$.

Therefore the ∠ at the center formed by the radii to the points of contact of the tangents from any point of the locus is constant and equal to the supplement of the given ∠. Q.E.D.

5. *Find the locus of the mid-point of a line drawn from a given point to a given line.*



Given the line AB and the external point P, and M, M', and M'' the mid-points of the lines PC, PD, and PE drawn from P to the line AB.

Required to find the locus of M.

Construction. Through M draw $HK \parallel$ to AB. § 233
Then HK is the locus required. Q.E.F.

Proof. Since HK is $\parallel$ to AB, by Const., and also bisects PC, it bisects all other lines from P to AB, by § 135. Q.E.D.

6. *Find the locus of the vertex of a triangle that has a given base and a given altitude.*

The locus of the vertex is evidently a line parallel to the given base at a distance from the base equal to the given altitude. Q.E.F.

7. *Find the locus of a point the sum of whose distances from two given parallel lines is constant.*

Let d be the distance between the given parallel lines, and l the given length.

If $l > d$, the locus consists of two lines parallel to the given parallel lines lying without the given lines, each at a distance $\frac{1}{2}(l - d)$ from the nearer line.

If $l = d$, every point between the lines satisfies the condition.

If $l < d$, there is no locus. Q.E.F.

8. *Find the locus of a point the difference of whose distances from two given parallel lines is constant.*

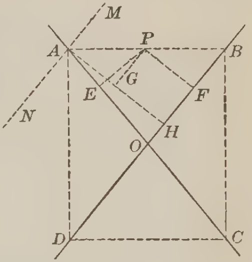
Let d be the distance between the given parallel lines, and l the given length.

If $l < d$, the locus consists of two lines between the given parallel lines and parallel to them each at a distance $\frac{1}{2}(d - l)$ from the nearer line.

If $l = d$, every point in either of the given parallel lines satisfies the condition, and every point outside the given parallel lines satisfies the given condition.

If $l > d$, there is no locus. Q.E.F

9. *Find the locus of a point the sum of whose distances from two given intersecting lines is constant.*



Given the lines AC and BD intersecting at O, and the length l.

Required to find the locus of a point the sum of whose distances from AC and BD is equal to l.

Construction. Draw $MN \parallel$ to BD at a distance equal to l. § 233

Let MN cut AC at A.

Take OB, OC, and OD equal to OA.

Draw AB, BC, CD, DA.

The sides of the quadrilateral $ABCD$ are the required locus. Q.E.F.

Proof. From any point P lying on $ABCD$ draw PE and $PF \perp$ to AC and BD.

Draw $AH \perp$ to BD, and $PG \perp$ to AH.

Then PG is $\parallel$ to BD, and PF is $\parallel$ to GH. § 95

$\therefore PF = GH$. § 127

Also $\angle APG = \angle ABO$. § 102

Since $OA = OB$, Const.

$\therefore \angle OAB = \angle ABO$. § 74

$\therefore \angle OAB = \angle APG$. Ax. 8

Furthermore $AP = AP$. Iden.

$\therefore$ the rt. $\triangle$ AEP and AGP are congruent. § 91

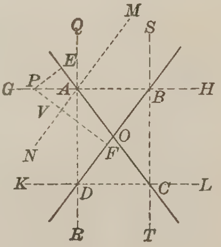
$\therefore PE = AG$. § 67

$\therefore PE + PF = AG + GH = AH$. Ax. 1

Since $AH = l$, § 128

$\therefore PE + PF = l$, by Ax. 9. Q.E.D.

10. *Find the locus of a point the difference of whose distances from two given intersecting lines is constant.*



Given the lines AC and BD intersecting at O, and the length l.

Required to find the locus of a point the difference of whose distances from AC and BD is equal to l.

Construction. Construct the quadrilateral $ABCD$ as in Ex. 9, and prolong each of its sides both ways.

The prolongations of the sides of $ABCD$ are the locus required. Q.E.F.

Proof. From any point P on the constructed locus draw the $\perp$s PE and PF to AC and BD.

Let PF intersect MN in V.

PV is $\perp$ to MN. § 97

Then $\angle PAN = \angle ABO$. § 102

$\angle EAP = \angle OAB$. § 60

But $\angle OAB = \angle ABO$. § 74

$\therefore \angle PAN = \angle EAP$. Ax. 8

Furthermore $PA = PA$. Iden.

$\therefore$ the rt. $\triangle$s PAE and PVA are congruent. § 91

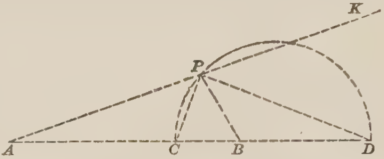
$\therefore PE = PV$. § 67

Now $PF - PV = VF$.

Since $VF = l$, § 128

$\therefore PF - PE = l$, by Ax. 9. Q. E. D.

11. *Find the locus of a point whose distances from two given points are in the ratio* $m : n$.



Given the points A and B, and $m : n$ the given ratio.

Required to find the locus of a point whose distances from A and B are in the ratio $m : n$.

Construction. Divide AB harmonically at C and D in the ratio $m : n$. § 306

Upon CD as diameter describe a $\odot$.

This $\odot$ is the locus required. Q. E. F.

Proof. Suppose P any point not in the same line as A and B, so that $AP : PB = m : n$.

Draw APK and PB.

Bisect the $\angle$s APB and BPK.

These bisectors must divide the line AB harmonically in the ratio $m : n$. § 280

$\therefore$ the bisectors go through C and D.

Now $\angle BPD = \frac{1}{2} \angle BPK$ Supp.

$= \frac{1}{2} (\angle A + \angle PBA)$. § 111

$\angle CPB = \frac{1}{2} \angle APB$. Supp.

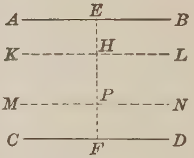
$\therefore \angle BPD + \angle CPB = \frac{1}{2} (\angle A + \angle PBA + \angle APB)$. Ax. 1

But $\angle A + \angle PBA + \angle APB = 180°$. § 107

$\therefore \angle CPD = 90°$.

That is, P must lie on the semicircle upon CD as diameter. Q. E. D.

12. *Find the locus of a point whose distances from two given parallel lines are in the ratio $m : n$.*



Given the $\parallel$ lines AB and CD, and the ratio $m : n$.

Required to find the locus of a point whose distances from AB and CD are in the ratio $m : n$.

Construction. Draw any line $EF \perp$ to AB and CD. § 228

Divide EF at H, so that $EH : HF = m : n$. § 306

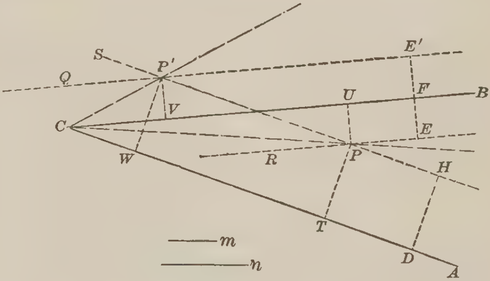
Through H draw $KL \parallel$ to AB. § 233

Again divide EF at P, so that $FP : PE = m : n$. § 306

Through P draw $MN \parallel$ to AB. § 233

Then the line KL or MN is the required locus. Q. E. F.

13. *Find the locus of a point whose distances from two given intersecting lines are in the ratio $m : n$.*



Given the intersecting lines CA and CB, and the ratio $m : n$.

Required to find the locus of a point whose distances from CB and CA are in the ratio $m : n$.

Construction. From any point on CA, as D, draw $DH \perp$ to CA and equal to n. § 228

Through H draw $HS \parallel$ to AC. § 233

From any point on CB, as F, draw $FE \perp$ to CB and equal to m, and also $FE' \perp$ to CB and equal to m.

Through E draw $ER \parallel$ to BC, intersecting HS in P, and through E' draw $E'Q \parallel$ to BC, intersecting HS in P'.

Then the locus required consists of the two straight lines CP and CP'. Q. E. F.

Proof. Draw $PT \perp$ to CA, and $PU \perp$ to CB.

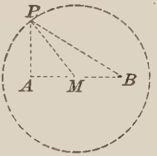
Since $PT = HD = n$, and $PU = EF = m$, the point P is a point of the locus required.

Any other point on CP will also satisfy the required condition; for $\perp$s from this point on CB and CA will, by similarity of $\triangle$s, be as $m:n$.

Therefore CP is one branch of the required locus.

In like manner, CP' may be shown to be another branch. Q. E. D.

14. *Find the locus of a point the sum of the squares of whose distances from two given points is constant.*



Given the points A and B, and P any point of the required locus.

Required to find the locus of P such that $\overline{PA}^2 + \overline{PB}^2$ is constant.

Construction. Draw AB and bisect AB at M. § 229

Draw PM.

With M as center and PM as radius draw a ⊙.

This ⊙ is the locus required. Q. E. F.

Proof. Draw PA and PB.

Now $\overline{PA}^2 + \overline{PB}^2 = 2\,\overline{AM}^2 + 2\,\overline{PM}^2$. § 343

Since AM and PM are constant,

$\therefore\ \overline{PA}^2 + \overline{PB}^2$ is constant. Q. E. D.

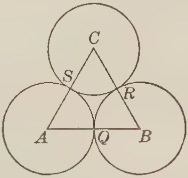
Discussion. Let the sum of the squares be equal to k^2.

If $k^2 < 2\,\overline{AM}^2$, that is, if $k < AM\sqrt{2}$, PM is an imaginary number and there is no locus.

If $k^2 = 2\,\overline{AM}^2$, that is, if $k = AM\sqrt{2}$, $PM = 0$, and the circle is reduced to the point M.

Exercise 67. Page 253

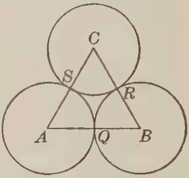
1. *Each side of a triangle is $2n$ centimeters, and about each vertex as a center, with a radius of n centimeters, a circle is described. Find the area bounded by the three arcs that lie outside the triangle, and the area bounded by the three arcs that lie within the triangle.*



Given the equilateral $\triangle ABC$, with each side equal to $2n$ centimeters, and three Ⓢ each with a radius equal to n centimeters described about A, B, C as centers respectively, meeting the sides in Q, R, and S.

Required to find the area bounded by the three arcs that lie outside the $\triangle$, and the area bounded by the three arcs that lie inside the $\triangle$.

Solution. Since the sides of the △ are $2n$ centimeters and the radius of each circle is n centimeters, the three Ⓢ are tangent each to the other two.



Each of the ⦣ of the △ is 60°.

∴ the area of the sector QAS

$= \frac{1}{6}$ of the ⊙ § 391

$= \frac{1}{6}$ of πn^2 § 389

$= \frac{1}{6}\pi n^2$.

∴ the area of the three sectors

$= 3 \times \frac{1}{6}\pi n^2 = \frac{1}{2}\pi n^2$.

Now the area bounded by the three arcs that lie outside the △ equals the area of the △ plus the area of the three Ⓢ less the area of half of one ⊙; that is, equals the area of the △ plus $2\frac{1}{2}$ times the area of one of the Ⓢ.

The square on the altitude of the △ $= (2n)^2 - n^2 = 3n^2$. § 338

∴ the altitude of the △ $= n\sqrt{3}$.

∴ the area of the △ $= \frac{1}{2}n\sqrt{3} \times 2n = n^2\sqrt{3}$. § 325

Again the area of a ⊙ $= \pi n^2$. § 389

∴ $2\frac{1}{2}$ times the area of a ⊙ $= \frac{5}{2}\pi n^2$.

∴ the area bounded by the three arcs that lie outside the △

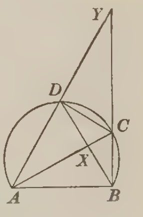
$= n^2\sqrt{3} + \frac{5}{2}\pi n^2 = \frac{1}{2}n^2(2\sqrt{3} + 5\pi)$.

The area bounded by the three arcs inside the △

= area of △ − area of the three sectors

$= n^2\sqrt{3} - \frac{1}{2}\pi n^2 = \frac{1}{2}n^2(2\sqrt{3} - \pi)$. Q. E. F.

2. *Upon a line AB a segment of a circle containing 240° is constructed, and in the segment any chord CD subtending an arc of 60° is drawn. Find the locus of the intersection of AC and BD, and also of the intersection of AD and BC.*



Given upon AB the segment of a ⊙ containing 240°, and in the segment any chord CD subtending an arc of 60°.

Required to find the locus of X, the point of intersection of AC and BD, and also of Y, the point of intersection of AD and BC produced.

Construction. Upon AB as diameter describe a semicircle.

This semicircle is the locus of X.

Again upon AB as chord describe the segment of a ⊙ in which an ∠ of 30° may be inscribed. § 247

This segment is the locus of Y. Q. E. F.

Proof. Arc $AB = 360° - 240° = 120°$.

$\angle AXB$ is measured by $\frac{1}{2}$ (arc AB + arc CD). § 219

$\therefore \angle AXB = \frac{1}{2}(120° + 60°) = 90°$. Ax. 9

$\therefore$ the locus of X is a semicircle upon AB as diameter.

Again $\angle AYB$ is measured by $\frac{1}{2}$ (arc AB − arc CD). § 221

$\therefore \angle AYB = \frac{1}{2}(120° - 60°) = 30°$. Ax. 9

$\therefore$ the locus of Y is the segment described upon AB as chord, in which an $\angle$ of 30° may be inscribed. Q.E.D.

3. *Three successive vertices of a regular octagon are A, B, and C. If the length AB is a, compute the length AC.*

AC is the side of the square inscribed in the same ⊙ in which the octagon may be inscribed.

$$\therefore a = \sqrt{2 - \sqrt{4 - \overline{AC}^2}}. \qquad \text{§ 403}$$

$$\therefore a^2 = 2 - \sqrt{4 - \overline{AC}^2}.$$

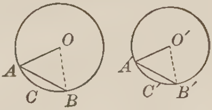
$$\sqrt{4 - \overline{AC}^2} = 2 - a^2.$$

$$4 - \overline{AC}^2 = 4 - 4a^2 + a^4.$$

$$\overline{AC}^2 = 4a^2 - a^4.$$

$$\therefore AC = a\sqrt{4 - a^2}.$$

4. *The areas of similar segments of circles are proportional to the squares on their radii.*



Given the two similar segments ABC and $A'B'C'$ of the ⓢ whose radii are OA and $O'A'$ respectively.

To prove that $ABC : A'B'C' = \overline{OA}^2 : \overline{O'A'}^2$.

Proof. Draw the radii OB and $O'B'$.

In angle measure arc ACB = arc $A'C'B'$. § 387

$\therefore$ sector AOB : sector $A'O'B' = \overline{OA}^2 : \overline{O'A'}^2$. § 390

The ▲ AOB and $A'O'B'$ are similar. § 288

$\therefore \triangle AOB : \triangle A'O'B' = \overline{OA}^2 : \overline{O'A'}^2$. § 333

$\therefore$ sector AOB : sector $A'O'B' = \triangle AOB : \triangle A'O'B'$. Ax. 8

$\therefore$ sector $AOB : \triangle AOB$ = sector $A'O'B' : \triangle A'O'B'$. § 265

$\therefore$ sector $AOB - \triangle AOB$: sector $A'O'B' - \triangle A'O'B' = \triangle AOB : \triangle A'O'B'$. § 268

$\therefore$ segment ABC : segment $A'B'C' = \triangle AOB : \triangle A'O'B'$.

$\therefore$ segment ABC : segment $A'B'C' = \overline{OA}^2 : \overline{O'A'}^2$, by Ax. 9. Q.E.D.

5. *An arc of a certain circle is 100 ft. long and subtends an angle of 25° at the center. Compute the radius of the circle correct to one decimal place.*

$$c = \tfrac{360}{25} \text{ of } 100 \text{ ft.} = 1440 \text{ ft.}$$

$$c = 2\pi r. \qquad \S\ 385$$

$$\therefore\ r = \frac{c}{2\pi} = \frac{1440}{2 \times 3.1416} \text{ ft.} = 229.2 \text{ ft.}$$

6. *Given a circle whose radius is 16, find the perimeter and the area of the regular inscribed octagon.*

Given the ⊙ with radius 16, AC the side of the inscribed square, and AB the side of the regular inscribed octagon.

Required to find the perimeter and the area of the regular inscribed octagon.

Solution. $\angle AOC = 90°$.

$$\therefore\ \overline{AC}^2 = 16^2 + 16^2. \qquad \S\ 337$$

$$\therefore\ AC = \sqrt{16^2 + 16^2} = \sqrt{256 + 256} = \sqrt{512}.$$

$$\begin{aligned} AB &= \sqrt{r\left(2r - \sqrt{4r^2 - s^2}\right)} \qquad \S\ 402 \\ &= \sqrt{16\left(2 \times 16 - \sqrt{4 \times 16^2 - 512}\right)} \\ &= \sqrt{16\left(32 - \sqrt{1024 - 512}\right)} \\ &= \sqrt{16\left(32 - \sqrt{512}\right)} \\ &= \sqrt{16\,(32 - 22.623)} \\ &= \sqrt{16 \times 9.377} \\ &= \sqrt{150.032} \\ &= 12.248 \end{aligned}$$

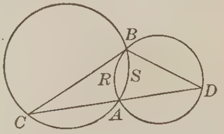
$$\therefore\ p = 8 \times 12.248 = 97.984.$$

$$\begin{aligned} \text{Apothem of octagon} &= \sqrt{r^2 - (\tfrac{1}{2}a)^2} \\ &= \sqrt{16^2 - 6.124^2} = \sqrt{256 - 37.503376} \\ &= \sqrt{218.496624} = 14.782. \end{aligned}$$

$$\begin{aligned} s &= \tfrac{1}{2}ap \qquad \S\ 386 \\ &= \tfrac{1}{2} \times 14.782 \times 97.984 \\ &= 724.199744, \text{ or } 724.2. \end{aligned}$$

Q. E. F.

7. *If two circles intersect at the points A and B, and through A a variable secant is drawn, cutting the circles in C and D, the angle CBD is constant for all positions of the secant.*



Given the two Ⓢ intersecting at A and B, the secant CD drawn through A, and the chords BC and BD drawn.

To prove that the $\angle CBD$ is constant.

Proof. Now $\angle C$ is measured by $\frac{1}{2}$ arc ASB,

and $\angle D$ is measured by $\frac{1}{2}$ arc ARB. § 214

Since the arcs ASB and ARB are constant, the $\angle C$ and $\angle D$ are constant.

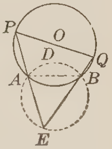
Now $\angle C + \angle D + \angle CBD = 180°$. § 107

$\therefore \angle CBD = 180° - \angle C - \angle D$. Ax. 2

Since $\angle C$ and $\angle D$ are constant,

$\therefore \angle CBD$ is constant. Q.E.D.

8. *If A and B are two fixed points on a given circle, and P and Q are the extremities of a variable diameter of the same circle, find the locus of the point of intersection of the lines AP and BQ.*



Given A and B, two fixed points on the ⊙ whose center is O; PQ, a variable diameter of the same ⊙; and the chords AP and BQ.

Required to find the locus of the point of intersection of the chords AP and BQ.

Construction. Draw the chord AB.

Produce PA and QB to intersect in E.

Circumscribe the ⊙ ABE about the △ ABE. § 240

Then the ⊙ ABE is the locus required. Q.E.F.

Proof. Case 1. When the lines AP and BQ intersect without the given ⊙.

The $\angle E$ is constant, being always measured by half the arc AB less than half a semicircumference. § 221

Therefore the vertex of $\angle E$ must lie on the arc AEB of the ⊙ constructed.

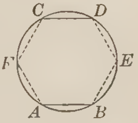
Case 2. When the lines AP and BQ intersect within the given ⊙.

The $\angle E$ is constant, being always measured by half the arc AB greater than half a semicircumference. § 219

Therefore the vertex of $\angle E$ must lie on the arc ADB of the ⊙ constructed.

Hence the ⊙ ABE is the locus of the point of intersection of the lines AP and BQ. Q.E.D.

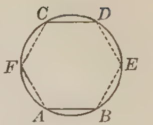
9. *The radius of a circle is 10 ft. Two parallel chords are drawn, each equal to the radius. Find that part of the area of the circle lying between the parallel chords.*



Given the ⊙ with radius 10 ft., and the ∥ chords AB and CD each equal to the radius.

To find the area of $ABDC$.

Solution. Since $AB = CD =$ the radius, AB and CD are sides of the regular inscribed hexagon. § 394



Complete the regular inscribed hexagon *CDEBAF*.

Then the six segments on the chords are all equal.

Now area of ⊙ = area of hexagon + area of 6 segments.

∴ area hexagon + area of 4 segments = area ⊙ − area of 2 segments.

Apothem of regular hexagon $= \sqrt{10^2 - 5^2}$ ft. § 338

$= \sqrt{75}$ ft. $= 8.66$ ft.

Area of hexagon $= \frac{1}{2}ap$ § 386

$= \frac{1}{2}$ of $8.66 \times 6 \times 10$ sq. ft.

$= 259.80$ sq. ft.

Area of circle $= \pi r^2$ § 389

$= 3.1416 \times 10^2$ sq. ft.

$= 314.16$ sq. ft.

∴ area 6 equal segments $= 314.16$ sq. ft. $- 259.80$ sq. ft.

$= 54.36$ sq. ft.

∴ area 2 equal segments $= \frac{1}{3}$ of 54.36 sq. ft. $= 18.12$ sq. ft.

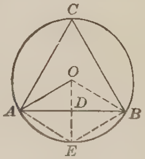
∴ area *ABDC* = area ⊙ − area of 2 segments

$= 314.16$ sq. ft. $- 18.12$ sq. ft.

$= 296.04$ sq. ft. Q. E. F.

Exercise 68. Page 254

1. *If r denotes the radius of a circle and s one side of an inscribed equilateral triangle, prove that* $s = r\sqrt{3}$, *and find the value of s to two decimal places when* $r = 1$.



Given the equilateral △*ABC* inscribed in the ⊙ whose center is *O*.

To prove that $s = r\sqrt{3}$.

Proof. Here $OA = r$, $AB = s$.

Draw the radius *OE* ⊥ to *AB*, and draw *OB*, *AE*, and *BE*.

Then *D* is the mid-point of the chord *AB*, § 174

and *E* is the mid-point of the arc *AB*. § 174

∴ $AO = AE = BO = BE$. § 394

∴ *AB* and *OE* bisect each other at *D*. § 151

∴ $\overline{AD}^2 = \overline{AO}^2 - \overline{OD}^2$. § 338

That is, $(\frac{1}{2}s)^2 = r^2 - (\frac{1}{2}r)^2$,

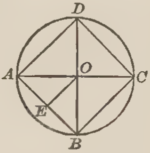
∴ $\frac{1}{4}s^2 = \frac{3}{4}r^2$.

∴ $s^2 = 3r^2$.

∴ $s = r\sqrt{3}$. Q. E. D.

When $r = 1$, $s = \sqrt{3} = 1.73$, correct to two decimal places.

2. *If r denotes the radius of a circle and s one side of an inscribed square, prove that $s = r\sqrt{2}$, and find the value of s to two decimal places when $r = 1$.*



Given the square $ABCD$ inscribed in the ⊙ whose center is O.

To prove that $s = r\sqrt{2}$.

Proof. Here $OA = r$, and $AB = s$.

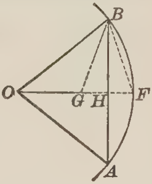
$OA = \frac{1}{2} AB\sqrt{2}$. § 339

$\therefore AB = OA\sqrt{2}$.

That is, $s = r\sqrt{2}$. Q. E. D.

When $r = 1$, $s = \sqrt{2} = 1.41$, correct to two decimal places.

3. *If r denotes the radius of a circle and s one side of a regular inscribed pentagon, prove that $s = \frac{1}{2} r\sqrt{10 - 2\sqrt{5}}$, and find the value of s to two decimal places when $r = 1$.*



Given the regular pentagon $ABCDE$ inscribed in the ⊙ whose center is O.

To prove that $s = \frac{1}{2} r\sqrt{10 - 2\sqrt{5}}$.

Proof. Let AB be one side of the pentagon.

Then $AB = s$, and $OB = r$.

Let FB be one side of the inscribed decagon.

Then $FB = s'$, and $OF = r$.

Let OF intersect AB in H.

On OF take $OG = FB$, and draw BG.

Then $OG = BG = FB = s'$. § 397

Now $\overline{BH}^2 = \overline{BG}^2 - \overline{GH}^2$. § 338

$\therefore (\frac{1}{2} s)^2 = s'^2 - \left(\frac{r - s'}{2}\right)^2$. Ax. 9

$\therefore s^2 = 4s'^2 - r^2 + 2rs' - s'^2$

$= 3s'^2 - r^2 + 2rs'$.

Since $s' = \frac{1}{2} r(\sqrt{5} - 1)$, Ex. 4, p. 245

$\therefore s^2 = \frac{3}{4} r^2(6 - 2\sqrt{5}) - r^2 + r^2(\sqrt{5} - 1)$ Ax. 9

$= r^2(\frac{5}{2} - \frac{1}{2}\sqrt{5})$

$= \frac{r^2(10 - 2\sqrt{5})}{4}$.

$\therefore s = \frac{1}{2} r\sqrt{10 - 2\sqrt{5}}$. Q. E. D.

When $r = 1$, $s = \frac{1}{2}\sqrt{10 - 2\sqrt{5}}$

$= \frac{1}{2}\sqrt{10 - 2 \times 2.236068}$

$= \frac{1}{2}\sqrt{10 - 4.472136} = \frac{1}{2}\sqrt{5.527864}$

$= \frac{1}{2}$ of 2.351

$= 1.18$, correct to two decimal places.

4. *If r denotes the radius of a circle and s one side of a regular inscribed hexagon, prove that $s = r$, and find the value of s when $r = 1$.*

Given the regular hexagon $ABCDEF$ inscribed in the ⊙ whose center is O.

To prove that $s = r$.

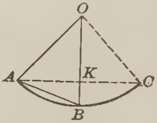
Proof. Here $OA = r$, and $AB = s$.

Now $AB = OA$. § 394

$\therefore s = r$, by Ax. 9. Q.E.D.

When $r = 1$, $s = 1$.

5. *If r denotes the radius of a circle and s one side of a regular inscribed octagon, prove that $s = r\sqrt{2 - \sqrt{2}}$, and find the value of s to two decimal places when $r = 1$.*



Given AB, the side of a regular octagon inscribed in the ⊙ whose center is O.

To prove that $s = r\sqrt{2 - \sqrt{2}}$.

Proof. Here $OA = r$, and $AB = s$.

Draw AC, the side of the inscribed square, cutting OB in K.

Then $AC = r\sqrt{2}$. Ex. 2

$\therefore AK = \frac{1}{2} r\sqrt{2}$.

Now $\overline{AB}^2 = \overline{OA}^2 + \overline{OB}^2 - 2\,OB \times OK$. § 341

$\therefore s^2 = r^2 + r^2 - 2\,r \times OK$,

or $s^2 = 2\,r^2 - 2\,r \times OK$.

But $\overline{OK}^2 = \overline{OA}^2 - \overline{AK}^2$. § 338

$\therefore \overline{OK}^2 = r^2 - (\frac{1}{2} r\sqrt{2})^2$

$= r^2 - \frac{1}{2} r^2 = \frac{1}{2} r^2$.

$\therefore OK = \sqrt{\frac{1}{2} r^2} = \frac{1}{2} r\sqrt{2}$.

$\therefore s^2 = 2\,r^2 - 2\,r(\frac{1}{2} r\sqrt{2})$

$= 2\,r^2 - r^2\sqrt{2}$

$= r^2(2 - \sqrt{2})$.

$\therefore s = r\sqrt{2 - \sqrt{2}}$. Q.E.D.

When $r = 1$, $s = \sqrt{2 - \sqrt{2}}$

$= \sqrt{2 - 1.4142} = \sqrt{0.5858}$

$= 0.77$, correct to two decimal places.

6. *If r denotes the radius of a circle and s one side of a regular inscribed decagon, prove that $s = \frac{1}{2} r(\sqrt{5} - 1)$, and find the value of s to two decimal places when $r = 1$.*

Given AB, the side of a regular decagon inscribed in the ⊙ whose center is O.

To prove that $s = \frac{1}{2} r(\sqrt{5} - 1)$.

Proof. Here $OA = r$, and $AB = s$.

$$OA : AB = AB : OA - AB. \qquad \text{§ 397}$$

That is, $r : s = s : r - s.$

$$\therefore\ s^2 = r^2 - rs. \qquad \text{§ 261}$$

$$\therefore\ s^2 + rs = r^2.$$

$$\therefore\ 4s^2 + 4rs + r^2 = 5r^2.$$

$$\therefore\ 2s + r = r\sqrt{5}.$$

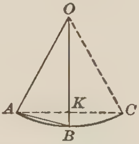
$$\therefore\ 2s = r\sqrt{5} - r.$$

$$\therefore\ s = \tfrac{1}{2}r(\sqrt{5} - 1). \qquad \text{Q.E.D.}$$

When $r = 1$,

$$\begin{aligned} s &= \tfrac{1}{2}(\sqrt{5} - 1) \\ &= \tfrac{1}{2}(2.236 - 1) \\ &= \tfrac{1}{2} \text{ of } 1.236 \\ &= 0.62, \text{ correct to two decimal places.} \end{aligned}$$

7. *If r denotes the radius of a circle and s one side of a regular inscribed dodecagon, prove that* $s = r\sqrt{2 - \sqrt{3}}$, *and find the value of s to two decimal places when* $r = 1$.



Given AB, the side of a regular dodecagon inscribed in a ⊙ whose center is O.

To prove that $s = r\sqrt{2 - \sqrt{3}}.$

Proof. Here $OA = r$, and $AB = s$.

Draw AC, the side of the regular inscribed hexagon, cutting OB in K.

Now $AC = r$, § 394

and $AK = \tfrac{1}{2}AC = \tfrac{1}{2}r.$ § 174

$$\overline{OK}^2 = \overline{OA}^2 - \overline{AK}^2. \qquad \text{§ 338}$$

That is, $\overline{OK}^2 = r^2 - (\tfrac{1}{2}r)^2 = \tfrac{3}{4}r^2$

$$\therefore\ OK = \tfrac{1}{2}r\sqrt{3}.$$

$$\begin{aligned} BK &= OB - OK \\ &= r - \tfrac{1}{2}r\sqrt{3} \\ &= \tfrac{1}{2}r(2 - \sqrt{3}). \end{aligned}$$

Again, $\overline{AB}^2 = \overline{AK}^2 + \overline{BK}^2.$ § 337

That is,

$$\begin{aligned} s^2 &= (\tfrac{1}{2}r)^2 + [\tfrac{1}{2}r(2 - \sqrt{3})]^2 \\ &= \tfrac{1}{4}r^2 + \tfrac{1}{4}r^2(7 - 4\sqrt{3}) \\ &= \tfrac{1}{4}r^2 + \tfrac{7}{4}r^2 - r^2\sqrt{3} \\ &= r^2(2 - \sqrt{3}). \end{aligned}$$

$$\therefore\ s = r\sqrt{2 - \sqrt{3}}. \qquad \text{Q.E.D.}$$

When $r = 1$,

$$\begin{aligned} s &= \sqrt{2 - \sqrt{3}} \\ &= \sqrt{2 - 1.73205} \\ &= \sqrt{0.26795} \\ &= 0.52, \text{ correct to two decimal places.} \end{aligned}$$

8. *A regular pentagon is inscribed in a circle whose radius is r. If the side is s, find the apothem.*

$$s = \tfrac{1}{2} r \sqrt{10 - 2\sqrt{5}}. \qquad \text{Ex. 3}$$

But

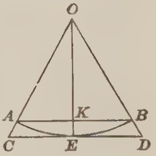
$$\begin{aligned} a^2 &= r^2 - (\tfrac{1}{2} s)^2 && \text{§ 338} \\ &= r^2 - \left(\tfrac{1}{4} r \sqrt{10 - 2\sqrt{5}}\right)^2 \\ &= r^2 - \tfrac{1}{16} r^2 (10 - 2\sqrt{5}) \\ &= \tfrac{1}{16} r^2 (6 + 2\sqrt{5}). \\ \therefore\ a &= \tfrac{1}{4} r \sqrt{6 + 2\sqrt{5}}. \end{aligned}$$

9. *A regular polygon is inscribed in a circle whose radius is r. If the side is s, show that the apothem is* $\frac{1}{2}\sqrt{4r^2 - s^2}$.

Now

$$\begin{aligned} a^2 &= r^2 - (\tfrac{1}{2} s)^2 && \text{§ 338} \\ &= r^2 - \tfrac{1}{4} s^2. \\ \therefore\ 4a^2 &= 4r^2 - s^2. \\ \therefore\ 2a &= \sqrt{4r^2 - s^2}. \\ \therefore\ a &= \tfrac{1}{2}\sqrt{4r^2 - s^2}. \end{aligned}$$

10. *If the radius of a circle is r, and the side of an inscribed regular polygon is s, show that the side of the similar circumscribed regular polygon is* $\dfrac{2sr}{\sqrt{4r^2 - s^2}}$.



Given AB, the side of a regular polygon inscribed in the ⊙ with center O, and CD, the side of the similar circumscribed regular polygon.

To prove that

$$CD = \frac{2sr}{\sqrt{4r^2 - s^2}}.$$

Proof. Here

$$\begin{aligned} AB &= s,\ OA = r. \\ \overline{OK}^2 &= \overline{OA}^2 - \overline{AK}^2. && \text{§ 338} \\ \therefore\ OK &= \sqrt{\overline{OA}^2 - \overline{AK}^2} \\ &= \sqrt{r^2 - (\tfrac{1}{2} s)^2} \\ &= \tfrac{1}{2}\sqrt{4r^2 - s^2}. \end{aligned}$$

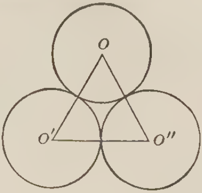
But $OK : OE = AB : CD.$ § 373

That is, $\frac{1}{2}\sqrt{4r^2 - s^2} : r = s : CD.$

$$\therefore\ CD = \frac{2sr}{\sqrt{4r^2 - s^2}}, \text{ by § 261.}$$

Q. E. D.

11. *Three equal circles are described, each tangent to the other two. If the common radius is r, find the area contained between the circles.*



Given three ⓢ of radius r, each tangent to the other two and with centers O, O', and O''.

To find the area contained between the ⓢ.

Solution. Draw OO', $O'O''$, and $O''O$.

The $\triangle OO'O''$ is equilateral, each side being $2r$.

$$\text{Altitude of } \triangle OO'O'' = \sqrt{(2r)^2 - r^2}$$
$$= \sqrt{3r^2} = r\sqrt{3}.$$

$$\therefore \text{ area of } \triangle OO'O'' = r \times r\sqrt{3} = r^2\sqrt{3}. \qquad \S\,325$$

The area of each sector cut out of each ⊙ by the $\triangle$

$$= \tfrac{1}{6} \text{ of the } \odot = \tfrac{1}{6}\pi r^2. \qquad \S\,391$$

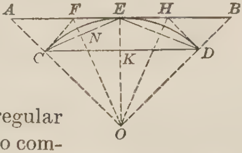
$$\therefore \text{ the area of the three sectors} = \tfrac{1}{2}\pi r^2.$$

But the area contained between the ⓢ is the area of the $\triangle$ diminished by the area of the three sectors.

That is, the area contained between the ⓢ

$$= r^2\sqrt{3} - \tfrac{1}{2}\pi r^2 = \tfrac{1}{2}r^2(2\sqrt{3} - \pi). \qquad \text{Q.E.F.}$$

12. *Given p, P, the perimeters of regular polygons of n sides inscribed in and circumscribed about a given circle. Find p', P', the perimeters of regular polygons of $2n$ sides inscribed in and circumscribed about the given circle.*



Given AB, one side of the circumscribed polygon, and CD, ∥ to AB, one side of the similar inscribed polygon.

To construct inscribed and circumscribed regular polygons of double the number of sides, and to compute their perimeters p', P' in terms of the perimeters p, P of the given polygons.

Construction. Draw $OE \perp$ to AB, and draw AO, BO, CE, DE.

Draw $CF \perp$ to AO, and $DH \perp$ to BO.

Then FH is a side of the circumscribed polygon required, and CE and DE sides of the inscribed polygon required. Q.E.F.

Proof. C and D lie in AO and BO respectively. § 369

$\therefore$ arc CE = arc DE. § 174

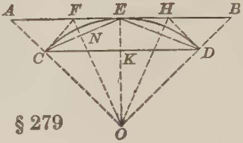
$\therefore$ $CE = DE$. § 170

$\therefore$ CE and DE are sides of the required inscribed polygon. § 370

CF and DH are tangent to the ⊙. § 184

$\therefore$ FH is a side of the required circumscribed polygon, by § 368. Q.E.D.

Computation. $\frac{P}{p}=\frac{OA}{OE}.$ § 375



Now the rt. $\triangle$s OFC and OFE are congruent. § 89

For $FO = FO$ and $OC = OE$. § 162

$\therefore FO$ bisects the $\angle COE$.

$\therefore \frac{OA}{OE}=\frac{AF}{EF}.$ § 279

$\therefore \frac{P}{p}=\frac{AF}{EF}.$ Ax. 8

$\therefore \frac{P+p}{p}=\frac{AF+EF}{EF}.$ § 267

$\therefore \frac{P+p}{2p}=\frac{AE}{2\,EF}=\frac{AE}{FH}.$

But AE is contained as often in P as FH is in P'.

$\therefore \frac{AE}{FH}=\frac{P}{P'}.$

$\therefore \frac{P+p}{2p}=\frac{P}{P'}.$ Ax. 1

Whence $P'=\frac{2pP}{P+p}.$

Also, the rt. $\triangle$s CKE and ENF are similar. § 287

For $\angle KCE = \angle FEN.$ § 100

$\therefore \frac{CK}{CE}=\frac{EN}{EF}.$ § 282

But $\frac{CK}{CE}=\frac{p}{p'},$ Ax. 3

and $\frac{EN}{EF}=\frac{p'}{P'}.$ Ax. 3

$\therefore \frac{p}{p'}=\frac{p'}{P'}.$ Ax. 8

Whence $p'=\sqrt{p\times P'}.$ § 261

$\therefore p'=\sqrt{\frac{2p^2P}{P+p}}=p\sqrt{\frac{2P}{P+p}}.$ Q. E. F.

13. *A circular plot of land d ft. in diameter is surrounded by a walk w ft. wide. Find the area of the circular plot and the area of the walk.*

The radius of the circular plot of land is $\frac{1}{2}$ of d ft., or $\frac{1}{2}d$ ft.

$\therefore$ the area of the circular plot of land is $\pi(\frac{1}{2}d)^2$ sq. ft., or $\frac{1}{4}\pi d^2$ sq. ft

The radius of the plot of land and walk together is $(\frac{1}{2}d + w)$ ft.

$\therefore$ the area of the plot of land and walk together is $\pi(\frac{1}{2}d+w)^2$ sq. ft., cr $\pi(\frac{1}{4}d^2 + dw + w^2)$ sq. ft.

$$\therefore \text{ the area of the walk} = \pi(\tfrac{1}{4}d^2 + dw + w^2) \text{ sq. ft.} - \tfrac{1}{4}\pi d^2 \text{ sq. ft.}$$
$$= \pi w(d + w) \text{ sq. ft.}$$

Exercise 69. Page 255

1. *The diameter of a bicycle wheel is 28 in. How many revolutions does the wheel make in going 10 mi.?*

$$c = 2\pi r = \pi d = 3.1416 \times 28 \text{ in.}$$
$$10 \text{ mi.} = 10 \times 5280 \text{ ft.} = 10 \times 5280 \times 12 \text{ in.}$$
$$\text{Number of revolutions} = \frac{10 \times 5280 \times 12}{3.1416 \times 28} = 7202.9.$$

2. *Find the diameter of a carriage wheel that makes 264 revolutions in going half a mile.*

$$264\,c = \tfrac{1}{2} \text{ of } 5280 \text{ ft.}$$
$$264\,c = 2640 \text{ ft.}$$
$$\therefore c = 10 \text{ ft.}$$
$$c = \pi d.$$
$$\therefore d = \frac{c}{\pi} = \frac{10}{3.1416} \text{ ft.} = 3.1831 \text{ ft.}$$

3. *A circular pond 100 yd. in diameter is surrounded by a walk 10 ft. wide. Find the area of the walk.*

$$r = \tfrac{1}{2}d = \tfrac{1}{2} \text{ of } 100 \text{ yd.} = 50 \text{ yd.} = 150 \text{ ft.}$$

Area of pond $= \pi r^2$ § 389

$= 3.1416 \times 150^2$ sq. ft.

$= 3.1416 \times 22{,}500$ sq. ft.

Area of pond and walk

$= 3.1416 \times 160^2$ sq. ft.

$= 3.1416 \times 25{,}600$ sq. ft.

$\therefore$ area of walk $= 3.1416 \times 25{,}600$ sq. ft. $- 3.1416 \times 22{,}500$ sq. ft.

$= 3.1416\,(25{,}600 - 22{,}500)$ sq. ft.

$= 3.1416 \times 3100$ sq. ft.

$= 9738.96$ sq. ft.

4. *The span (chord) of a bridge in the form of a circular arc is 120 ft., and the highest point of the arch is 15 ft. above the piers. Find the radius of the arc.*

Here $AB = 120$ ft. and $CD = 15$ ft.

Now $\overline{AC}^2 = 2\,OC \times CD.$ § 298

$$\begin{aligned} \text{But} \qquad \overline{AC}^2 &= \overline{AD}^2 + \overline{CD}^2 && \S\ 337 \\ &= 60^2 + 15^2 \\ &= 3825. \\ \therefore\ 2\,OC \times CD &= 3825. \\ \text{That is,} \qquad 2\,r \times 15 &= 3825. \\ 2\,r &= 255. \\ \therefore\ r &= 127.5. && 127.5 \text{ ft. } \textit{Ans.} \end{aligned}$$

5. *Two branch water pipes lead into a main pipe. It is necessary that the cross section area of the main pipe shall equal the sum of the cross sections of the two branch pipes. The diameters of the branch pipes are respectively* 3 *in. and* 4 *in. Required the diameter of the main pipe.*

Area of cross section of one pipe $= \pi \times (\frac{3}{2})^2$,

and area of cross section of other pipe $= \pi \times (\frac{4}{2})^2$. § 389

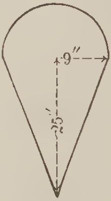
Again, area of cross section of main pipe $= \pi r^2 = \frac{1}{4}\pi d^2$.

$$\begin{aligned} \therefore\ \tfrac{1}{4}\pi d^2 &= \pi \times (\tfrac{3}{2})^2 + \pi \times (\tfrac{4}{2})^2 \\ &= \tfrac{9}{4}\pi + \tfrac{16}{4}\pi \\ &= \tfrac{25}{4}\pi. \\ \therefore\ d^2 &= \sqrt{25}. \\ \therefore\ d &= 5. && 5 \text{ in. } \textit{Ans.} \end{aligned}$$

6. *A kite is made as here shown, the semicircle having a radius of* 9 *in. and the triangle a height of* 25 *in. Find the area of the kite.*

Area of whole circle $= \pi r^2$. § 389

$$\begin{aligned} \text{Area of semicircle} &= \tfrac{1}{2}\pi r^2 = \tfrac{1}{2} \times 3.1416 \times 9^2 \text{ sq. in.} \\ &= \tfrac{1}{2} \times 3.1416 \times 81 \text{ sq. in.} \\ &= 127.2348 \text{ sq. in.} \\ \text{Area of triangle} &= \tfrac{1}{2}\,ab && \S\ 325 \\ &= \tfrac{1}{2} \text{ of } 25 \times 18 \text{ sq. in.} \\ &= 225 \text{ sq. in.} \\ \therefore\ \text{area kite} &= 127.23 \text{ sq. in.} + 225 \text{ sq. in.} \\ &= 352.23 \text{ sq. in.} \end{aligned}$$



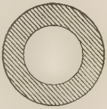
7. *In making a drawing for an arch it is required to mark off on a circle drawn with a radius of* 5 *in. an arc that shall be* 8 *in. long. This is best done by finding the angle at the center. How many degrees are there in this angle?*

$$\begin{aligned} c &= 2\,\pi r && \S\ 385 \\ &= 2 \times 3.1416 \times 5 \text{ in.} = 31.416 \text{ in.} \end{aligned}$$

$$31.416 : 8 = 360^\circ : \angle \text{ required.} \qquad \S\ 212$$

$$\therefore\ \angle \text{required} = \frac{8 \times 360^\circ}{31.416} = 91.67^\circ.$$

8. *In an iron washer here shown, the diameter of the hole is $1\frac{3}{8}$ in. and the width of the washer is $\frac{3}{8}$ in. Find the area of one face of the washer.*



$$\begin{aligned} \text{Area of circle} &= \pi r^2 && \S\ 389 \\ &= \pi\left(\tfrac{1}{2}\,d\right)^2 = \tfrac{1}{4}\,\pi d^2. \\ \text{Area of hole} &= \tfrac{1}{4}\,\pi \times \left(1\tfrac{3}{8}\right)^2 \text{ sq. in.} \\ \text{Diameter of washer} &= 1\tfrac{3}{8}\text{ in.} + 2 \times \tfrac{3}{8}\text{ in.} = 2\tfrac{1}{8}\text{ in.} \\ \text{Area of hole and washer} &= \tfrac{1}{4}\,\pi \times \left(2\tfrac{1}{8}\right)^2 \text{ sq. in.} \\ \therefore \text{ area of washer} &= \tfrac{1}{4}\,\pi \times \left(2\tfrac{1}{8}\right)^2 \text{ sq. in.} - \tfrac{1}{4}\,\pi \times \left(1\tfrac{3}{8}\right)^2 \text{ sq. in.} \\ &= \tfrac{1}{4}\,\pi\left(\tfrac{289}{64} - \tfrac{121}{64}\right) \text{ sq. in.} \\ &= \tfrac{1}{4} \times 3.1416 \times \tfrac{168}{64} \text{ sq. in.} \\ &= 0.7854 \times \tfrac{21}{8} \text{ sq. in.} \\ &= 2.0617 \text{ sq. in.} && 2.06 \text{ sq. in. } \textit{Ans.} \end{aligned}$$

9. *Find the area of a fan that opens out into a sector of 120°, the radius being $9\frac{3}{8}$ in.*

$$\begin{aligned} \text{Area of circle} &= \pi r^2 && \S\ 389 \\ &= 3.1416 \times \left(9\tfrac{3}{8}\right)^2 \text{ sq. in.} \\ \text{Area of sector} &= \tfrac{120}{360} \text{ of } 3.1416 \times \left(9\tfrac{3}{8}\right)^2 \text{ sq. in.} && \S\ 391 \\ &= 92.039 \text{ sq. in.} \end{aligned}$$

10. *The area of a fan that opens out into a sector of 111° is 96.866 sq. in. What is the radius? (Take $\pi = 3.1416$.)*

$$\begin{aligned} \text{Area of circle} &= \pi r^2. && \S\ 389 \\ \text{Area of sector} &= \tfrac{111}{360}\,\pi r^2. && \S\ 391 \\ \therefore 96.866 &= \tfrac{111}{360}\,\pi r^2. \\ \therefore r^2 &= \frac{96.866}{\frac{111}{360} \times 3.1416} = 100. \\ \therefore r &= \sqrt{100} = 10. && 10 \text{ in. } \textit{Ans.} \end{aligned}$$

Exercise 71. Page 257

1. *Write a classification of the different kinds of lines.*

A *straight* line is a line such that if any part is placed with its ends on any other part, it must lie wholly in the line.

A *broken* line is a line made up of two or more different straight lines.

A *curve* line is a line no part of which is straight.

2. *Write a classification of the different kinds of angles.*

A *right* angle is an angle formed by one straight line meeting another straight line so as to make the adjacent angles equal.

A *straight* angle is an angle whose sides extend in opposite directions so as to make a straight line.

A *reflex* angle is an angle greater than a straight angle and less than two straight angles.

An *acute* angle is an angle less than a right angle.

An *obtuse* angle is an angle greater than a right angle.

An *oblique* angle is an angle that is either acute or obtuse.

3. *Write a classification of the different kinds of triangles.*

A *scalene* triangle is a triangle no two sides of which are equal.

An *isosceles* triangle is a triangle two sides of which are equal.

An *equilateral* triangle is a triangle all of whose sides are equal.

A *right* triangle is a triangle one angle of which is a right angle.

An *obtuse* triangle is a triangle one angle of which is an obtuse angle.

An *acute* triangle is a triangle all of whose angles are acute.

An *equiangular* triangle is a triangle all of whose angles are equal.

4. *Write a classification of the different kinds of quadrilaterals.*

A *trapezoid* is a quadrilateral that has two sides parallel.

A *trapezium* is a quadrilateral that has no two sides parallel.

A *parallelogram* is a quadrilateral that has its opposite sides parallel

5. *Write a classification of the different kinds of polygons.*

A *triangle* is a polygon that has three sides.

A *quadrilateral* is a polygon that has four sides.

A *pentagon* is a polygon that has five sides.

A *hexagon* is a polygon that has six sides.

A *heptagon* is a polygon that has seven sides.

An *octagon* is a polygon that has eight sides.

A *nonagon* is a polygon that has nine sides.

A *decagon* is a polygon that has ten sides.

An *undecagon* is a polygon that has eleven sides.

A *dodecagon* is a polygon that has twelve sides.

A *pentadecagon* is a polygon that has fifteen sides.

An *equilateral* polygon is a polygon that has all its sides equal.

An *equiangular* polygon is a polygon that has all its angles equal.

A *convex* polygon is a polygon each angle of which is less than a straight angle.

A *concave* polygon is a polygon that contains an angle greater than a straight angle.

A *regular* polygon is a polygon that is both equiangular and equilateral.

An *inscribed* polygon is a polygon whose sides are all chords of a circle.

A *circumscribed* polygon is a polygon whose sides are all tangent to a circle.

6. *Write a classification of the different kinds of parallelograms.*

A *rhomboid* is a parallelogram that has all its angles oblique.
A *rhombus* is a parallelogram that has all its sides equal.
A *rectangle* is a parallelogram that has all its angles right angles.
A *square* is a rectangle that has all its sides equal.
A *square* is a rhombus that has all its angles right angles.

7. *State the conditions under which two triangles are congruent.*

Two triangles are congruent:

(i) If two sides and the included angle of the one are equal respectively to two sides and the included angle of the other.

(ii) If two angles and the included side of the one are equal respectively to two angles and the included side of the other.

(iii) If the three sides of the one are equal respectively to the three sides of the other.

8. *State the conditions under which two parallelograms are congruent.*

Two parallelograms are congruent if two sides and the included angle of the one are equal respectively to two sides and the included angle of the other.

9. *State the conditions under which two triangles are similar.*

Two triangles are similar:

(i) If they are mutually equiangular.

(ii) If two angles of the one are equal respectively to two angles of the other.

(iii) If an angle of the one is equal to an angle of the other, and the including sides are proportional.

(iv) If their sides are respectively proportional.

(v) If their sides are respectively parallel.

(vi) If their sides are respectively perpendicular.

10. *State the conditions under which two straight lines are parallel.*

Two lines in the same plane are parallel:

(i) If they cannot meet however far produced.

(ii) If they are perpendicular to the same line.

(iii) If they are each parallel to a third line.

(iv) If, when cut by a transversal, the alternate-interior angles are equal.

(v) If, when cut by a transversal, the exterior-interior angles are equal.

(vi) If, when cut by a transversal, the two interior angles on the same side of the transversal are supplementary.

(vii) If, when cut by a transversal, the alternate-exterior angles are equal.

11. *State the conditions under which two parallelograms are equivalent.*

Parallelograms having equal bases and equal altitudes are equivalent.

12. *State the conditions under which two polygons are similar.*

Two polygons are similar:

(i) If they have their corresponding angles equal and their corresponding sides proportional.

(ii) If they are composed of the same number of triangles, similar each to each and similarly placed.

13. *Complete the following statement in the most general manner: In any triangle the square on the side opposite ⋯.*

In any triangle the square on any side is equivalent to the sum of the squares on the other two sides diminished, if the side is opposite an acute angle, and increased, if the side is opposite an obtuse angle, by twice the product of one of those sides by the projection of the other upon that side.

14. *Complete the following statement in the most general manner: If two parallel lines are cut by a transversal, ⋯.*

If two parallel lines are cut by a transversal, the alternate-interior angles are equal, the exterior-interior angles are equal, the two interior angles on the same side of the transversal are supplementary, and the alternate-exterior angles are equal.

15. *Complete the following statement in the most general manner: If four quantities are in proportion, they are in proportion by ⋯.*

If four quantities are in proportion, they are in proportion by alternation, by inversion, by composition, and by division.

16. *Complete the following statement in the most general manner: If two secants of a circle intersect, the angle formed is measured by ⋯.*

The angle included between two lines that cut a circle is measured by half the sum of the intercepted arcs.

17. *Complete the following statement in the most general manner: The perimeters of two similar polygons are to each other as ⋯.*

The perimeters of two similar polygons are to each other as any two corresponding lines.

18. *Complete the following statement in the most general manner: The areas of two similar polygons are to each other as* $\cdots$.

The areas of two similar polygons are to each other as the squares on any two corresponding lines.

19. *Complete the following statement in the most general manner: The area of a circle is equal to* $\cdots$.

The area of a circle is equal to π times the square on its radius.

20. *Complete the following statement in the most general manner: In the same circle or in equal circles equal chords* $\cdots$.

In the same circle or in equal circles equal chords subtend equal arcs and are equidistant from the center.

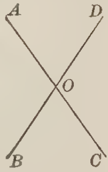
21. *Complete the following statement in the most general manner: In the same circle or in equal circles the central angles subtended by two arcs are* $\cdots$.

In the same circle or in equal circles the central angles subtended by two arcs are equal if the arcs are equal, and are unequal if the arcs are unequal, the greater central angle being subtended by the greater arc.

22. *Complete the following statement in the most general manner: If two secants of a circle intersect within, on, or outside the circle, the product of* $\cdots$.

If two secants of a circle intersect within, on, or outside the circle, the product of the segments of the one is equal to the product of the segments of the other.

23. *If four lines meet in a point so that the opposite angles are equal, these lines form two intersecting straight lines.*



Given the four lines OA, OB, OC, and OD intersecting at O so that $\angle AOB = \angle COD$, and $\angle BOC = \angle DOA$.

To prove that AOC and BOD are straight lines.

Proof. Now $\angle AOB = \angle COD$,

and $\angle BOC = \angle DOA$. Given

$\therefore \angle AOB + \angle BOC = \angle COD + \angle DOA$. Ax. 1

But $\angle AOB + \angle BOC + \angle COD + \angle DOA = 4$ rt. $\angle$s. § 41

$\therefore \angle AOB + \angle BOC + \angle AOB + \angle BOC = 4$ rt. $\angle$s. Ax. 9

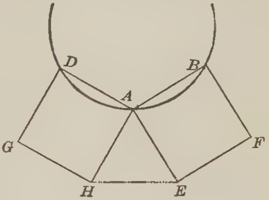
$\therefore 2\angle AOB + 2\angle BOC = 4$ rt. $\angle$s.

$\therefore \angle AOB + \angle BOC = 2$ rt. $\angle$s. Ax. 4

$\therefore AO$ and OC form one straight line. § 43

Similarly BO and OD form one straight line. Q.E.D.

24. *If squares are constructed outwardly on the six sides of a regular hexagon, the exterior vertices of these squares are the vertices of a regular dodecagon.*



Given DA and AB adjacent sides of an inscribed regular hexagon and $ADGH$ and $AEFB$ squares constructed outwardly on DA and AB.

To prove that G, H, E, F are the vertices of a regular dodecagon.

Proof. Draw HE.

Now the $\angle BAD = \frac{2(6-2)}{6}$ rt. $\angle$s. § 145

$= \frac{4}{3}$ rt. $\angle$s $= 120°$.

$\angle HAE = 360° - (\angle BAD + \angle DAH + \angle EAB)$. § 41

$\therefore \angle HAE = 360° - (120° + 90° + 90°) = 60°$.

Again, $AH = AE$, since they are sides of congruent squares.

$\therefore \angle AEH = \angle EHA$. § 74

But $\angle HAE = 60°$.

$\therefore \angle HAE = \angle AEH = \angle EHA$. § 107

$HE = AH = AE$. § 77

$\therefore GH = HE = EF$, and so on. Ax. 8

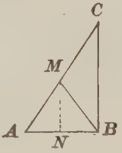
That is, the dodecagon is equilateral.

Now $\angle FEH = \angle EHG$, and so on, since each is the sum of a rt. $\angle$ and an angle of 60°. Ax. 1

That is, the dodecagon is equiangular.

Hence G, H, E, F, and so on, are the vertices of a regular dodecagon, by § 357. Q.E.D.

25. *In a right triangle the line joining the vertex of the right angle to the mid-point of the hypotenuse is equal to half the hypotenuse.*



Given the rt. $\triangle ABC$ and the line MB drawn from M, the mid-point of the hypotenuse.

To prove that $MB = \frac{1}{2} AC$.

Proof. Draw $MN \parallel$ to CB.

Since MN bisects AC, Given

MN bisects AB. § 135

Also MN is $\perp$ to AB, § 97

In the rt. $\triangle$s ANM and BNM,

$MN = MN$, Iden.

and $AN = BN$.

$\therefore$ the rt. $\triangle$s ANM and BNM are congruent. § 90

$\therefore AM = BM.$ § 67

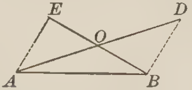
$2\,BM = 2\,AM.$ Ax. 3

But $AM = MC.$ Given

$\therefore 2\,BM = AM + MC = AC.$

$\therefore BM = \frac{1}{2} AC$, by Ax. 4. Q.E.D.

26. *No two lines drawn from the vertices of the base angles of a triangle to the opposite sides can bisect each other.*



Given the line AB and the lines AD and BE drawn from the extremities of AB and bisecting each other at O.

To prove that AB cannot be the base of a $\triangle$ in which E and D are points of the other two sides.

Proof. Draw AE and BD.

In the $\triangle$s AOE and DOB,

$AO = DO,$

$EO = BO,$ Given

and $\angle EOA = \angle BOD.$ § 60

$\therefore \triangle AOE$ is congruent to $\triangle DOB.$ § 68

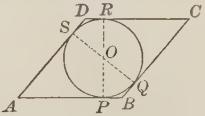
$\therefore \angle AEO = \angle DBO.$ § 67

$\therefore AE$ is $\parallel$ to $BD.$ § 101

Therefore AE and BD can never meet however far produced. § 93

Therefore AB cannot be the base of a $\triangle$ in which E and D are points of the other two sides. Q.E.D.

27. *The rhombus is the only parallelogram that can be circumscribed about a circle.*



Given the $\square\ ABCD$ circumscribed about the $\odot$, the points of contact being P, Q, R, S.

To prove that $ABCD$ is a rhombus.

Proof. $AP = AS$, $BP = BQ$, $CQ = CR$, and $DR = DS.$ § 192

$\therefore AP + BP + CR + DR = AS + DS + BQ + CQ.$ Ax. 1

That is, $AB + CD = DA + BC.$

But $AB = CD$, and $BC = DA.$ § 125

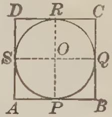
$\therefore AB + AB = BC + BC.$ Ax. 9

$\therefore AB = BC.$ Ax. 4

$\therefore AB = BC = CD = DA.$ Ax. 8

$\therefore ABCD$ is a rhombus, by § 119. Q.E.D.

28. *The square is the only rectangle that can be circumscribed about a circle.*



Given the □ $ABCD$ circumscribed about the ⊙, the points of contact being P, Q, R, S.

To prove that $ABCD$ is a square.

Proof. $AP = AS, BP = BQ, CQ = CR$, and $DR = DS$. § 192

$\therefore AP + BP + CR + DR = AS + DS + BQ + CQ$. Ax. 1

That is, $AB + CD = DA + BC$.

But $AB = CD$, and $BC = DA$. § 125

$\therefore AB + AB = BC + BC$. Ax. 9

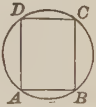
$\therefore AB = BC$. Ax. 4

$\therefore AB = BC = CD = DA$. Ax. 8

Now $\angle$s A, B, C, and D are rt. $\angle$s. § 119

Therefore $ABCD$, having four equal sides and four right angles, is a square, by § 65. Q.E.D.

29. *No oblique parallelogram can be inscribed in a circle.*



Given the ▱ $ABCD$ inscribed in a ⊙.

To prove that $ABCD$ is a □.

Proof. $AB = DC$, and $BC = DA$ § 125

$\therefore$ arc AB = arc DC,

and arc BC = arc DA. § 172

$\therefore$ arc AB + arc DA = arc BC + arc DC. Ax. 1

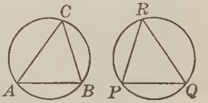
$\therefore$ arc DAB = arc $BCD = 180°$.

$\therefore \angle C = \angle A = 90°$. § 215

Similarly $\angle B = \angle D = 90°$.

$\therefore ABCD$ is a □, by § 119. Q.E.D.

30. *If two triangles have equal bases and equal vertical angles, the two circumscribing circles have equal diameters.*



Given the $\triangle$s ABC and PQR, with base AB equal to base PQ and vertical $\angle C$ equal to vertical $\angle R$, and also the Ⓢ ABC and PQR circumscribed about the $\triangle$s ABC and PQR respectively.

To prove that the Ⓢ ABC and PQR have equal diameters.

Proof. The $\angle C$ is measured by $\frac{1}{2}$ arc AB,

and $\angle R$ is measured by $\frac{1}{2}$ arc PQ. § 214

Since $\angle C = \angle R$,

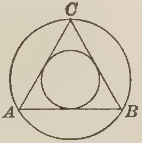
and chord AB = chord PQ, Given

$\therefore$ arc ACB = arc PRQ. § 217

$\therefore$ the Ⓢ ABC and PQR are equal.

$\therefore$ the Ⓢ ABC and PQR have equal diameters. Q.E.D.

31. *If the inscribed and circumscribed circles of a triangle are concentric, the triangle is equilateral.*



Given the $\triangle ABC$ and the inscribed and circumscribed ⓢ concentric.

To prove that $\triangle ABC$ is equilateral.

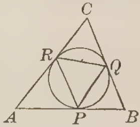
Proof. The sides of the $\triangle ABC$ are tangent to the inner ⊙,

$\therefore$ they are equidistant from the common center. § 162

$\therefore AB = BC = CA$. § 178

$\therefore \triangle ABC$ is equilateral, by § 62. Q.E.D.

32. *If the three points of contact of a circle inscribed in a triangle are joined, the angles of the resulting triangle are all acute.*



Given the $\odot PQR$ inscribed in the $\triangle ABC$, and the $\triangle PQR$ formed by joining P, Q, and R, the three points of contact.

To prove that $\measuredangle RQP$, QPR, and PRQ are all acute.

Proof. Now $CQ = CR$. § 192

$\therefore \angle CQR = \angle QRC$. § 74

$\therefore \angle CQR$ is acute. § 109

Now $\angle CQR$ is measured by $\frac{1}{2}$ arc QR, § 220

and $\angle QPR$ is measured by $\frac{1}{2}$ arc QR. § 214

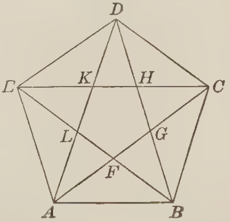
$\therefore \angle QPR = \angle CQR$.

But $\angle CQR$ is acute.

$\therefore \angle QPR$ is acute.

Similarly $\measuredangle RQP$ and PRQ are acute. Q.E.D.

33. *The diagonals of a regular pentagon intersect at the vertices of another regular pentagon.*



Given the regular pentagon $ABCDE$, with its diagonals AC, AD, BD, BE, CE intersecting in the points F, G, H, K, L.

To prove that $FGHKL$ is a regular pentagon.

Proof. In the $\triangle$s ABC, BCD, CDE, DEA, and EAB,

$AB = BC = CD = DE = EA$, § 357

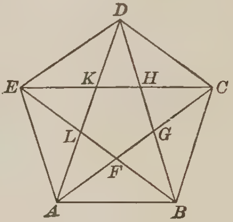
and $\angle CBA = \angle DCB = \angle EDC = \angle AED = \angle BAE$. § 357

$\therefore \triangle$s ABC, BCD, CDE, DEA, and EAB are congruent. § 68

$\therefore AC = BD = CE = DA = EB$. § 67

Circumscribe a ⊙ about the pentagon $ABCDE$.

Then arc $AB =$ arc $BC =$ arc $CD =$ arc $DE =$ arc EA. § 172



In the $\triangle$ ABF and BCG,

$\angle BAF$ is measured by $\frac{1}{2}$ arc BC,

$\angle CBG$ is measured by $\frac{1}{2}$ arc CD,

$\angle FBA$ is measured by $\frac{1}{2}$ arc EA,

and $\angle GCB$ is measured by $\frac{1}{2}$ arc AB, § 214

Since these arcs are all equal,

$\angle BAF = \angle CBG = \angle FBA = \angle GCB$.

$\therefore$ $\triangle$ ABF and BCG are isosceles, § 76

and the $\triangle$ ABF and BCG are congruent. § 72

$\therefore AF = BF = BG = CG$. § 67

Similarly, $AF = BF = BG = CG = CH = DH = DK = EK = EL = AL$.

$\therefore FG = GH = HK = KL = LF$. Ax. 2

$\therefore$ pentagon $FGHKL$ is equilateral.

Also $\triangle$ BGF, CHG, DKH, ELK, and AFL are congruent. § 80

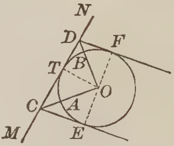
$\therefore \angle BFG = \angle CGH = \angle DHK = \angle EKL = \angle ALF$. § 76

$\therefore \angle GFL = \angle HGF = \angle KHG = \angle LKH = \angle FLK$. § 58

$\therefore$ pentagon $FGHKL$ is equiangular.

$\therefore$ pentagon $FGHKL$ is regular, by § 357. Q.E.D.

34. *If two perpendicular radii of a circle are produced to intersect a tangent to the circle, the other tangents from the two points of intersection are parallel.*



Given the $\perp$ radii OA and OB produced to meet at C and D the tangent MN, and CE and DF the second tangents from C and D.

To prove that CE is $\parallel$ to DF.

Proof. Draw the radii OT, OE, and OF to the points of contact.

In the $\triangle$ EOC and TOC, $OE = OT$, § 162

$CE = CT$, § 192

$CO = CO$. Iden.

$\therefore \triangle EOC$ is congruent to $\triangle TOC$. § 80

$\therefore \angle COE = \angle TOC$. § 67

Similarly $\angle FOD = \angle DOT$.

$\therefore \angle COE + \angle FOD = \angle TOC + \angle DOT$. Ax. 1

That is $\angle COE + \angle FOD = \angle COD$.

But $\angle COD = 1$ rt. $\angle$. Given

$\therefore \angle COE + \angle FOD = 1$ rt. $\angle$. Ax. 8

$\therefore \angle EOC + \angle COD + \angle FOD = 2$ rt. $\angle$s. Ax. 1

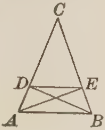
$\therefore EOF$ is a st. line. § 43

But OE is $\perp$ to CE,

and OF is $\perp$ to DF. § 185

$\therefore$ CE is $\parallel$ to DF, by § 95. Q.E.D.

35. *The line that joins the feet of the perpendiculars drawn from the extremities of the base of an isosceles triangle to the equal sides is parallel to the base.*



Given the isosceles $\triangle ABC$ with AE and BD drawn $\perp$ to the equal sides BC and AC, and DE drawn.

To prove that DE is $\parallel$ to AB.

Proof. In the rt. $\triangle$s ABD and ABE,

$AB = AB$, Iden.

and $\angle BAD = \angle EBA$. § 74

$\therefore$ the $\triangle$s ABD and ABE are congruent. § 72

$\therefore$ $AD = BE$. § 67

$\therefore$ $AC - AD = BC - BE$. Ax. 2

That is, $CD = CE$.

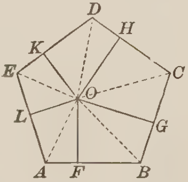
$\therefore$ the $\triangle DEC$ is isosceles.

Now the $\angle C$ is common to the $\triangle$s DEC and ABC.

$\therefore$ $\angle EDC = \angle BAC$.

$\therefore$ DE is $\parallel$ to AB, by § 103. Q.E.D.

36. *The sum of the perpendiculars drawn to the sides of a regular polygon from any point within the polygon is equal to the apothem multiplied by the number of sides.*



Given the regular polygon $ABCDE$ of n sides, and OF, OG, OH, OK, OL $\perp$s drawn from any point O within the polygon to the sides.

Let r denote the apothem and P the perimeter of the polygon.

To prove that $OF + OG + OH + OK + OL = nr$.

Proof. Join O to each of the vertices of the polygon.

The area of the $\triangle AOB = \frac{1}{2} AB \times OF$, § 325

the area of the $\triangle BOC = \frac{1}{2} BC \times OG$,

the area of the $\triangle COD = \frac{1}{2} CD \times OH$, and so on.

But $AB = BC = CD$, and so on. § 357

Now the area of the polygon is the sum of the areas of the $\triangle$s AOB, BOC, COD, etc.

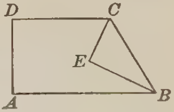
$\therefore$ the area of the polygon $= \frac{1}{2} AB \times (OF + OG + OH + OK + OL)$.

But the area of the polygon $= \frac{1}{2} P \times r = n \times \frac{1}{2} AB \times r$. § 386

$\therefore$ $\frac{1}{2} AB \times (OF + OG + OH + OK + OL) = n \times \frac{1}{2} AB \times r$. Ax. 8

$\therefore$ $OF + OG + OH + OK + OL = nr$. Q.E.D.

37. *If two consecutive angles of a quadrilateral are right angles, the bisectors of the other two angles are perpendicular.*



Given the quadrilateral $ABCD$, with $\angle$s D and A rt. $\angle$s, and BE bisecting $\angle B$, and CE bisecting $\angle C$.

To prove that BE is $\perp$ to CE.

Proof. $\angle A + \angle B + \angle C + \angle D = 4$ rt. $\angle$s. § 144

But $\angle A + \angle D = 2$ rt. $\angle$s. Given

$\therefore \angle B + \angle C = 2$ rt. $\angle$s. Ax. 2

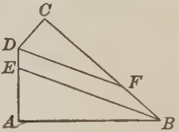
$\therefore \frac{1}{2}\angle B + \frac{1}{2}\angle C = 1$ rt. $\angle$. Ax. 4

$\therefore \angle CBE + \angle ECB = 1$ rt. $\angle$.

$\therefore \angle BEC = 1$ rt. $\angle$. Ax. 2

$\therefore$ BE is $\perp$ to CE. Q.E.D.

38. *If two opposite angles of a quadrilateral are right angles, the bisectors of the other two angles are parallel.*



Given the quadrilateral $ABCD$, with $\angle$s A and C rt. $\angle$s, and BE bisecting the $\angle CBA$, and DF bisecting the $\angle ADC$.

To prove that BE and FD are $\parallel$.

Proof. $\angle A + \angle CBA + \angle C + \angle ADC = 4$ rt. $\angle$s. § 144

But $\angle A + \angle C = 2$ rt. $\angle$s. Given

$\therefore \angle CBA + \angle ADC = 2$ rt. $\angle$s. Ax. 2

$\therefore \frac{1}{2}\angle CBA + \frac{1}{2}\angle ADC = 1$ rt. $\angle$. Ax. 4

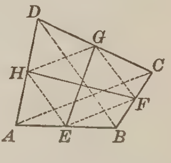
$\therefore \angle EBA + \angle ADF = 1$ rt. $\angle$.

But $\angle EBA + \angle AEB = 1$ rt. $\angle$. § 107

$\therefore \angle AEB = \angle ADF$. § 58

$\therefore$ BE and FD are $\parallel$, by § 103. Q.E.D.

39. *The two lines that join the mid-points of opposite sides of a quadrilateral bisect each other.*



Given the quadrilateral $ABCD$, with EG and FH lines joining the mid-points of the opposite sides.

To prove that EG and FH bisect each other.

Proof. Draw EF, FG, GH, HE, and draw AC and BD.

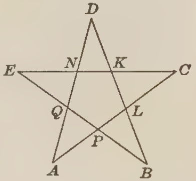
Then EF and HG are $\parallel$ to AC,

and EH and FG are $\parallel$ to BD. § 136

$\therefore$ $EFGH$ is a $\square$. § 118

$\therefore$ EG and FH bisect each other, by § 131. Q.E.D.

40. *The sum of the angles at the vertices of a five-pointed star is equal to two right angles.*



Given A, B, C, D, E, the $\angle$s at the vertices of the pentagram.

To prove that $A + B + C + D + E = 2$ rt. $\angle$s.

Proof. $\angle DNK + \angle CKL + \angle BLP + \angle APQ + \angle EQN = 4$ rt. $\angle$s. § 146

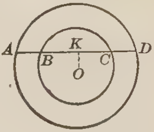
Also $\angle DKN + \angle CLK + \angle BPL + \angle AQP + \angle ENQ = 4$ rt. $\angle$s. § 146

Hence sum of base $\angle$s of the $\triangle$s DNK, CKL, etc. $= 8$ rt. $\angle$s. Ax. 1

But sum of all the $\angle$s of the $\triangle$s DNK, CKL, etc. $= 10$ rt. $\angle$s. § 107

$\therefore A + B + C + D + E = 2$ rt. $\angle$s, by Ax. 2. Q.E.D.

41. *The segments of any line intercepted between two concentric circles are equal.*



Given AD, cutting the outer of two concentric $\odot$s in A and D and the inner $\odot$ in B and C.

To prove that $AB = DC$.

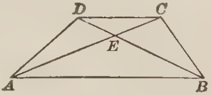
Proof. Draw OK from the center $O \perp$ to AD.

Then $KA = KD$, and $KB = KC$. § 174

$\therefore KA - KB = KD - KC$. Ax. 2

That is, $AB = DC$. Q.E.D.

42. *The diagonals of a trapezoid divide each other into segments which are proportional.*



Given the trapezoid $ABCD$, with diagonals AC and BD intersecting at E.

To prove that $EA : EB = EC : ED$.

Proof. In the $\triangle$s AEB and CED,

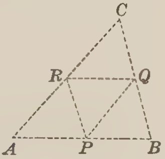
$\angle AEB = \angle CED$, § 60

and $\angle EBA = \angle EDC$. § 100

$\therefore$ the $\triangle$s AEB and CED are similar. § 286

$\therefore EA : EB = EC : ED$, by § 282. Q.E.D.

43. *Given the mid-points of the sides of a triangle, to construct the triangle.*



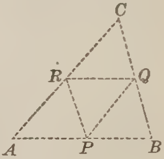
Given the three points P, Q, and R.

Required to construct the $\triangle$ the mid-points of the sides of which are P, Q, and R.

Construction. Draw PQ, QR, and RP.

Through P draw a line ∥ to RQ, through Q draw a line ∥ to PR, and through R draw a line ∥ to PQ, these three lines forming the $\triangle ABC$.

Then ABC is the △ required. Q.E.F.



Proof. In the quadrilateral $PBQR$,

PB is ∥ to RQ,

and BQ is ∥ to PR. Const.

∴ the quadrilateral $PBQR$ is a parallelogram. § 118

$\therefore PB = RQ.$ § 125

Again in the quadrilateral $APQR$,

AP is ∥ to RQ,

and AR is ∥ to PQ. Const.

∴ the quadrilateral $APQR$ is a parallelogram. § 118

$\therefore AP = RQ.$ § 125

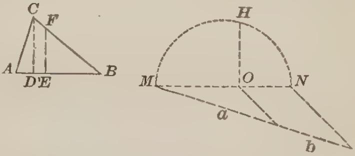
$\therefore AP = PB.$ Ax. 8

Similarly, $BQ = QC$,

and $AR = RC.$

That is, P, Q, and R are the mid-points of the sides of the $\triangle ABC$. Q.E.D.

44. *To divide a given triangle into two equivalent parts by a line perpendicular to the base.*



Given the $\triangle ABC$.

Required to divide the $\triangle ABC$ into two equivalent parts by a line ⊥ to AB.

Construction. Draw CD, the altitude of the $\triangle ABC$.

Take a equal to $2\,DB$, b equal to AB, MO equal to BC.

Find OH, so that $\overline{MO}^2 : \overline{OH}^2 = a : b.$ § 355

Cut off BF equal to OH, and draw FE ⊥ to AB.

Then FE is the line required. Q.E.F.

Proof. Now FE is ∥ to CD. § 95

∴ △ DBC and EBF are similar. § 285

$$\therefore \frac{\triangle DBC}{\triangle EBF} = \frac{\overline{BC}^2}{\overline{BF}^2} \quad \text{§ 333}$$

$$= \frac{\overline{MO}^2}{\overline{OH}^2} = \frac{a}{b} = \frac{2\,DB}{AB}. \quad \text{Const.}$$

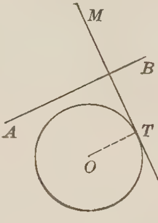
That is, $$\frac{\triangle DBC}{\triangle EBF}=\frac{2\,DB}{AB}.$$

But $$\frac{\triangle ABC}{\triangle DBC}=\frac{AB}{DB}.\qquad \S\ 327$$

Multiply the corresponding members of these equations.

Then $$\frac{\triangle ABC}{\triangle EBF}=\frac{2}{1},\ \text{by Ax. 3.}\qquad \text{Q.E.D.}$$

45. *To draw a tangent to a given circle that shall also be perpendicular to a given line.*



Given the ⊙ with center O, and the line AB.

Required to draw a line tangent to the ⊙ and ⊥ to AB.

Construction. Draw the radius $OT \parallel$ to AB.

Through T draw $MN \perp$ to OT.

Then MN is the tangent required. Q.E.F.

Proof. MN is ⊥ to OT. Const.

∴ MN is tangent to the ⊙. § 184

Also MN is ⊥ to AB. § 97

∴ MN is the tangent required. Q.E.D.

46. *To divide a given line into two segments such that the square on one shall be double the square on the other.*

Given the line AB with length a.

Required to divide AB into two parts $a-x$, and x such that $(a-x)^2 : x^2 = 2 : 1$.

Solution. If $$(a-x)^2 : x^2 = 2 : 1,$$
then
$$\begin{aligned} a^2-2\,ax+x^2&=2\,x^2,\\ x^2+2\,ax&=a^2,\\ x^2+2\,ax+a^2&=2\,a^2,\\ x+a&=a\sqrt{2},\\ x&=a\left(\sqrt{2}-1\right)\\ &=0.41\,a.\end{aligned}$$

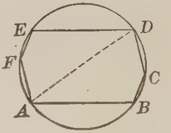
Draw a line AX from A, making any convenient ∠ with AB.

On AX lay off AP equal to 41 units, and PC equal to 59 units.

Draw CB, and draw $PD \parallel$ to CB.

Then $$AD : DB = AP : PC = 41 : 59.$$
$$\therefore\ \overline{AD}^2 : \overline{DB}^2 = 41^2 : 59^2 = 1 : 2.\qquad \text{Q.E.F.}$$

47. *If any two consecutive sides of an inscribed hexagon are respectively parallel to their opposite sides, the remaining two sides are parallel.*



Given the inscribed hexagon $ABCDEF$, with the consecutive sides AB and BC ∥ to the opposite sides ED and FE.

To prove that CD is ∥ to FA.

Proof. Draw AD.

Since AB is ∥ to ED,

and BC is ∥ to FE, Given

$\therefore \angle B = \angle E$. § 123

In the quadrilateral $ABCD$,

$\angle B + \angle ADC = 180°$. § 218

In the quadrilateral $ADEF$,

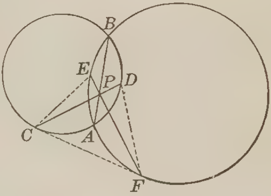
$\angle E + \angle DAF = 180°$. § 218

But $\angle B = \angle E$.

$\therefore \angle ADC = \angle DAF$.

$\therefore$ CD is ∥ to FA, by § 101. Q.E.D.

48. *If through any given point in the common chord of two intersecting circles two other chords are drawn, one in each circle, their four extremities will all lie on a third circle.*



Given the common chord AB of the two intersecting Ⓢ and two chords CD and EF drawn through any point P in the chord AB.

To prove that C, D, E, and F lie on a ⊙.

Proof. Draw CE, FD.

Then $PA \times PB = PC \times PD$,

and $PA \times PB = PF \times PE$. § 299

$\therefore PC \times PD = PF \times PE$. Ax. 8

$\therefore PC : PE = PF : PD$. § 264

Now $\angle EPC = \angle FPD$. § 60

$\therefore$ the $\triangle$s CPE and PFD are similar. § 288

$\therefore \angle CEP = \angle PDF$. § 282

Draw CF.

$\therefore$ the points E and D lie in the segment of a ⊙ described upon CF as chord in which an ∠ equal to $\angle CEF$ may be inscribed. § 217

$\therefore$ the points C, D, E, and F lie on a ⊙. Q.E.D.

49. *If two chords intersect at right angles within a circle, the sum of the squares on their segments equals the square on the diameter. Investigate the case in which the chords intersect outside the circle; also the case in which they intersect on the circle.*

Given the chords AB and $CD \perp$ to each other at E, and squares described on AE, EB, CE, ED.

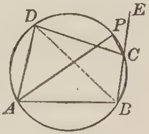
To prove that $AE^2 + EB^2 + CE^2 + ED^2$ equals the square on the diameter.

Proof. This is the same as in Ex. 14, p. 209. (See p. 228 of this book.)

Discussion. If the chords intersect outside the circle, it becomes a case of two secants, and the square on the diameter is equal to the sum of the squares on the secants and the squares on the external segments.

If the chords intersect on the circle, we have the case of a right triangle.

50. *The lines bisecting any angle of an inscribed quadrilateral and the opposite exterior angle intersect on the circle.*



Given the inscribed quadrilateral $ABCD$, with PA bisecting the $\angle A$, and CP bisecting the exterior $\angle ECD$.

To prove that P lies on the $\odot$.

Proof. Let AP intersect the $\odot$ at Q.

Draw BD.

Now $\angle ECD = \angle BDC + \angle CBD$. § 111

But $\angle BDC$ is measured by $\frac{1}{2}$ arc BC,

and $\angle CBD$ is measured by $\frac{1}{2}$ arc CD. § 214

$\therefore \angle ECD$ is measured by $\frac{1}{2}$(arc BC + arc CD);

that is, $\angle ECD$ is measured by $\frac{1}{2}$ arc BD.

But $\angle BAD$ is measured by $\frac{1}{2}$ arc BD.

$\therefore \angle BAD = \angle ECD$.

$\therefore \angle QAD = \angle PCD$.

But $\angle PCD$ is measured by $\frac{1}{2}$ arc PD,

and $\angle QAD$ is measured by $\frac{1}{2}$ arc QD. § 214

Since the $\angle$s PCD and QAD are equal,

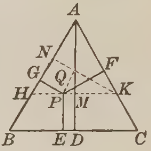
$\therefore$ arc DQ = arc DP.

$\therefore Q$ must coincide with P.

But Q is on the $\odot ABCD$.

$\therefore P$ is on the $\odot ABCD$. Q.E.D.

51. *The sum of the perpendiculars from any point in an equilateral triangle to the three sides is constant.*



Given the equilateral $\triangle ABC$, BC the base, AD the altitude, P any point within the $\triangle$, PE, PF, PG $\perp$s from P to the sides BC, AC, AB respectively.

To prove that $PE + PF + PG = AD$.

Proof. Through P draw $HK \parallel$ to BC, meeting AD at M. Draw $KN \perp$ to AB, and $PQ \perp$ to KN.

Then the rt. $\triangle$s PKF and PKQ are congruent. § 91

$\therefore PF = KQ$. § 67

Also $PG = QN$. §§ 95, 127

$\therefore PF + PG = KQ + QN = KN$. Ax. 1

Now the rt. $\triangle$s HKN and MKA are congruent. § 91

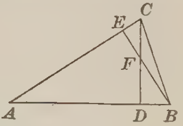
$\therefore KN = AM$. § 67

Now $PE = MD$. §§ 95, 127

$\therefore PE + KN = MD + AM = AD$. Ax. 1

That is, $PE + PF + PG = AD$, by Ax. 9 Q.E.D.

52. *The perpendiculars from the vertices of a triangle upon the opposite sides cut one another into segments that are reciprocally proportional to each other.*



Given the $\triangle ABC$, CD and BE $\perp$, respectively, to AB and AC, and meeting at F.

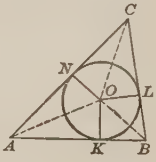
To prove that $FB : FC = FD : FE$.

Proof. $\angle DFB = \angle CFE$. § 60

$\therefore$ the rt. $\triangle$s BDF and CEF are similar. § 287

$\therefore FB : FC = FD : FE$, by § 282. Q.E.D.

53. *The area of a triangle is equal to half the product of its perimeter by the radius of the inscribed circle.*



Given the $\triangle ABC$, and NLK the inscribed $\odot$ having the radius OK.

To prove that

area of $\triangle ABC = \frac{1}{2}(AB + BC + CA) \times OK$.

Proof. The $\perp$s OK, OL, and ON are all equal. § 162

Draw AO, BO, CO.

Then area of $\triangle AOB = AB \times \frac{1}{2} OK$,

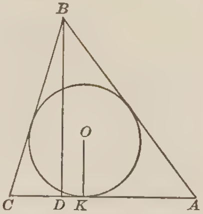
area of $\triangle BOC = BC \times \frac{1}{2} OL$,

and area of $\triangle AOC = AC \times \frac{1}{2} ON$.

$\therefore$ area of $(\triangle AOB + \triangle BOC + \triangle AOC) = \frac{1}{2}(AB + BC + AC) \times OK$. Ax. 1

$\therefore$ area of $\triangle ABC = \frac{1}{2}(AB + BC + AC) \times OK$. Q.E.D.

54. *The perimeter of a triangle is to one side as the perpendicular from the opposite vertex is to the radius of the inscribed circle.*



Given the △ with perimeter p, and $BD \perp$ to CA, OK being the radius of the inscribed ⊙.

To prove that

$$p : AC = BD : OK.$$

Proof. area of $\triangle = \frac{1}{2} p \times OK$. Ex. 53

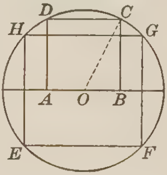
Also area of $\triangle = \frac{1}{2} AC \times BD$. § 325

$\therefore \frac{1}{2} p \times OK = \frac{1}{2} AC \times BD$. Ax. 8

$\because p \times OK = AC \times BD$. Ax. 3

$\therefore p : AC = BD : OK$, by § 264. Q. E. D.

55. *The area of a square inscribed in a semicircle is equal to two fifths of the area of the square inscribed in the circle.*



Given the square $ABCD$ inscribed in the semicircle, and $EFGH$ a square inscribed in the ⊙ whose center is O.

To prove that $ABCD : EFGH = 2 : 5$.

Proof. Now area of $ABCD = \overline{BC}^2$,

and area of $EFGH = \overline{EF}^2$. § 320

Draw OC.

$\overline{BC}^2 = \overline{OC}^2 - \overline{OB}^2$. § 338

$\therefore \overline{BC}^2 = \overline{OC}^2 - (\frac{1}{2} \overline{BC})^2$.

$\therefore \overline{BC}^2 = \overline{OC}^2 - \frac{1}{4} \overline{BC}^2$.

$\therefore \overline{BC}^2 = \frac{4}{5} \overline{OC}^2$.

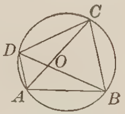
$EF = OC \sqrt{2}$. Ex. 2, p. 254

$\therefore \overline{EF}^2 = 2 \overline{OC}^2$.

$\therefore \overline{BC}^2 : \overline{EF}^2 = \frac{4}{5} \overline{OC}^2 : 2 \overline{OC}^2 = \frac{4}{5} : 2 = 2 : 5$.

That is, $ABCD : EFGH = 2 : 5$. Q. E. D.

56. *The diagonals of any inscribed quadrilateral divide it into two pairs of similar triangles.*



Given the inscribed quadrilateral $ABCD$, with the diagonals AC and BD intersecting at O.

To prove that the △ ABO and DCO are similar, and that the △ BCO and DAO are similar.

Proof. In the △ ABO and DCO,

$\angle BAC$ is measured by $\frac{1}{2}$ arc BC,

and $\angle BDC$ is measured by $\frac{1}{2}$ arc BC. § 214

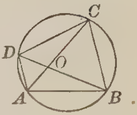
$\therefore \angle BAC = \angle BDC$.

Also $\angle DBA$ is measured by $\frac{1}{2}$ arc DA,

and $\angle DCA$ is measured by $\frac{1}{2}$ arc DA. § 214

$\therefore \angle DBA = \angle DCA$.

$\therefore$ $\triangle$ ABO and DCO are similar. § 286



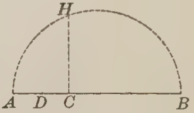
Similarly, in the $\triangle$ BCO and DAO,

$\angle CBD = \angle CAD$,

and $\angle ACB = \angle ADB$.

$\therefore$ $\triangle$ BCO and DAO are similar, by § 286. Q.E.D.

57. *To draw a line whose length is* $\sqrt{7\frac{1}{2}}$ *in.*



Given $AD = 1$ in.

Required to draw a line $= \sqrt{7\frac{1}{2}}$ in.

Construction. Draw any line AB, and on AB take $AC = 2\,AD$ and $CB = 3\frac{3}{4}\,AD$.

On AB as a diameter describe a semicircle.

At C erect the $\perp$ CH, meeting the $\odot$ at H.

Then $CH = \sqrt{7\frac{1}{2}}$ in. Q.E.F.

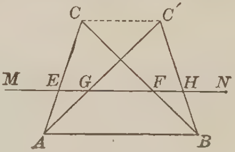
Proof. $AC : CH = CH : CB$. § 297

$\therefore 2 : CH = CH : 3\frac{3}{4}$. Ax. 9

$\therefore \overline{CH}^2 = 2 \times 3\frac{3}{4} = 7\frac{1}{2}$. § 261

$\therefore CH = \sqrt{7\frac{1}{2}}$ in., by Ax. 5. Q.E.D.

58. *If two equivalent triangles are on the same base and the same side of the base, any line cutting the triangles, and parallel to the base, cuts off equal areas from the triangles.*



Given two equivalent $\triangle$ ABC and ABC' on the same base AB and the same side of AB, and the line $MN \parallel$ to AB, cutting AC and BC in E and F, and AC' and BC' in G and H respectively.

To prove that $ABFE = ABHG$,

and $\triangle EFC = \triangle GHC'$.

Proof. Since the $\triangle$ ABC and EFC are similar, § 285

$\therefore AC : EC = AB : EF$. § 282

Similarly $AC' : GC' = AB : GH$.

Draw CC'.

Then since the $\triangle$ ABC and ABC' are equivalent and have the same base, Given

$\therefore$ C and C' are equally distant from AB. § 325

$\therefore$ CC' is $\parallel$ to AB. § 128

$\therefore$ CC' is $\parallel$ to MN. § 96

$\therefore AC : EC = AC' : GC'$. § 273

$\therefore AB : EF = AB : GH$. Ax. 8

$\therefore EF = GH$. § 263

Let d be the distance between the $\parallel$s AB and MN.

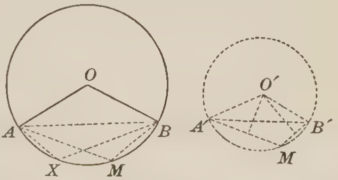
Then area of $ABFE = \frac{1}{2}(AB + EF)d$,

and area of $ABHG = \frac{1}{2}(AB + GH)d$. § 329

$\therefore$ area of $ABFE$ = area of $ABHG$. Ax. 8

$\therefore$ area of $\triangle EFC$ = area of $\triangle GHC'$, by Ax. 2. Q. E. D.

59. *To divide a given arc of a circle into two parts such that their chords shall be in a given ratio.*



Given the arc AB in the ⊙ whose center is O, and the ratio $m : n$.

Required to divide AB into two parts such that their chords shall be in the ratio $m : n$.

Construction. Take any point X in the arc AB and draw AX and BX.

Construct an $\angle B'M'A' = \angle BXA$. § 233

Take $M'A' = m$, and $M'B' = n$.

Draw the $\perp$ bisectors of $M'A'$ and $M'B'$, intersecting at O'. §§ 229, 228

Draw $A'O'$ and $B'O'$.

Construct $\angle MAO = \angle M'A'O'$, letting AM meet the ⊙ in M. § 233

Then point M divides the arc AB as required. Q. E. F.

Proof. Draw MB, AB, $A'B'$.

Describe the ⊙ with center O' and radius $O'A'$.

$\angle BXA = \angle BMA$. § 217

$\angle B'M'A' = \angle BXA$. Const.

$\therefore \angle B'M'A' = \angle BMA$. Ax. 8

Now $\angle BMA$ is measured by $\frac{1}{2}$ arc BA,

and $\angle B'M'A'$ is measured by $\frac{1}{2}$ arc $B'A'$. § 214

$\therefore$ arc $B'A'$ = arc BA in angular measure.

$\therefore \angle A'O'B' = \angle AOB$.

$\therefore \angle B'A'O' + \angle O'B'A' = \angle BAO + \angle OBA$. § 107

Since the $\triangle$s ABO and $A'B'O'$ are isosceles,

$\therefore \angle B'A'O' = \angle BAO$. Ax. 4

Now $\angle M'A'O' = \angle MAO$. Const.

$\therefore \angle M'A'B' = \angle MAB$. Ax. 2

$\therefore$ the $\triangle$s $A'M'B'$ and AMB are similar. § 285

$\therefore A'M' : B'M' = AM : BM$.

But $A'M' : B'M' = m : n$. Const.

$\therefore AM : BM = m : n$, by Ax. 8. Q. E. D.

60. *The areas between two concentric circles may be found by multiplying half the sum of the two circumferences by the difference between the radii.*

Given two concentric Ⓢ with the circumferences c and c' and the radii r and r'.

To prove that the area between the Ⓢ is equal to $\frac{1}{2}(c + c')(r - r')$.

Proof. The area of the larger $\odot = \frac{1}{2} rc$,

and the area of the smaller $\odot = \frac{1}{2} r'c'$.

Now
$$\frac{\frac{1}{2} rc}{\frac{1}{2} r'c'} = \frac{r^2}{r'^2}. \qquad \S\ 390$$

$$\frac{\frac{1}{2} rc - \frac{1}{2} r'c'}{\frac{1}{2} r'c'} = \frac{r^2 - r'^2}{r'^2}. \qquad \S\ 268$$

$$\begin{aligned}\frac{1}{2} rc - \frac{1}{2} r'c' &= (r^2 - r'^2)\frac{\frac{1}{2} r'c'}{r'^2} \\ &= \frac{1}{2}(r - r')(r + r')\frac{c'}{r'} \\ &= \frac{1}{2}(r - r')\left(\frac{rc'}{r'} + c'\right).\end{aligned}$$

Now
$$\frac{c}{c'} = \frac{r}{r'}. \qquad \S\ 382$$

$$\therefore\ c = \frac{rc'}{r'}.$$

$$\therefore\ \tfrac{1}{2} rc - \tfrac{1}{2} r'c' = \tfrac{1}{2}(r - r')(c + c'). \qquad \text{Ax. 8}$$

But the area between the Ⓢ $= \frac{1}{2} rc - \frac{1}{2} r'c'$.

$\therefore$ the area between the Ⓢ $= \frac{1}{2}(c + c')(r - r')$. Q.E.D.

61. *Find the length of the belt connecting two wheels of the same size, if the radius of each wheel is* 18 *in., the distance between the centers* 6 *ft., and* 4 *in. is allowed for sagging.*

The length of the belt will be equal to the sum of two semicircumferences of a wheel, or one circumference, twice the distance between the centers of the wheels, and the allowance for sagging.

$$4 \text{ in.} = \tfrac{1}{3} \text{ ft. and } 18 \text{ in.} = 1\tfrac{1}{2} \text{ ft.}$$

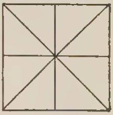
$$\begin{aligned} l &= 2\pi \times 1\tfrac{1}{2} + 2 \times 6 + \tfrac{1}{3} \text{ ft.} \\ &= 9.4248 + 12 + \tfrac{1}{3} \text{ ft.} \\ &= 21.76 \text{ ft.}\end{aligned}$$

62. *To construct a regular inscribed heptagon draftsmen sometimes use for a side half the side of an inscribed equilateral triangle. Construct such a figure with the compasses, and state whether the rule seems exact or only approximate.*

The rule is seemingly exact. But calculation shows that the side of the heptagon thus found is somewhat too small.

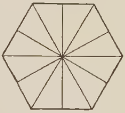
APPENDIX TO PLANE GEOMETRY

Exercise 72. Page 264



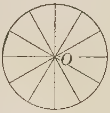
1. *Draw a figure showing the number of axes of symmetry possessed by a square.*

A square possesses four axes of symmetry, as shown in this figure.



2. *Draw a figure showing the number of axes of symmetry possessed by a regular hexagon.*

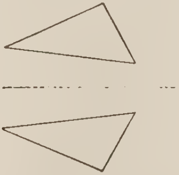
A regular hexagon possesses six axes of symmetry, as shown in this figure.

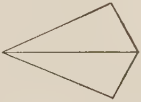


3. *Draw a figure showing six of the unlimited number of axes of symmetry of a circle, and showing the center of symmetry.*

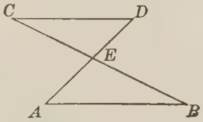
Any diameter in a $\odot$ is an axis of symmetry, and the center O of the $\odot$ is its center of symmetry.

4. *Show by drawings that two congruent triangles may be placed in a position of symmetry with respect to an axis. In one of the drawings let a common side be the axis.*



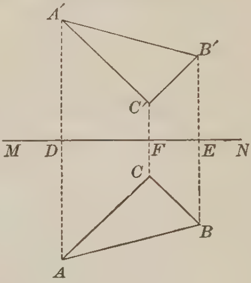


5. *Show by a drawing that two congruent triangles may be placed in a position of symmetry with respect to a center.*



When the two congruent $\triangle ABE$ and CDE are so placed that two sides of the one lie in the produced corresponding sides of the other, then their common vertex E is their center of symmetry.

6. *Two figures symmetric with respect to an axis are congruent.*



Given the two $\triangle$ ABC and $A'B'C'$ which are symmetric with respect to the axis MN.

To prove that $\triangle ABC$ is congruent to $\triangle A'B'C'$.

Proof. Draw AA', BB', CC', cutting MN in D, E, F respectively.

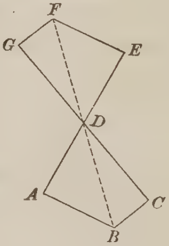
Then AA', BB', and CC' are $\perp$ to MN and are bisected by it. § 406

Hence we can turn that part of the figure which lies above MN on MN as axis until $A'D$ falls on DA, $B'E$ on EB, and $C'F$ on FC.

The points A, B, C will then coincide with the points $A'B'C'$.

Hence $\triangle ABC$ is congruent to $\triangle A'B'C'$, by § 66. Q. E. D.

7. *Two figures symmetric with respect to a center are congruent.*



Given the two quadrilaterals $ABCD$ and $DEFG$ which are symmetric with respect to point D.

To prove that $ABCD$ is congruent to $DEFG$.

Proof. Draw BDF.

In the $\triangle$ ABD and DEF,

$AD = DE$,

and $BD = DF$. § 406

$\angle ADB = \angle EDF$. § 60

$\therefore$ the $\triangle ABD$ is congruent to $\triangle DEF$. § 68

Similarly the $\triangle BCD$ is congruent to $\triangle DFG$.

Hence $ABCD$ is congruent to $DEFG$. Q. E. D.

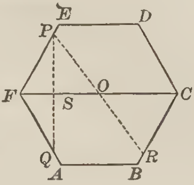
8. *Make a list of quadrilaterals that are symmetric with respect to an axis.*

The following are the quadrilaterals that are symmetric with respect to an axis: the square, the rectangle, the rhombus, and any quadrilateral having two adjacent sides and the other two sides equal.

9. *Make a list of quadrilaterals that are symmetric with respect to a center.*

All the kinds of parallelograms are symmetric with respect to a center. This is the point of intersection of the diagonals.

10. *What kinds of regular polygons are symmetric with respect both to a center and to an axis? Prove this for the hexagon.*



All regular polygons with an even number of sides are symmetric with respect both to a center and to an axis.

Given the regular hexagon $ABCDEF$ with center O, and the diagonal FC.

To prove that $ABCDEF$ is symmetric with respect both to FC and to the center O.

Proof. Let P be any point in the perimeter.

Draw $PQ \perp$ to FC.

Let PQ cut FC in S.

In the rt. $\triangle$ FQS and FSP,

$FS = FS$, Iden.

and $\angle QFS = \angle SFP$.

$\therefore$ the rt. $\triangle$ FQS and FSP are congruent. § 72

$\therefore SP = SQ$. § 67

$\therefore$ FC bisects PQ, any $\perp$ to FC, and hence bisects every $\perp$ to FC terminated by the perimeter.

$\therefore$ FC is an axis of symmetry. § 407

Draw PR through the center O.

In the $\triangle$ FOP and OCR,

$\angle POF = \angle ROC$, § 60

$\angle OFP = \angle OCR$, § 101

and $OF = OC$. § 162

$\therefore$ the $\triangle$ FOP and OCR are congruent. § 72

$\therefore OP = OR$. § 68

$\therefore$ O bisects PR, any line, and hence bisects every line drawn through O and terminated by the perimeter.

$\therefore$ O is the center of symmetry, by § 407. Q. E. D.

11. *A circle is symmetric with respect to its center as a center of symmetry, and is also symmetric with respect to any diameter as an axis.*

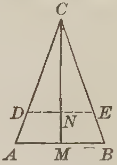
All chords in a ⊙ drawn perpendicular to a diameter are bisected by it. § 174

Hence a diameter is an axis of symmetry.

All diameters of a ⊙ are bisected by the center.

Hence the center of a ⊙ is its center of symmetry.

12. *An isosceles triangle is symmetric with respect to an axis, and therefore the angles opposite the equal sides are equal.*



Given the isosceles $\triangle ABC$ with $CM \perp$ to AB.

To prove that CM is the axis of symmetry and that $\angle MAC = \angle CBM$.

Proof. Through any point N in CM draw $DE \perp$ to CM.

Then DE is $\parallel$ to AB. § 95

In the rt. $\triangle$ DNC and NEC, $CD = CE$, § 275

and $CN = CN$. Iden.

$\therefore$ the rt. $\triangle$ DNC and NEC are congruent. § 89

$\therefore DN = NE$. § 60

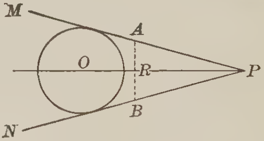
$\therefore CM$ bisects DE, any line, and hence bisects every line drawn $\perp$ to it and terminated by the perimeter.

$\therefore CM$ is an axis of symmetry. § 407

Since the $\triangle$ AMC and MBC are congruent, § 89

$\therefore \angle MAC = \angle CBM$. Q. E. D.

13. *Two tangents drawn to a circle from the same point are symmetric with respect to an axis.*



Given the ⊙ with center O, the point P outside the ⊙, PM and PN two tangents drawn to the ⊙, and the line OP.

To prove that the lines PM and PN are symmetric with respect to OP as axis.

Proof. Through R, any point in OP, draw $AB \perp$ to OP, cutting PM and PN in A and B respectively.

In the rt. $\triangle$ RBP and RPA, $RP = RP$, Iden.

and $\angle RPB = \angle APR$. § 192

$\therefore$ the rt. $\triangle$ RBP and RPA are congruent. § 72

$\therefore RB = RA$. § 67

$\therefore OP$ bisects AB, any line, and hence bisects every line drawn $\perp$ to it and terminated by the tangents PM and PN.

$\therefore OP$ is an axis of symmetry, by § 407. Q. E. D.

14. *The four common tangents to two given circles form, together with the circles, a figure symmetric with respect to the line of centers as an axis.*

Given the two Ⓢ with centers O and O', the four common tangents MN, QR, AB, and CD, and the line of centers OO'.

To prove that the figure formed by the tangents is symmetric with respect to OO' as axis.

Proof. Let AB and CD intersect in P.

Draw PO and PO'.

The lines PA and PC are symmetric with respect to PO' as axis. Ex. 13

The lines PB and PD are symmetric with respect to PO' as axis. Ex. 13

Now $\angle CPO = \frac{1}{2} \angle CPA$,

and $\angle DPO' = \frac{1}{2} \angle DPB$. § 192

But $\angle CPA = \angle DPB$. § 60

$\therefore \angle CPO = \angle DPO'$. Ax. 8

Since CP and PD are one straight line,

$\therefore$ PO and PO' are one straight line, and hence coincide with OO'.

$\therefore$ OO' is the axis of symmetry of the lines AB and CD.

Since O and O' are equidistant from the lines MN and QR,

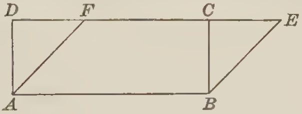
$\therefore$ OO' is the bisector of the $\angle$ formed by MN and QR produced. § 152

$\therefore$ OO' is the axis of symmetry of the lines MN and QR. Ex. 13

$\therefore$ the figure formed by the four common tangents to the two given Ⓢ is symmetric with respect to OO', the line of centers, as axis. Q. E. D.

Exercise 73. Page 272

1. *Of all equivalent parallelograms that have equal bases, the rectangle has the minimum perimeter.*



Given the rectangle $ABCD$, and the ▱ $ABEF$ on the same base AB and with the same altitude.

To prove that

$$AB + BC + CD + DA < AB + BE + EF + FA.$$

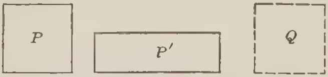
Proof. $DA < FA$,

and $BC < BE$. § 114

$CD = AB = EF$. § 125

$\therefore AB + BC + CD + DA < AB + BE + EF + FA$, by Ax. 6. Q. E. D.

2. *Of all equivalent rectangles, the square has the minimum perimeter.*



Given the square P equivalent to any rectangle P'.

To prove that the perimeter of P is less than the perimeter of P'.

Proof. Construct the square Q, having the same perimeter as P'.

Then $Q > P'$. § 418

But $P = P'$. Given

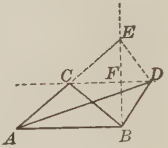
$\therefore Q > P$. Ax. 6

Since $Q > P$, and they are both squares, the perimeter of Q must be greater than the perimeter of P.

But the perimeter of Q is equal to the perimeter of P'. Const.

$\therefore$ the perimeter of P is less than the perimeter of P'. Q. E. D

3. *Of all triangles that have the same base and the same altitude, the isosceles has the minimum perimeter.*



Given an isosceles $\triangle ABC$, and the $\triangle ABD$ on the same base AB and with the same altitude.

To prove that $AB + BC + CA < AB + BD + DA$.

Proof. Draw CD.

Then CD is $\parallel$ to AB, being equidistant from it.

Draw $BE \perp$ to AB, meeting AC produced in E and CD in F, and draw DE.

Then BE is $\perp$ to CD. § 97

Now $\angle FCE = \angle BAC$, § 102

and $\angle BCF = \angle CBA$. § 100

But $\angle BAC = \angle CBA$. § 74

$\therefore \angle FCE = \angle BCF$. Ax. 8

In the rt. ⊿s ECF and BCF,

$CF = CF$. Iden.

$\angle FCE = \angle BCF$. Proved

$\therefore$ the rt. ⊿s ECF and BCF are congruent. § 72

$\therefore CE = BC$,

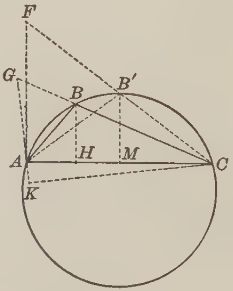
and $DE = BD$. § 83

But $AE < DA + DE$; § 112

that is, $BC + CA < DA + BD$. Ax. 9

$\therefore AB + BC + CA < AB + BD + DA$, by Ax. 1. Q.E.D.

4. *Of all triangles that can be inscribed in a given circle, the equilateral is the maximum and has the maximum perimeter.*



Given the $\triangle ABC$ inscribed in a ⊙.

To prove that

1. $\triangle ABC$ is the maximum when equilateral.

2. $\triangle ABC$ has the maximum perimeter when equilateral.

Proof. 1. If $\triangle ABC$ is not equilateral, at least two sides are unequal.

Let BC be greater than BA.

Draw $BH \perp$ to AC.

Draw MB', the $\perp$ bisector of AC, meeting the ⊙ at B', and draw AB' and CB'.

Then MB' passes through the center of the ⊙. § 177

$\therefore$ the tangent through B' is $\perp$ to MB', § 185

and therefore is $\parallel$ to AC. § 95

B' lies in this tangent and B lies between the tangent and AC. § 183

$\therefore B'M > BH.$ § 327

$\therefore$ of all $\triangle$s inscribed in the $\odot$ having a base equal to AC, the isosceles is the maximum.

If now AB' and AC are unequal, considering $B'C$ the base of the $\triangle$, the maximum $\triangle$ is isosceles.

That is, the $\triangle ABC$ is the maximum when all three sides are equal; that is, when equilateral. Q.E.D.

2. Produce CB' to F, making $B'F = B'A$; and produce CB to G, making $BG = BA$.

Draw AF and AG, and draw $CK \perp$ to GA produced.

Then FA is $\perp$ to AC. Ex. 7, p. 82

$\angle F = \angle B'AF,$

and $\angle G = \angle BAG.$ § 74

$\therefore 2\angle F = \angle AB'C,$

and $2\angle G = \angle ABC.$ § 111

But $\angle AB'C = \angle ABC.$ § 217

$\therefore \angle F = \angle G.$ Ax. 8

$\therefore$ the rt. $\triangle$s FAC and GKC are similar. § 287

$\therefore CF : CG = CA : CK.$ § 282

That is, $CB' + B'A : CB + BA = CA : CK.$ Ax. 9

But $CA > CK.$ § 114

$\therefore CB' + B'A > CB + BA.$

$\therefore$ the perimeter of the $\triangle AB'C$ is greater than the perimeter of the $\triangle ABC$.

That is, the perimeter of a $\triangle$ inscribed in the $\odot$ on the base AC is the maximum when the $\triangle$ is isosceles.

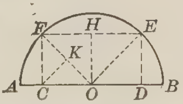
If now AB' and AC are unequal, considering $B'C$ the base of the $\triangle$, the $\triangle$ with the maximum perimeter is isosceles.

That is, the perimeter of the $\triangle ABC$ is the maximum when the $\triangle$ is equilateral. Q.E.D.

5. *To inscribe in a semicircle the maximum rectangle.*

Given the semicircle described upon AB as diameter.

Required to inscribe in the semicircle the maximum rectangle.



Construction. Draw the radii OE and OF, making $\angle$s of 45° with AB.

Draw ED and $FC \perp$ to AB, and draw EF.

Then $CDEF$ is the ▭ required. Q.E.F.

Proof. $\angle OFE = \angle FEO$. § 74

But $\angle EOF = 90°$. Const.

$\therefore \angle OFE = 45°$. § 107

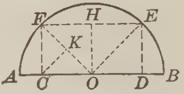
But $\angle CFO = 45°$. § 107

$\therefore \angle CFE = 90°$.

Similarly $\angle FED = 90°$.

$\therefore$ $CDEF$ is a rectangle.

Draw $OH \perp$ to FE.



Now the rt. $\triangle$s COF, FOH, HOE, and ODE are congruent. § 91

$\therefore \triangle COF = \frac{1}{4}$ of the rectangle $CDEF$.

Draw CK.

Now $\angle CFO = \angle FOC = 45°$.

$\therefore CO = CF$. § 72

$\therefore$ in the rt. $\triangle COF$, CK is the maximum.

Since the hypotenuse OF is determined,

$\therefore$ the rt. $\triangle COF$ is the maximum. § 327

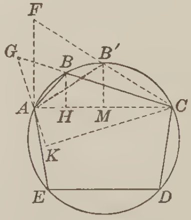
$\therefore$ the rectangle $CDEF$ is the maximum, by Ax. 3. Q. E. D.

6. *Find the area of the maximum triangle inscribed in a semicircle whose radius is 3 in.*

Since the $\triangle$ is inscribed in a semicircle, it must be a rt. $\triangle$. § 215
The $\triangle$ will be the maximum when its altitude is the maximum.
The altitude is the maximum when it is equal to the radius.
$\therefore$ the area of the maximum $\triangle$ is equal to one half the product of the diameter and the radius.

$$\therefore \text{area of } \triangle = \tfrac{1}{2} \times 6 \times 3 \text{ sq. in.}$$
$$= 9 \text{ sq. in.}$$

7. *Of all polygons of a given number of sides that can be inscribed in a given circle, that which is regular has the maximum area and the maximum perimeter.*



Given the inscribed polygon $ABCDE$.

To prove that

1. $ABCDE$ is the maximum when regular.

2. The perimeter of $ABCDE$ is the maximum when $ABCDE$ is regular.

Proof. 1. If $ABCDE$ is not a regular polygon, at least two adjacent sides are unequal.

Let BC be greater than AB.

Draw AC, and draw $BH \perp$ to AC.

Draw MB', the $\perp$ bisector of AC, meeting the $\odot$ in B', and draw AB' and CB'.

MB' passes through the center of the $\odot$. § 177

$\therefore$ the tangent through B' is $\perp$ to MB', § 185

and therefore is $\parallel$ to AC. § 95

B' lies in this tangent, and B lies between the tangent and AC. § 183

$\therefore B'M > BH.$

$\therefore$ of all $\triangle$s inscribed in the $\odot$, having a base equal to AC, the isosceles is the maximum.

$\therefore AB'CDE > ABCDE.$ Ax. 6

But each $\triangle$ constructed in a similar manner to $\triangle ABC$ is the maximum when isosceles.

$\therefore ABCDE$ is the maximum when equilateral.

But an equilateral inscribed polygon is regular. § 365

$\therefore ABCDE$ is the maximum when regular. Q.E.D.

2. Produce CB' to F, making $B'F = B'A$, and produce CB to G, making $BG = BA$.

Draw AF and AG, and draw $CK \perp$ to GA produced.

Then FA is $\perp$ to AC. Ex. 7, p. 82

$\angle F = \angle B'AF,$

and $\angle G = \angle BAG.$ § 74

$\therefore 2\angle F = \angle AB'C,$

and $2\angle G = \angle ABC.$ § 111

But $\angle AB'C = \angle ABC.$ § 217

$\therefore \angle F = \angle G.$ Ax. 8

$\therefore$ the rt. $\triangle$s FAC and GKC are similar. § 287

$\therefore CF : CG = CA : CK.$ § 282

That is, $CB' + B'A : CB + BA = CA : CK.$ Ax. 9

But $CA > CK.$ § 114

$\therefore CB' + B'A > CB + BA.$

$\therefore$ the perimeter of the $\triangle AB'C$ is greater than the perimeter of the $\triangle ABC$.

$\therefore$ the perimeter of $AB'CDE >$ the perimeter of $ABCDE$. Ax. 6

But each $\triangle$ constructed in a similar manner to ABC has the maximum perimeter when isosceles.

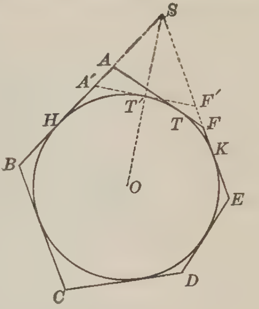
$\therefore ABCDE$ has the maximum perimeter when equilateral.

But an inscribed equilateral polygon is regular. § 365

$\therefore ABCDE$ has the maximum perimeter when regular. Q. E. D.

8. *Of all polygons of a given number of sides that can be circumscribed about a given circle, that which is regular has the minimum area and the minimum perimeter.*

Given the polygon *ABCDEF* circumscribed about a ⊙ whose center is *O*.



To prove that

1. *ABCDEF* is the minimum when regular.

2. *ABCDEF* has the minimum perimeter when regular.

Proof. 1. If *ABCDEF* is not a regular polygon, at least two adjacent arcs between the points of tangency are unequal.

Let *H*, *T*, *K* be the points of tangency of the sides *BA*, *AF*, *FE* respectively.

Suppose arc $HT >$ arc *TK*.

Through *T′*, the mid-point of arc *HK*, draw the tangent *A′F′*, intersecting the side *AB* in *A′* and the side *EF* produced in *F′*.

Produce *HA* and *KF* to meet in *S*.

Draw *OT′S*.

Suppose the base of the △ *AFS* variable so that its point of tangency *T* moves along the arc *KH*. In moving from *K* to *T′* the area of the △ is increasing, and also when moving from *H* to *T′*.

Since *T′* lies on *OT′S*, the axis of symmetry of the tangents *SH* and *SK*, Ex. 13, p. 264

∴ a limit must be reached when *AF* is in the position *A′F′*.

∴ the △ *A′F′S* is the maximum of the △ *AFS*.

∴ $A'BCDEF' < ABCDEF$. Ax. 6

∴ the circumscribed polygon is not the minimum when any two adjacent arcs formed by the points of tangency are unequal.

But if all these arcs are equal, the polygon is regular. § 367

∴ the polygon is the minimum when regular. Q. E. D.

2. Suppose again the line *AF* variable. While the point *T* moves from *K* to *T′*, the line *AF* diminishes, and also when *T* moves from *H* to *T′*.

Since the tangents *SH* and *SK* are symmetric with respect to *OS* as axis, Ex. 13, p. 264

∴ a limit must be reached when *AF* is in the position *A′F′*.

∴ *A′F′* is the minimum of *AF*.

That is, as long as the circumscribed polygon is not regular, its perimeter is not the minimum.

∴ the perimeter of the circumscribed polygon is the minimum when the polygon is regular. Q. E. D.

9. *In a given line required to find a point such that the sum of its distances from two given points on the same side of the line shall be the minimum.*

This is the same as Ex. 25, p. 147 (see p. 155 of this book).

10. *To divide a given line into two segments such that the sum of the squares on these segments shall be the minimum.*

Since the hypotenuse of a right △, the sum of whose sides is given, is the minimum when the △ is isosceles, therefore the sum of the squares on the sides of such an isosceles △ is the minimum.

Therefore bisect the given line, and the sum of the squares on the two parts is the minimum.

11. *To divide a given line into two segments such that their product shall be the maximum.*

Since the maximum of isoperimetric polygons of the same number of sides is a regular polygon (§ 418), the maximum quadrilateral whose perimeter is equal to twice the given line is a regular quadrilateral, that is, a square; and each side of the square is equal to half the given line.

Therefore bisect the given line, and the product of the two segments is the maximum.

SOLID GEOMETRY

TEACHERS' EDITION

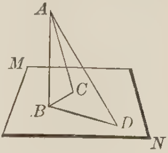
BOOK VI

Exercise 74. Page 283

1. *Why does folding a sheet of paper give a straight edge?*

For the reason, see § 429.

2. *If equal oblique lines are drawn from a given external point to a plane, they make equal angles with lines drawn from the points where the oblique lines meet the plane to the foot of the perpendicular from the given point.*



Given AB the $\perp$, and AC and AD any two equal oblique lines drawn to the plane MN from an external point A; also lines BC and BD.

To prove that $\angle ACB = \angle ADB$.

Proof. The $\angle$s CBA and DBA are rt. $\angle$s. § 430

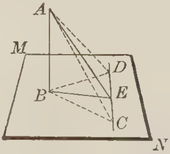
In the rt. $\triangle$s CBA and DBA, $AC = AD$, Given

and $AB = AB$. Iden.

$\therefore$ rt. $\triangle$s CBA and DBA are congruent. § 89

$\therefore \angle ACB = \angle ADB$, by § 67. Q. E. D.

3. *If from the foot of a perpendicular to a plane a line is drawn at right angles to any line in the plane, the line drawn from its intersection with the line in the plane to any point of the perpendicular is perpendicular to the line of the plane.*



Given AB a $\perp$ to the plane MN, BE a $\perp$ from B to any line CD in MN, and EA any line from E to AB.

To prove that AE is $\perp$ to CD.

Proof. On the line CD take $ED = EC$.

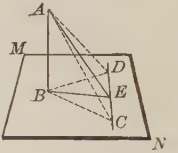
Draw BC, BD, AC, and AD.

Then $BC = BD$. § 150

$\therefore AC = AD$. § 439

$\therefore AE$ is $\perp$ to CD, by § 151. Q. E. D.

4. *If two perpendiculars are drawn from a point to a plane and to a line in that plane respectively, the line joining the feet of the perpendiculars is perpendicular to the given line.*



Given $AB \perp$ to the plane MN, and $AE \perp$ to CD in the plane MN; also the line BE.

To prove that BE is $\perp$ to CD.

Proof. On CD take $ED = EC$.

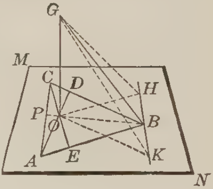
Draw AC, AD, BC, and BD.

Then $AC = AD$. § 150

$\therefore BC = BD$. § 440

$\therefore BE$ is $\perp$ to CD, by § 151. Q.E.D.

5. *From two vertices of a triangle perpendiculars are let fall on the opposite sides. From the intersection of these perpendiculars a perpendicular is drawn to the plane of the triangle. Prove that a line drawn to any vertex of the triangle, from any point on this perpendicular, is perpendicular to the line drawn through that vertex parallel to the opposite side.*



Given the $\triangle ABC$, lying in the plane MN, and the $\perp$s AD and CE from the vertices A and C to the opposite sides BC and AB respectively; $GO \perp$ to plane MN at O, the point of intersection of AD and CE; also HK drawn $\parallel$ to CA through B, and the line GB.

To prove that GB is $\perp$ to HK.

Proof. Draw $BP \perp$ to AC.

Then BP passes through O. Ex. 4, p. 78

$\therefore BP$ is $\perp$ to HK. § 97

On HK take $BK = BH$.

Draw OH, OK, GH, and GK.

Then $OH = OK$. § 150

$\therefore GH = GK$. § 439

$\therefore GB$ is $\perp$ to HK, by § 151. Q.E.D.

6. *Find the point in a plane to which lines may be drawn from two given external points on the same side of the plane so that their sum shall be the least possible.*

Given A and B, two points on the same side of the plane MN.

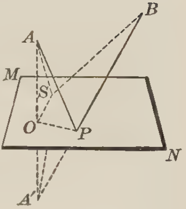
Required to find a point P in MN, so that $PA + PB$ is the least possible.

Construction. Let fall $AO \perp$ to MN.

Produce AO to A', making $OA' = AO$.

Draw BA', cutting MN in P.

Then P is the required point in MN. Q.E.F.



Proof. Draw PA.

Take any other point S in MN, and draw SA, SB, and SA'. Also draw SO and PO.

Pass a plane through line AA' and point P; also pass a plane through line AA' and point S.

In the planes APA' and ASA', PO and SO are each $\perp$ to AA' at its mid-point. § 430

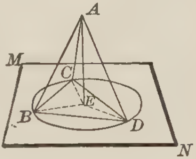
$\therefore PA = PA'$, and $SA = SA'$. § 150

Now $PB + PA' < SB + SA'$. Post. 3

$\therefore PB + PA < SB + SA$, by Ax. 9. Q.E.D.

7. *If three equal oblique lines are drawn from an external point to a plane, the perpendicular from the point to the plane meets the plane at the center of the circle circumscribed about the triangle having for its vertices the feet of the oblique lines.*

Given AE the $\perp$, and AB, AC, and AD three equal oblique lines drawn to the plane MN from an external point A; also lines BD, DC, and CB inclosing the $\triangle BCD$.



To prove that E is the center of the $\odot$ circumscribed about $\triangle BCD$.

Proof. Draw EB, EC, and ED.

The $\angle s$ AEB, AEC, and AED are rt. $\angle s$. § 430

In the rt. $\triangle s$ AEB, AEC, and AED,

$AE = AE = AE$, Iden.

and $AB = AC = AD$. Given

Therefore the rt. $\triangle s$ AEB, AEC, and AED are congruent. § 89

$\therefore EB = EC = ED$. § 67

Then a $\odot$ may be circumscribed about the $\triangle BCD$ with E as center and EB as radius.

Therefore E is the center of the $\odot$ circumscribed about $\triangle BCD$. Q.E.D.

8. *State and prove the propositions of plane geometry corresponding to §§ 444, 445, and 446. Why do not the proofs of those propositions apply to these sections?*

The following propositions of plane geometry correspond to §§ 444, 445, and 446 respectively:

Two lines in the same plane perpendicular to the same line are parallel. § 95

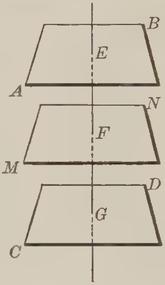
If a line is perpendicular to one of two parallel lines, it is perpendicular to the other also. § 97

Two lines in the same plane parallel to a third line are parallel to each other. § 96

The proofs are not directly applicable to these sections, since these deal with lines that are not given as lying in the same plane.

Exercise 75. Page 285

1. *What is the locus of a point in a plane equidistant from two parallel lines? What is the corresponding locus in space, given two parallel planes instead of two parallel lines? Draw the figure, without proof.*



The locus of a point in a plane equidistant from two parallel lines is a line parallel to either of them and midway between them.

The locus of a point in space equidistant from two parallel planes is a plane parallel to either of them and midway between them.

Given the ∥ planes AB and CD.

Required to find the locus of a point in space equidistant from AB and CD.

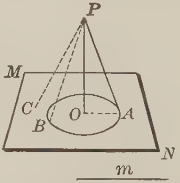
Construction. From E, any point in AB, let fall $EG \perp$ to CD. § 437

Bisect EG in F. § 229

Through F pass the plane MN ∥ to AB.

Then MN is the required locus. Q.E.F.

2. *Find the locus in a plane of a point at a given distance from a given external point. What is the corresponding case of plane geometry?*



Given MN any plane, and P any external point; also PO the $\perp$ to the plane MN from P, and line $m > PO$.

Required to find the locus of a point in MN at the distance m from P.

Construction. Draw the line $PA = m$.

Draw OA.

From O as a center, with a radius equal to OA, describe a ⊙ in MN.

Then the ⊙ in MN whose center is O is the required locus. Q.E.F.

Proof. Take any point B other than A in the ⊙ whose center is O, and any point C within or without this ⊙.

Draw PB and PC.

Then $PB = PA$,

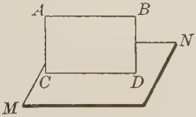
and $PC \neq PA$. § 439

Therefore any point, and hence every point, in the ⊙ whose center is O is the distance PA, or m, from P. § 439

Hence the ⊙ whose center is O is the locus of a point in MN at the distance m from P. Q.E.D.

The corresponding case of plane geometry is: The locus of a point in a plane at a given distance from a fixed point is a circle with the fixed point as center and the given distance as radius.

3. *If a given line is parallel to a given plane, the intersection of the plane with any plane passed through the given line is parallel to that line.*



Given the line AB ∥ to the plane MN, and the plane CB passed through AB, intersecting MN in CD.

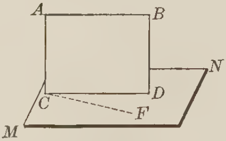
To prove that CD is ∥ to AB.

Proof. The lines CD and AB lie in the plane CB.

CD and AB cannot meet, however far they are produced, for AB is ∥ to MN. § 447

$\therefore$ CD is ∥ to AB, by § 93. Q.E.D.

4. *If a given line is parallel to a given plane, a line parallel to the given line drawn through any point of the plane lies in the plane.*



Given the line AB ∥ to the plane MN, and the line CD drawn through any point C of the plane MN ∥ to AB.

To prove that CD lies in MN.

Proof. Pass a plane through the line AB and the point C, and let it intersect the plane MN in the line CF. § 426

Then, since CF and AB lie in the same plane and cannot meet, however far produced, CF and AB are ∥. § 93

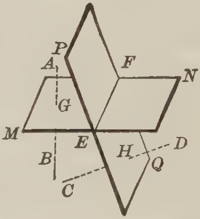
But CD is ∥ to AB. Given

$\therefore$ CF coincides with CD. § 94

Since CF lies in the plane MN, CD lies in MN. Q.E.D.

Exercise 76. Page 286

1. *What is the locus of a point equidistant from two given points A, B, and also equidistant from two other given points C, D?*



Given the points A, B, C, and D.

Required to find the locus of a point equidistant from A and B, and also equidistant from C and D.

Construction. Draw the lines AB and CD.

Bisect AB and CD, and denote the respective points of bisection by G and H. § 229

Through G and H pass the planes MN and $PQ \perp$ to AB and CD respectively.

Let the planes MN and PQ intersect in EF. § 429

Then EF is the required locus. Q. E. F.

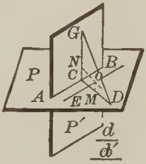
Proof. MN is the locus of a point equidistant from A and B.

Also PQ is the locus of a point equidistant from C and D. § 443

Then the straight line EF, the line of intersection of the planes MN and PQ, is the locus of a point equidistant from A and B, and also equidistant from C and D. Q. E. D.

Discussion. This problem is impossible for the general case where AB and CD are $\parallel$ lines. If the plane bisecting AB perpendicularly bisects also the line CD perpendicularly, that plane is the locus of a point fulfilling the conditions.

2. *What is the locus of a point at the distance d from a given plane P and at the distance d′ from a given plane P′?*



Given the planes P and P'; also the lines d and d'.

Required to find the locus of a point at the given distance d from the plane P, and at the given distance d' from the plane P'.

Construction. Produce the planes P and P', letting them intersect in AB. § 429

Take any point C in AB.

Through C pass the plane $GCD \perp$ to the line AB, cutting the planes P and P' in the lines CD and CG respectively.

Through a point in the plane GCD at the distance d from the plane P draw the line $NO \parallel$ to CD; through a point in the plane GCD at the distance d' from plane P' draw the line $MO \parallel$ to CG.

Let NO and MO intersect in O.

Construct $EO \perp$ to the plane GCD at O.

Also construct similarly the three other possible lines corresponding to EO.

Then EO and the three other lines constructed similarly constitute the required locus. Q.E.F.

Proof. NO is $\parallel$ to CD,

and MO is $\parallel$ to CG. Const.

$\therefore$ NO is $\parallel$ to the plane P,

and MO is $\parallel$ to the plane P'. § 448

Then any point on NO is at the distance d from the plane P, and any point on MO is at the distance d' from the plane P'. § 128

$\therefore$ O is at the distance d from P, and at the distance d' from P'.

Now AC and EO are $\perp$ to the plane GCD. Const.

$\therefore$ EO is $\parallel$ to AC. § 444

$\therefore$ EO is $\parallel$ both to P and to P'. § 448

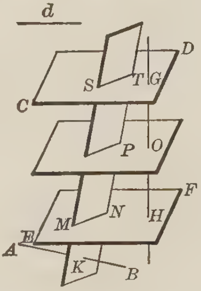
Since O, one point on EO, is at the distance d from P, and at the distance d' from P', every point on EO is at the distance d from P, and at the distance d' from P'.

Similarly, any point on each of the other three lines constructed in like manner to EO fulfills the same conditions.

Therefore EO and the three other lines constructed similarly constitute the required locus. Q.E.D.

Discussion. When the planes P and P' are $\parallel$, the given problem is possible if the distance between them is $d + d'$, or $d' - d$; otherwise it is impossible.

3. *What is the locus of a point at the distance d from a given plane P, and equidistant from two given points A, B?*



Given the plane P, points A and B, and line d.

Required to find the locus of a point at the distance d from the plane P, and equidistant from the points A and B.

Construction. At any point O in the plane P erect the $\perp$ OG.

On OG take $OG = OH = d$.

Through G and H pass the planes CD and EF $\parallel$ to the plane P.

Draw the line AB, and bisect AB at K. § 229

Through K pass the plane $MT \perp$ to AB, intersecting the planes CD and EF in the lines ST and MN respectively. § 429

Then the lines MN and ST constitute the locus of a point fulfilling the required conditions. Q.E.F.

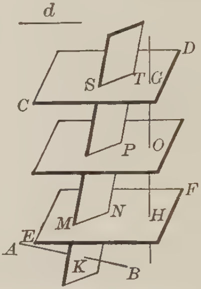
Proof. The planes CD and EF are each $\parallel$ to the plane P, and at the distance d from it. Const.

Therefore the planes CD and EF constitute the locus of a point at the distance d from the plane P.

The plane MT is $\perp$ to AB at its mid-point K. Const.

Therefore MT is the locus of a point equidistant from A and B. § 443

Then MN and ST, the respective intersections of the plane MT with the planes EF and CD, constitute the locus of a point at the distance d from P, and equidistant from A and B. Q. E. D.

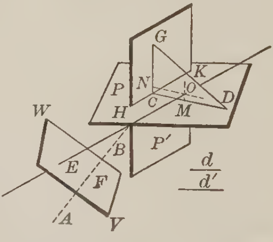


Discussion. This problem is impossible when the points A and B have positions such that the locus of a point equidistant from A and B is $\parallel$ to the given plane. An exception to this is noted when the line AB is $\perp$ to the plane P, and K, the mid-point of AB, is at the given distance d from the plane P.

4. *Find a point at the distance d from a given plane P, at the distance d' from a given plane P', and equidistant from two given points A, B. Can there be more than one such point? Draw the figure, without proof.*

Given the planes P and P', the lines d and d', and the points A and B.

Required to find a point at the distances d and d' from the planes P and P' respectively, and equidistant from A and B.



Construction. Produce the planes P and P', letting them intersect in HK. § 429

Take any point C in HK.

Through C pass the plane GCD $\perp$ to the line HK, cutting the planes P and P' in the lines CD and CG respectively. § 429

Through a point in the plane GCD at the distance d from the plane P draw the line NO $\parallel$ to CD; through a point in the plane GCD at the distance d' from the plane P' draw the line MO $\parallel$ to CG.

Let NO and MO intersect in O.

Then construct EO $\perp$ to the plane GCD at O.

Next draw the line AB, and bisect it at F. § 229

Through F pass the plane WV $\perp$ to AB.

Produce the plane WV, and the line OE to intersect in E.

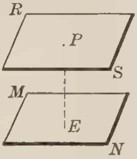
Then E is a point fulfilling the given conditions. Q. E. F.

Discussion. If the constructed plane WV is not ∥ to, or does not contain, the line EO or any of the three other possible lines constructed similarly, there are three other points that fulfill the given conditions.

If the plane WV contains EO, or any one of the three other possible lines constructed similarly, or any two of these four lines, every point on that line or those lines fulfills the given conditions; and if the plane WV is ∥ to the line EO and to each of the three other possible lines constructed similarly, the required point does not exist.

Exercise 77. Page 292

1. *Find the locus of a line drawn through a given point, parallel to a given plane.*



Given the plane MN, and the point P.

Required to find the locus of a line drawn through P ∥ to the plane MN.

Construction. Through P draw the line $PE \perp$ to MN.
Through P pass the plane $RS \perp$ to PE.
Then the plane RS is the required locus. Q. E. F.

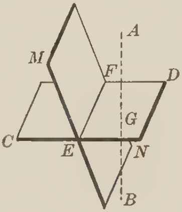
Proof. The planes MN and RS are ∥. § 452

Therefore any line, and hence every line, drawn in the plane RS through P is ∥ to the plane MN. § 447

Moreover, any line, and hence every line, drawn through P and not lying in the plane RS is not ∥ to the plane MN.

Therefore the plane RS is the locus of all lines drawn through P ∥ to the plane MN. Q. E. D.

2. *Find the locus of a point in a given plane that is equidistant from two given points not in the plane.*



Given the plane MN, and the points A and B not in the plane MN.

Required to find the locus of a point in the given plane MN equidistant from the points A and B.

Construction. Draw the line AB.
Bisect the line AB in G. § 229

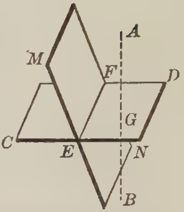
Through G pass a plane $\perp$ to AB, cutting the given plane MN in EF.

Then EF is the required locus. Q. E. F.

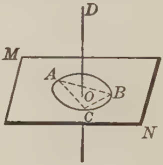
Proof. The plane CD is the locus of a point equidistant from A and B. § 443

Therefore EF, the intersection of the planes MN and CD, is the locus of a point in MN equidistant from A and B. Q.E.D.

Discussion. This problem is impossible when the line AB is $\perp$ to the plane MN.



3. *Find the locus of a point equidistant from three given points not in a straight line.*



Given the three points A, B, and C not in a straight line.

Required to find the locus of a point in space equidistant from A, B, and C.

Construction. Pass the plane MN through the points A, B, and C. § 427

Draw the lines AB, AC, and BC.

Circumscribe about $\triangle ABC$ the $\odot ABC$. § 240

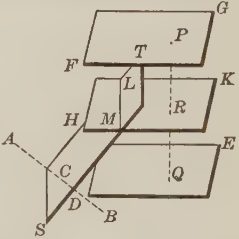
Through O, the center of the $\odot ABC$, draw $OD \perp$ to the plane MN.

Then OD is the required locus. Q.E.F.

Proof. The line OD is $\perp$ to the plane MN of the $\triangle ABC$ at the center O of the $\odot$ circumscribed about $\triangle ABC$. Const.

Therefore the line OD is the locus of a point equidistant from A, B, and C, by § 442. Q.E.D.

4. *Find the locus of a point equidistant from two given parallel planes and also equidistant from two given points.*



Given the two $\parallel$ planes DE and FG, and the two points A and B.

Required to find the locus of a point equidistant from the $\parallel$ planes DE and FG, and also equidistant from A and B.

Construction. From any point P in the plane FG let fall $PQ \perp$ to the plane DE.

Bisect PQ in R. § 229

Through R pass the plane $HK \parallel$ to the plane DE.

Draw the line AB.

Bisect AB in C. § 229

Through C pass the plane $ST \perp$ to line AB, cutting the plane HK in ML.

Then ML is the required locus. Q.E.F.

Proof. PQ is $\perp$ to plane FG. § 456

Therefore the plane HK is the locus of a point equidistant from the two $\parallel$ planes FG and DE. § 458

But the plane ST is the locus of a point equidistant from the points A and B. § 443

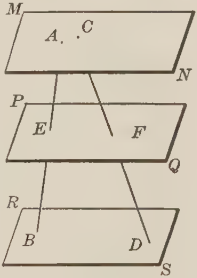
Therefore ML, the line of intersection of the planes HK and ST, is the locus of a point equidistant from the $\parallel$ planes FG and DE, and also equidistant from A and B. Q.E.D.

5. *What is the locus of a point in a plane at a given distance from a given line in the plane? What is the locus of a point at a given distance from a given plane?*

The locus of a point in a plane at a given distance from a given line consists of two lines each $\parallel$ to the given line, the one lying at the given distance on one side of the given line, the other at the given distance on the opposite side of the given line.

The locus of a point in space at a given distance from a given plane consists of two planes each $\parallel$ to the given plane, the one lying at the given distance on one side of the given plane, the other at the given distance on the opposite side of the given plane.

6. *The line AB cuts three parallel planes in the points A, E, B; and the line CD cuts these planes in the points C, F, D. If $AE = 6$ in., $EB = 8$ in., and $CD = 12$ in., compute CF and FD.*



Given the three parallel planes MN, PQ, and RS cut by the lines AB and CD in the points A, E, B, and C, F, D respectively. $AE = 6$ in., $EB = 8$ in., and $CD = 12$ in.

Required to compute the lengths of CF and FD.

Solution. $AE : EB = CF : FD$. § 462

$\therefore AE + EB : EB = CF + FD : FD$. § 267

$\therefore AE + EB : EB = CD : FD$. Ax. 11

$\therefore 6 + 8 : 8 = 12 : FD$. Ax. 9

That is, $14 : 8 = 12 : FD$.

$\therefore 14\,FD = 96$. § 261

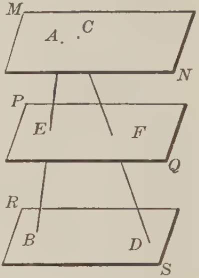
$\therefore FD = 6\frac{6}{7}$. Ax. 4

But $CF = CD - FD$.

$\therefore CF = 12 - 6\frac{6}{7}$, or $5\frac{1}{7}$. Ax. 9

$\therefore$ the lengths of CF and FD are $5\frac{1}{7}$ in. and $6\frac{6}{7}$ in. Q.E.F.

7. *The line AB cuts three parallel planes in the points A, E, B; and the line CD cuts these planes in the points C, F, D. If $AB = 8$ in., $CF = 5$ in., and $CD = 9$ in., compute AE and EB.*



Given the three $\parallel$ planes MN, PQ, and RS cut by the lines AB and CD in the points A, E, B, and C, F, D respectively. $AB = 8$ in., $CF = 5$ in., and $CD = 9$ in.

Required to compute the lengths of AE and EB.

Solution. $AE : EB = CF : FD$. § 462

$\therefore AE + EB : AE = CF + FD : CF$. § 267

$\therefore AB : AE = CD : CF$. Ax. 11

$\therefore 8 : AE = 9 : 5$. Ax. 9

$\therefore 9\,AE = 5 \times 8$. § 261

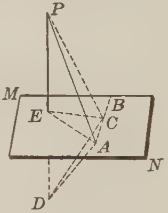
$\therefore AE = \frac{40}{9} = 4\frac{4}{9}$. Ax. 4

But $EB = AB - AE$.

$\therefore EB = 8 - 4\frac{4}{9} = 3\frac{5}{9}$. Ax. 9

$\therefore$ the lengths of AE and EB are $4\frac{4}{9}$ in. and $3\frac{5}{9}$ in. Q.E.F.

8. *To draw a perpendicular to a given plane from a given point without the plane.*



Given P any point without the plane MN.

Required to let fall a $\perp$ from the point P to the plane MN.

Construction. Draw any line AB in the plane MN.

Through the point P and the line AB pass a plane PAB. § 426

In the plane PAB draw the line $PC \perp$ to AB. § 227

Draw EC in the plane $MN \perp$ to AB. § 228

Through PC and EC pass the plane PEC. § 425

Let fall the $\perp$ PE to EC in the plane PEC. § 227

Then PE is the required line. Q.E.F.

Proof. Draw EA to any point A in AB except C.

Then draw PA.

In the plane PEC produce PE to D, making $ED = PE$, and draw CD.

Pass a plane through PE and PA, and in the plane PEA draw AD.

The line EC is $\perp$ to PD at its mid-point E. Const.

$\therefore PC = CD$. § 150

Since AC is $\perp$ to EC and PC, Const.

AC is $\perp$ to the plane PEC. § 431

$\therefore AC$ is $\perp$ to CD. § 430

Then in the rt. $\triangle$s ACP and ACD,

$PC = CD$, Proved

and $AC = AC$. Iden.

$\therefore$ the rt. $\triangle$s ACP and ACD are congruent. § 90

$\therefore PA = AD$. § 67

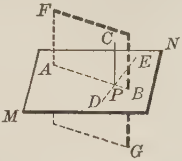
Then in the plane PEA, A is equidistant from P and D.

$\therefore EA$ is $\perp$ to PE. § 151

Also PE is $\perp$ to EC. Const.

$\therefore PE$ is $\perp$ to the plane MN, by § 431. Q. E. D.

9. *To erect a perpendicular to a given plane at a given point in the plane.*



Given P any point in the plane MN.

Required to erect a $\perp$ to the plane MN at P.

Construction. Draw DE, any line through P in the plane MN.

Through the point P pass the plane $FG \perp$ to DE, cutting the plane MN in AB.

In the plane FG draw $CP \perp$ to the line AB at P. § 228

Then CP is the required line. Q. E. F.

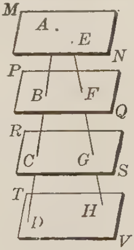
Proof. DE is $\perp$ to the plane FG. Const.

$\therefore DE$ is $\perp$ to CP. § 430

Also CP is $\perp$ to AB. Const.

$\therefore CP$ is $\perp$ to the plane MN, by § 431. Q. E. D.

10. *It is proved in plane geometry that if three or more parallels intercept equal segments on one transversal, they intercept equal segments on every transversal. State and prove a corresponding proposition in solid geometry.*



If three or more parallel planes intercept equal segments on one line, they intercept equal segments on every line passing through them.

Given the $\parallel$ planes MN, PQ, RS, and TV intercepting the equal segments AB, BC, and CD on the line AD, and cutting any other line EH in E, F, G, and H.

To prove that $EF = FG = GH$.

Proof. $AB : BC = EF : FG$,

and $BC : CD = FG : GH$. § 462

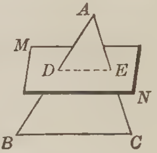
But $AB = BC = CD$. Given

$$\therefore \frac{EF}{FG} = 1, \text{ and } \frac{FG}{GH} = 1.$$ Ax. 9

$\therefore EF = FG$, and $FG = GH$. Ax. 3

$\therefore EF = FG = GH$, by Ax. 8. Q. E. D.

11. *It is proved in plane geometry that the line joining the mid-points of two sides of a triangle is parallel to the third side. State and prove a corresponding proposition in solid geometry, referring to a plane passing through the mid-points of two sides of a triangle.*



A plane passing through the mid-points of two sides of a triangle, and not containing the third side, is parallel to the third side.

Given the plane MN passing through D and E, the mid-points of the sides AB and AC respectively of the $\triangle ABC$, but not containing the third side BC.

To prove that the plane MN is $\parallel$ to BC.

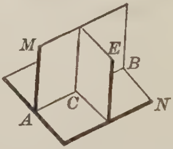
Proof. Now the plane MN contains the line DE. § 422

But DE is $\parallel$ to BC. § 136

$\therefore$ the plane MN is $\parallel$ to BC, by § 448. Q. E. D.

Exercise 78. Page 299

1. *A plane perpendicular to the edge of a dihedral angle is perpendicular to each of its faces.*



Given the plane $CE \perp$ to the edge of the dihedral $\angle N\text{-}AB\text{-}M$ at C.

To prove that the plane CE is $\perp$ to the planes NA and MB.

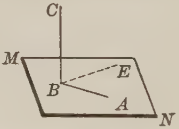
Proof. The plane CE is $\perp$ to AB. Given

That is, AB is $\perp$ to the plane CE.

$\therefore$ the planes NA and MB are each $\perp$ to the plane CE. § 477

Therefore the plane CE is $\perp$ to the planes NA and MB. Q. E. D.

2. *If one line is perpendicular to another, is any plane passed through the first line perpendicular to the second? Prove it.*



If one line is perpendicular to another, only one plane can be passed through the first line perpendicular to the second.

Given the line $AB \perp$ to the line BC.

To prove that only one plane can be passed through $AB \perp$ to BC.

Proof. Draw BE any other line $\perp$ to BC.

Through AB and BE pass the plane MN. § 425

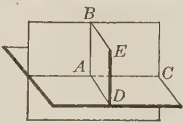
Then CB is $\perp$ to the plane MN. § 431

Only one plane can be drawn $\perp$ to BC at B. § 433

Therefore MN is the only plane that can be passed through $AB \perp$ to BC at B. Q. E. D.

3. *If three lines are perpendicular to one another at a common point, what is the relation to one another of the three planes determined by the three pairs of lines? Prove it.*

If three lines are perpendicular to one another at a common point, the three planes determined by the three pairs of lines are perpendicular each to the other two.



Given the three lines AB, AC, and AD ⊥ to one another, and the planes BC, DC, and AE determined by AB and AC, AD and AC, and AB and AD respectively.

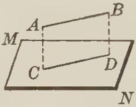
To prove that the planes BC, DC, and AE are ⊥ each to the other two.

Proof. AB is ⊥ to AC and AD. Given
$\therefore$ AB is ⊥ to the plane DC. § 431
$\therefore$ the planes AE and BC are each ⊥ to the plane DC. § 477
Again, AD is ⊥ to AB and AC. Given
$\therefore$ AD is ⊥ to the plane BC. § 431
$\therefore$ the plane AE is ⊥ to the plane BC. § 477

Therefore the planes BC, DC, and AE are ⊥ each to the other two. Q.E.D.

Exercise 79. Page 305

1. *Describe the position of a segment of a line relative to a given plane if the projection of the segment on the plane is equal to its own length.*



Let CD be the projection of AB, a segment of a line, on the plane MN, and let $CD = AB$.

Draw AC and BD.

Then AC and BD are each ⊥ to the plane MN, § 481
and hence to the line CD. § 430
Consequently AC and BD are ∥ lines. § 95
Pass a plane AD through the ∥ lines AC and BD. § 428
Then CD and AB will lie in the plane AD. § 422

The previous discussion shows that the quadrilateral $CABD$ is either an isosceles trapezoid or a parallelogram. In either case the sum of the $\angle s$ ACD and ABD would equal 180°.

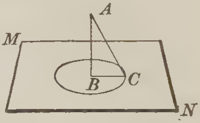
Since $\angle ACD = 90°$,
$\therefore \angle ABD = 90°$.

Consequently AB and CD, each being ⊥ to the line BD, are ∥. § 95
Therefore AB is ∥ to the plane MN, by § 448. Q.E.F.

2. *From a point A, 4 in. from a plane MN, an oblique line AC 5 in. long is drawn to the plane and made to turn around the perpendicular AB dropped from A to the plane. Find the area of the circle described by the point C.*

Given AB and AC, a ⊥ and an oblique line respectively, drawn to the plane MN from an external point A, and the ⊙ in the plane MN having for its center B and for its radius BC. $AC = 5$ in., and $AB = 4$ in.

Required to find the area of the given ⊙.



Solution. AB is ⊥ to MN. Given

∴ AB is ⊥ to BC. § 430

In the rt. △ ABC, $\overline{BC}^2 = \overline{AC}^2 - \overline{AB}^2$. § 338

∴ $\overline{BC}^2 = 5^2 - 4^2$, or 3^2. Ax. 9

But the area of the given ⊙ $= \pi \times \overline{BC}^2$ § 389

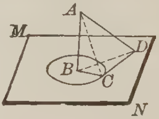
$= 3^2 \cdot \pi = 9\,\pi$, or 28.2744.

Therefore the area of the ⊙ = 28.2744 sq. in. Q. E. F.

3. *From a point A, 8 in. from a plane MN, a perpendicular AB is drawn to the plane; with B as a center and a radius equal to 6 in., a circle is described in the plane; at any point C on this circle a tangent CD is drawn 24 in. in length. Find the distance from A to D.*

Given AB ⊥ to the plane MN from an external point A; a ⊙ in the plane MN, having B as a center and BC as a radius; CD a tangent to the given ⊙; and the line AD. $AB = 8$ in., $BC = 6$ in., and $CD = 24$ in.

Required to compute the length of AD.



Solution. Draw AC and BD.

The line CD is ⊥ to BC. § 185

Also the line AB is ⊥ to BD. § 430

In the rt. △ ABD, $\overline{AD}^2 = \overline{AB}^2 + \overline{BD}^2$. § 337

But in the rt. △ BCD, $\overline{BD}^2 = \overline{BC}^2 + \overline{CD}^2$. § 337

∴ $\overline{AD}^2 = \overline{AB}^2 + \overline{BC}^2 + \overline{CD}^2$. Ax. 9

∴ $\overline{AD}^2 = 8^2 + 6^2 + 24^2$. Ax. 9

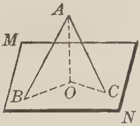
∴ $\overline{AD}^2 = 676$, or 26^2.

∴ $AD = 26$. Ax. 5

Hence the length of AD is 26 in. Q. E. F.

4. *Equal lines drawn from a given external point to a given plane are equally inclined to the plane.*

Given AB and AC, equal straight lines drawn to the plane MN from the external point A; also BO and CO, the projections of AB and AC respectively on the plane MN.



To prove that $\angle ABO = \angle ACO$.

Proof. Draw AO.

Then AO is $\perp$ to the plane MN. § 481

$\therefore AO$ is $\perp$ to BO and CO. § 430

In the rt. ◬ AOB and AOC,

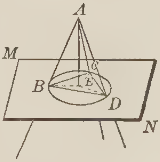
$AB = AC$, Given

and $AO = AO$. Iden.

$\therefore$ the rt. ◬ AOB and AOC are congruent. § 89

$\therefore \angle ABO = \angle ACO$, by § 67. Q. E. D.

5. *If three equal lines are drawn to a plane from an external point, the perpendicular from the point to the plane determines the center of the circle circumscribed about the triangle determined by the planes of the three lines.*



Given the three straight lines AB, AC, and AD, not in the same plane, cut by the plane MN, such that $AB = AC = AD$; AE the $\perp$ to the plane MN from A; and the $\odot BCD$ circumscribed about the $\triangle BCD$ inclosed by the intersections of the plane MN with the planes ABC, ACD, and ABD.

To prove that E is the center of the $\odot BCD$.

Proof. Draw EB, EC, and ED.

Then AE is $\perp$ to EB, EC, and ED. § 430

In the rt. ◬ AEB, AEC, and AED,

$AB = AC = AD$, Given

and $AE = AE = AE$. Iden.

$\therefore$ the rt. ◬ AEB, AEC, and AED are congruent. § 89

$\therefore EB = EC = ED$. § 67

Then EB is the radius of the $\odot BCD$.

Therefore E is the center of the $\odot BCD$. Q. E. D.

6. *Three lines not in the same plane meet in a point. How shall a line be drawn so as to make equal angles with all three of these lines?*

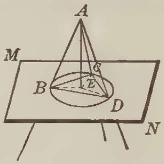
Given the three straight lines AB, AC, and AD, not lying in the same plane, cut by the plane MN in the points B, C, and D, such that $AB = AC = AD$.

Required to draw a line so as to make equal ∡ with all three of these lines.

Construction. Draw the lines BC, BD, and CD.

Find the center E of the ⊙ circumscribed about the $\triangle BCD$. § 246

Then draw AE.



The line AE makes equal $\angle$s with AB, AC, and AD. Q.E.F.

Proof. In the $\triangle$s BEA, CEA, and DEA,

$AB = AC = AD$, Given

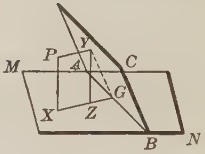
$EB = EC = ED$, § 162

$AE = AE = AE$. Iden.

$\therefore$ the $\triangle$s BEA, CEA, and DEA are congruent. § 80

$\therefore \angle BAE = \angle CAE = \angle DAE$, by § 67. Q.E.D.

7. *From a point P two perpendiculars PX and PY are drawn to two planes MN and AC which intersect in AB. From Y a perpendicular YZ is drawn to MN. Prove that the line XZ is perpendicular to AB.*



Given PX and PY, $\perp$s drawn from the point P to the planes MN and AC, intersecting in AB; also YZ, a $\perp$ from Y to the plane MN, and the line XZ.

To prove that XZ is $\perp$ to AB.

Proof. The lines PX and YZ are each $\perp$ to the plane MN. Given

$\therefore PX$ is $\parallel$ to YZ. § 444

Through PX and YZ pass a plane intersecting the plane AC in YG.

The line XZ is the intersection of the planes PG and MN, and meets AB in G.

Since PX is $\perp$ to MN, and PY to AC, Given

$\therefore$ the plane PG is $\perp$ both to MN and to AC. § 477

$\therefore AB$ is $\perp$ to the plane PG. § 478

$\therefore AB$ is $\perp$ to XZ. § 430

That is, XZ is $\perp$ to AB. Q.E.D.

8. *If the length of the shadow of a tree standing on level ground exceeds the height of the tree, the angle made by the sun above the horizon must be less than what known angle?*

The angle in question must be less than 45°. See §§ 107, 113.

9. *Find the locus of a point at a given distance from a given plane and equidistant from two given points not in the plane.*

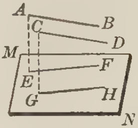
The proof is the same as for Ex. 3, p. 286 (see p. 349 of this book).

Exercise 80. Page 307

1. *Parallel lines have parallel projections on a plane.*

Given EF and GH, the projections of the $\parallel$ lines AB and CD respectively on the plane MN.

To prove that EF and GH are $\parallel$.



Proof. Pass the planes AF and CH through the lines AB and EF and CD and GH respectively. § 425

Draw AE and CG.

Then AE and CG are each $\perp$ to the plane MN. § 481

$\therefore$ AE is $\parallel$ to CG. § 444

$\therefore$ the planes AF and CH are $\parallel$. § 461

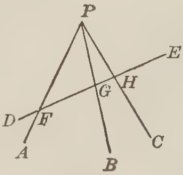
Now the plane MN cuts the $\parallel$ planes AF and CH in the lines EF and GH respectively.

Therefore EF and GH are $\parallel$, by § 453. Q.E.D.

2. *If two planes are perpendicular to each other, any line perpendicular to one of them is how related to the other?*

If two planes are $\perp$ to each other, any line $\perp$ to one of them is $\parallel$ to the other, for the $\perp$ to the first plane cannot meet the second plane, however far they are produced.

3. *If three lines passing through a given point P are cut by a fourth line that does not pass through P, the four lines all lie in the same plane.*



Given the three lines PA, PB, and PC intersected by the line DE in the points F, G, and H respectively.

To prove that the lines PA, PB, PC, and DE all lie in the same plane.

Proof. Pass the plane PFG through the lines PA and PB. § 425

Since F and G lie in the plane PFG, the line FG lies in the plane PFG. § 422

That is, the line DE lies in the plane PFG.

Then the point H lies in the plane PFG.

Since P and H both lie in the plane PFG, the line PH, or PC, lies in the plane PFG. § 422

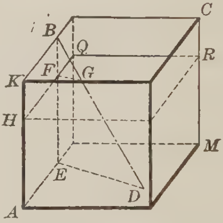
Therefore the lines PA, PB, PC, and DE all lie in the plane PFG. Q.E.D.

4. *Seven lines, no three of which lie in the same plane, pass through the same point. How many planes are determined by these lines?*

The number of planes determined by the seven given lines is twenty-one. To determine this, we reason as follows:

Each of the seven lines with the other six lines taken one at a time determines six planes. Then 7 × 6, or 42, is twice the number of planes determined by the seven given lines. Hence $\frac{1}{2}$ of 42, or 21, is the required number of planes.

5. *A cubical tank* 10 *in. deep contains water to a depth of* 7 *in. A foot rule is placed obliquely on the bottom so as just to reach the top edge of the tank. Make a sketch of the tank, and compute the length of the rule covered by water.*



Given the figure AC representing a cubical tank filled with water to the depth of AH; also BD representing a foot rule in a position oblique to the bottom of AC, the point B being on the top edge. $AK = 10$ in., $AH = 7$ in., and $BD = 12$ in.

Required to find the length of that part of the rule that is in the water.

Solution. Draw $BE \perp$ to the line AE. § 227

Connect D and E by the line DE.

Then draw FG joining the points on BE and BD which lie in the plane HR, the plane HR being the plane of the surface of the water in the tank.

The line BE is $\perp$ to the plane AM. § 474

Since the planes AM and HR are $\parallel$, the line BF is $\perp$ to the plane HR. § 456

$\therefore$ the $\angle$s BED and BFG are rt. $\angle$s. § 430

Moreover, $\angle EBD = \angle FBG$. Iden.

$\therefore$ the rt. $\triangle$s BED and BFG are similar. § 287

$\therefore BE : BF = BD : BG$. § 282

$\therefore BE - BF : BE = BD - BG : BD$. § 268

But $BE - BF = FE$, and $BD - BG = GD$.

$\therefore FE : BE = GD : BD$. Ax. 9

Now $FE = AH$, and $BE = AK$. § 454

$\therefore AH : AK = GD : BD$. Ax. 9

$\therefore 7 : 10 = GD : 12$. Ax. 9

$\therefore 10\,GD = 84$. § 261

$\therefore GD = 8.4$. Ax. 4

$\therefore$ the length of GD is 8.4 in. Q. E. F.

Exercise 81. Page 314

1. *Find the locus of a point in a space of three dimensions equidistant from two given intersecting lines.*

Given the lines AB and CD intersecting at O.

Required to find the locus of a point in space equidistant from AB and CD.

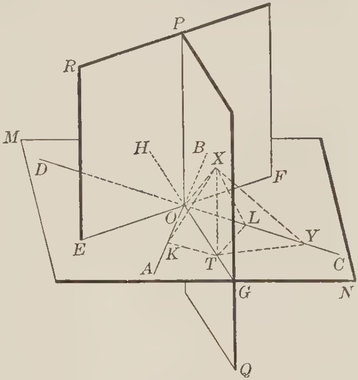


FIG. 1

Construction. Pass through AB and CD the plane MN determined by these lines. § 425

In the plane MN draw EF and GH, the bisectors of the $\angle s$ DOA and AOC respectively. § 231

Through EF and GH pass the planes RF and PQ respectively $\perp$ to the plane MN, intersecting each other in PO. § 429

Then the planes PQ and RF constitute the locus of a point in space equidistant from the given lines AB and CD. Q. E. F.

Proof. Take any point X in the plane PQ, and draw $XT \perp$ to GH. § 227

Draw TK and $TL \perp$ to AB and CD respectively. § 227

Then draw XK and XL.

Furthermore, on LC take $LY = OL$, and draw the lines OX, YX, and TY.

The line XT is $\perp$ to the plane MN. § 474

Since TL is $\perp$ to OY at its mid-point L, Const.

$\therefore TO = TY$. § 150

$\therefore OX = YX$. § 439

$\therefore XL$ is $\perp$ to the line CD. § 151

Similarly, it may be shown that XK is $\perp$ to AB.

Now $TK = TL$. § 152

$\therefore XK = XL$. § 439

Next take any point Z (Fig. 2) in space without the plane PQ such that the foot of the $\perp$ ZZ' to the plane MN falls within the $\angle AOC$.

Draw $Z'K'$ and $Z'L'$ $\perp$ to AB and CD respectively. § 227

Also draw ZK' and ZL'.

Then $Z'K' \neq Z'L'$. § 152

Since ZZ' is $\perp$ to the plane MN, reasoning as above, it may be shown that ZK' and ZL' are $\perp$ to AB and CD respectively.

Then $ZK' \neq ZL'$. § 439

Likewise it may be shown that any point in the plane RF is equally distant from the lines AB and CD, and any point without RF is unequally distant from the lines AB and CD.

Therefore the planes PQ and RF constitute the locus of a point in space equidistant from the intersecting lines AB and CD. Q.E.D.

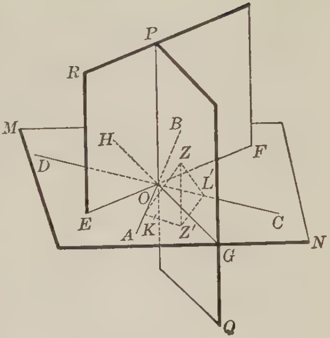
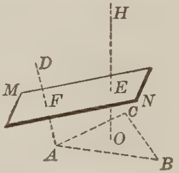


FIG. 2

2. *Find a point at equal distances from four points not all in the same plane.*



Given the points A, B, C, and D, not all in the same plane.

Required to find a point equidistant from A, B, C, and D.

Construction. Draw AB, AC, BC, and AD.

Find O, the point in the $\triangle ABC$ equidistant from A, B, and C. § 240

At O erect $HO \perp$ to the plane of $\triangle ABC$.

Bisect the line AD at F. § 229

Through F pass the plane $MN \perp$ to the line AD, cutting HO in E.

Then E is the required point. Q.E.F.

Proof. HO is the locus of a point equidistant from A, B, and C. § 442

Also the plane MN is the locus of a point equidistant from A and D. § 443

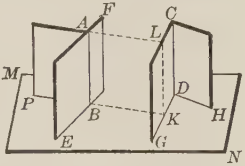
Therefore E, the point of intersection of the line HO with the plane MN, is equidistant from A, B, C, and D. Q.E.D.

3. *Two dihedral angles which have their edges parallel and their faces perpendicular are equal or supplementary.*

Given the planes PA and EF intersecting in AB, and the planes GC and HC intersecting in CD, PA being $\perp$ to GC and $EF \perp$ to HC, and AB being $\parallel$ to CD.

To prove that the dihedral $\measuredangle$ formed by the given intersecting planes are either equal or supplementary.

Proof. Pass the plane MN through $B \perp$ to AB, cutting CD in D and intersecting the planes PA, EF, GC, and HC in the lines PB, EB, GD, and HD respectively. § 429



Now since AB is $\perp$ to MN, Hyp.

$\therefore$ CD is $\perp$ to MN. § 445

$\therefore$ AB is $\perp$ to PB and EB, and CD is $\perp$ to GD and HD. § 430

That is, PB and EB are each $\perp$ to AB, and GD and HD are each $\perp$ to CD.

$\therefore$ the $\angle$s included by the lines PB and EB, and the lines GD and HD, are the plane $\angle$s of the given dihedral $\angle$s. § 468

Produce the plane PA to meet the plane GC in LK. § 429

Now since AB and CD are each $\perp$ to MN, the planes PA and GC are each $\perp$ to MN. § 477

Then LK is $\perp$ to MN. § 478

$\therefore$ LK is $\perp$ to PK. § 430

$\therefore$ PK is $\perp$ to the plane GC. § 474

$\therefore$ PK is $\perp$ to GD.

That is, PB is $\perp$ to GD. § 430

Similarly, it may be shown that

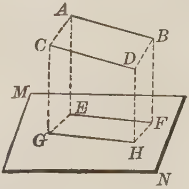
EB is $\perp$ to HD.

Now it is proved in plane geometry that if any two $\angle$s have their sides respectively $\perp$, they are either equal or supplementary. Ex. 17, p. 79

Therefore the $\angle$s included by the lines PB and EB, and the lines GD and HD, are either equal or supplementary.

Therefore the dihedral $\angle$s formed by the given intersecting planes are either equal or supplementary, by § 471. Q. E. D.

4. *The projections on a plane of equal and parallel line-segments are equal and parallel.*



Given EF and GH, the projections of the equal and $\parallel$ lines AB and CD respectively on the plane MN.

To prove that EF and GH are equal and $\parallel$.

Proof. Pass planes through AB and EF, and CD and GH, and draw AE, BF, CG, and DH.

Also draw AC, EG, BD, and FH.

Then AE and CG are each $\perp$ to MN. § 481

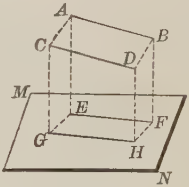
$\therefore$ AE and CG are $\parallel$. § 444

But AB and CD are $\parallel$. Given

∴ the planes AF and CH are ∥. § 461

∴ EF and GH are ∥. § 453

Now EG and FH are the projections of AC and BD respectively on the plane MN. § 482



Since AB and CD are equal and ∥, Given

∴ AC and BD are ∥. § 130

Since CG and DH are each ⊥ to MN, § 481

∴ CG and DH are ∥. § 444

∴ the planes AG and BH are ∥. § 461

∴ EG and FH are ∥. § 453

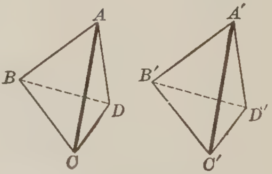
∴ the figure $EGHF$ is a ▱. § 118

∴ $EF = GH$. § 125

∴ EF and GH are equal and ∥. Q.E.D.

5. *Two trihedral angles are equal when two dihedral angles and the included face angle of the one are equal respectively to two dihedral angles and the included face angle of the other, and are similarly placed.*

Given, in the trihedral ∠s A and A', dihedral ∠s AB and AD equal to dihedral ∠s $A'B'$ and $A'D'$ respectively, and the face $\angle BAD$ equal to the face $\angle B'A'D'$; also the equal parts similarly placed.



To prove that the trihedral ∠s A and A' are equal.

Proof. Superpose the trihedral $\angle A'$ on the trihedral $\angle A$ so that $\angle B'A'D'$ coincides with its equal, $\angle BAD$.

Then, since the dihedral ∠s $A'B'$ and $A'D'$ equal the dihedral ∠s AB and AD respectively, the plane $B'A'C'$ will fall on the plane BAC, and the plane $D'A'C'$ will fall on the plane DAC.

Then $A'C'$ will fall along AC. § 429

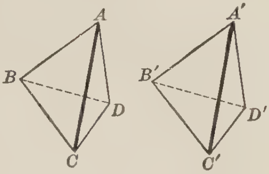
Therefore the trihedral ∠s A and A' coincide throughout and are equal, by § 493. Q.E.D.

6. *Two trihedral angles are equal when two face angles and the included dihedral angle of the one are equal respectively to two face angles and the included dihedral angle of the other, and are similarly placed.*

Given, in the trihedral ∠s A and A', dihedral $\angle AB$ equal to dihedral $\angle A'B'$, and the face ∠s BAC and BAD equal respectively to the face ∠s $B'A'C'$ and $B'A'D'$; also the equal parts similarly placed.

To prove that the trihedral ∠s A and A' are equal.

Proof. Superpose the trihedral $\angle A'$ on the trihedral $\angle A$ so that dihedral $\angle A'B'$ coincides with its equal, dihedral $\angle AB$.

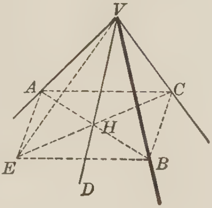


Then, since the face $\angle$s $B'A'C'$ and $B'A'D'$ equal the face $\angle$s BAC and BAD respectively, line $A'C'$ will fall along the line AC, and line $A'D'$ along AD.

Hence the face $\angle C'A'D'$ will coincide with the face $\angle CAD$. § 425

Therefore the trihedral $\angle$s A and A' coincide throughout and are equal, by § 493. Q. E. D.

7. *If the face angle AVB of the trihedral angle V-ABC is bisected by the line VD, the angle CVD is less than, equal to, or greater than half the sum of the angles AVC and BVC, according as $\angle CVD$ is less than, equal to, or greater than* 90°.



Case 1. Given VD bisecting the face $\angle AVB$ of the trihedral $\angle V$-ABC, the $\angle CVD$ being less than 90°.

To prove that $\angle CVD < \frac{1}{2}(\angle AVC + \angle BVC)$.

Proof. Through H, any point in VD, pass a plane $\perp$ to VD, cutting the edges VA, VB, and VC in the points A, B, and C respectively.

Then VH is $\perp$ to the plane ABC.

$\therefore VH$ is $\perp$ to AB. § 430

In the rt. $\triangle$s VHA and VHB,

$VH = VH$, Iden.

and $\angle AVH = \angle HVB$. Given

$\therefore$ the rt. $\triangle$s VHA and VHB are congruent. § 72

$\therefore VA = VB$, and $AH = BH$. § 67

Produce CH to E, making $HE = CH$.

Draw VE, AE, and BE.

Then $ACBE$ is a ▱. § 131

$\therefore AE = CB$. § 125

In the $\triangle$s VAE and VBC,

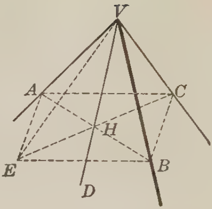
$AE = CB$, Proved

$VA = VB$, § 83

and $VE = VC$. § 83

$\therefore$ the $\triangle$s VAE and VBC are congruent. § 80

$\therefore \angle AVE = \angle BVC$. § 67



Now $\angle EVH = \angle CVH$. § 83

$\therefore \angle CVE = 2\angle CVH$

$= 2\angle CVD$.

In the trihedral $\angle V\text{-}AEC$,

$\angle CVE < \angle AVC + \angle AVE$. § 494

That is,

$2\angle CVD < \angle AVC + \angle BVC$. Ax. 9

$\therefore \angle CVD < \frac{1}{2}(\angle AVC + \angle BVC)$, by Ax. 6. Q.E.D.

CASE 2. Given VD bisecting the face $\angle AVB$ of the trihedral $\angle V\text{-}ABC$, the $\angle DVC$ being equal to 90°.

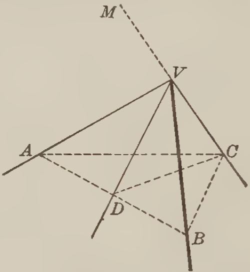
To prove that $\angle DVC = \frac{1}{2}(\angle AVC + \angle BVC)$.

Proof. In the trihedral $\angle$s $V\text{-}DBC$ and $V\text{-}DAM$,

the face $\angle DVC =$ the face $\angle MVD$, § 56

the face $\angle DVB =$ the face $\angle AVD$, Given

and the dihedral $\angle B\text{-}VD\text{-}C =$ the dihedral $\angle A\text{-}VD\text{-}M$. § 470



Moreover the equal parts occur in the same order.

$\therefore$ the trihedral $\angle$s $V\text{-}DBC$ and $V\text{-}DAM$ are equal. § 493

$\therefore \angle BVC = \angle AVM$. § 67

But $\angle AVM$ is supplementary to $\angle AVC$. § 43

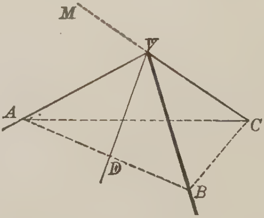
$\therefore \angle AVC$ is supplementary to $\angle BVC$. § 58

That is, $\angle AVC + \angle BVC = 180°$.

$\therefore \angle DVC = \frac{1}{2}(\angle AVC + \angle BVC)$, by Ax. 8. Q.E.D.

CASE 3. Given VD bisecting the face $\angle AVB$ of the trihedral $\angle V\text{-}ABC$, the $\angle DVC$ being greater than 90°.

To prove that $\angle DVC > \frac{1}{2}(\angle AVC + \angle BVC)$.



Proof. Produce CV to M.

Then $\angle MVD + \angle DVC = 180°$. § 43, 1

But $\angle DVC > 90°$. Given

$\therefore \angle MVD < 90°$. Ax. 7

In the trihedral $\angle V\text{-}ABM$,

$\angle MVD < \frac{1}{2}(\angle AVM + \angle BVM)$. Case 1

Now $\angle AVC$ is supplementary to $\angle AVM$,

and $\angle BVC$ is supplementary to $\angle BVM$. § 43

$\therefore \angle AVM = 180° - \angle AVC$,

and $\angle BVM = 180° - \angle BVC$. Ax. 2

$\therefore \angle MVD < \frac{1}{2}(180^\circ - \angle AVC + 180^\circ - \angle BVC)$ Ax. 9

$< \frac{1}{2}(360^\circ - [\angle AVC + \angle BVC])$

$< 180^\circ - \frac{1}{2}(\angle AVC + \angle BVC).$

Since $\angle MVD + \angle DVC = 180^\circ$,

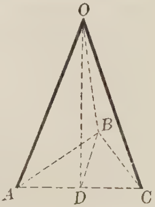
$\therefore \angle MVD = 180^\circ - \angle CVD.$ Ax. 2

$\therefore 180^\circ - \angle DVC < 180^\circ - \frac{1}{2}(\angle AVC + \angle BVC).$ Ax. 9

$\therefore -\angle DVC < -\frac{1}{2}(\angle AVC + \angle BVC).$ Ax. 6

$\therefore \angle DVC > \frac{1}{2}(\angle AVC + \angle BVC).$ Q.E.D.

8. *If two face angles of a trihedral angle are equal, the dihedral angles opposite them are equal.*



Given the equal face $\angle$s AOB and BOC of the trihedral $\angle$ O-ABC.

To prove that the dihedral $\angle$ OA equals the dihedral $\angle$ OC.

Proof. Draw OD, the bisector of the face $\angle$ AOC.

Through OD and OB pass a plane. § 425

Then in the two trihedral $\angle$s O-ABD and O-BCD,

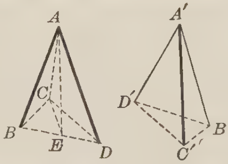
the face $\angle$ AOB = the face $\angle$ BOC, Given

the face $\angle$ AOD = the face $\angle$ DOC, Hyp.

and the face $\angle$ BOD = the face $\angle$ BOD. Iden.

$\therefore$ the dihedral $\angle$ OA equals the dihedral $\angle$ OC, by § 499. Q.E.D.

9. *A trihedral angle having two of its face angles equal is superposable on its symmetric trihedral angle.*



Given the equal face $\angle$s BAC and CAD of the trihedral $\angle A$, which is symmetric to the trihedral $\angle A'$.

To prove that the trihedral $\angle A$ is superposable on the trihedral $\angle A'$.

Proof. Draw AE, the bisector of the face $\angle BAD$. § 231

Pass a plane through AC and AE. § 425

Then the trihedral $\angle$s A-BEC and A-DEC have the face $\angle$s of the one equal respectively to the face $\angle$s of the other.

Therefore the dihedral $\angle AB$ equals the dihedral $\angle AD$. § 499

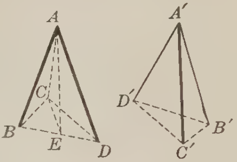
Similarly, the dihedral $\angle A'B'$ equals the dihedral $\angle A'D'$.

But the dihedral $\angle AB$ equals the dihedral $\angle A'B'$, and the dihedral $\angle AD$ equals the dihedral $\angle A'D'$. § 496

Therefore the dihedral $\angle AB$ equals the dihedral $\angle A'D'$, and the dihedral $\angle AD$ equals the dihedral $\angle A'B'$. Ax. 8

Superpose the trihedral $\angle A$ on the trihedral $\angle A'$ so that the dihedral $\angle AB$ coincides with the dihedral $\angle A'D'$.

Since the face ∡ BAD and $B'A'D'$ are corresponding face ∡ of symmetric trihedral ∡, they are equal. § 496



Therefore the face ∠ BAD will coincide with the face ∠ $B'A'D'$, AD falling on $A'B'$.

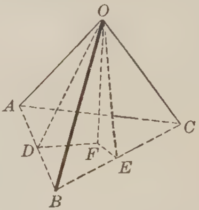
Since the dihedral ∠ AD equals the dihedral ∠ $A'B'$, Proved

the plane of the face ∠ CAD will fall on the plane of the face ∠ $C'A'B'$.

Then AC will fall on $A'C'$. § 425

Therefore the trihedral ∠ A is superposable on its symmetric trihedral ∠ A'. Q. E. D.

10. *Find the locus of a point equidistant from the three edges of a trihedral angle.*



Given the trihedral ∠ O-ABC.

Required to find the locus of a point equidistant from the edges OA, OB, and OC.

Construction. Draw OD and OE, the bisectors of the face ∡ AOB and BOC respectively. § 231

Through OD and OE pass planes ⊥ to the planes of the face ∡ AOB and BOC respectively.

Let these planes intersect in OF. § 429

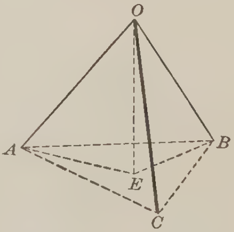
Then OF is the required locus. Q. E. F.

Proof. The plane through OD ⊥ to the plane of the face ∠ AOB is the locus of a point equidistant from OA and OB. Ex. 1, p. 314

Also the plane through OE ⊥ to the plane of the face ∠ BOC is the locus of a point equidistant from OB and OC. Ex. 1, p. 314

Therefore OF, the intersection of these two planes, is the locus of a point equidistant from OA, OB, and OC. Q. E. D.

11. *Find the locus of a point equidistant from the three faces of a trihedral angle.*



Given the trihedral ∠ O-ABC.

Required to find the locus of a point equidistant from the three faces AOB, AOC, and BOC.

Construction. Draw the planes bisecting the dihedral ∡ OA and OB.

Let these planes intersect in OE. § 429

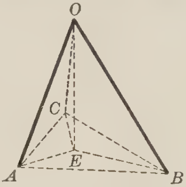
Then OE is the required locus. Q. E. F.

Proof. The plane bisecting the dihedral ∠ OA is the locus of a point equidistant from the faces AOB and AOC. § 479

Also the plane bisecting the dihedral ∠ *OB* is the locus of a point equidistant from the faces *AOB* and *BOC*. § 479

Therefore *OE*, the intersection of these two planes, is the locus of a point equidistant from the faces *AOB*, *AOC*, and *BOC*. Q. E. D.

12. *The planes that bisect the dihedral angles of a trihedral angle meet in a straight line.*



Given the trihedral ∠ *O-ABC*.

To prove that the planes which bisect the dihedral ⦣ *OA*, *OB*, and *OC* intersect in the same straight line.

Proof. Let the planes *OAE* and *OBE*, bisecting the dihedral ⦣ *OA* and *OB* respectively, intersect in *OE*. § 429

The plane *OAE* is the locus of a point equidistant from the faces *OAC* and *OAB*, and the plane *OBE* is the locus of a point equidistant from the faces *OAB* and *OBC*. § 479

Therefore *OE* is equidistant from the faces *OAC* and *OBC*.

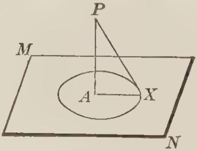
But the plane bisecting the dihedral ∠ *OC* is the locus of a point equidistant from the planes *OAC* and *OBC*. § 479

Therefore the plane bisecting the dihedral ∠ *OC* contains *OE*; that is, the plane bisecting the dihedral ∠ *OC* passes through *OE*.

Hence the planes bisecting the dihedral ⦣ *OA*, *OB*, and *OC* intersect in the same straight line *OE*. Q. E. D.

Exercise 82. Page 315

1. *From a point P, 4 in. from a plane, a line PX is drawn meeting the plane at X. If PX is 5 in., what is the length of the locus of X in the plane?*



Given *PA* the ⊥, and *PX* an oblique line drawn to the plane *MN* from an external point *P*, so that *PA* = 4 in., and *PX* = 5 in.; also the locus of *X*, which is a ⊙ in *MN* with *A* as a center and *AX* as a radius.

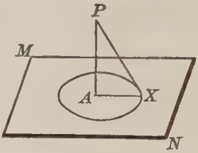
Required to find the length of the ⊙ which is the locus of *X*.

Solution.	PA is $\perp$ to AX.	Given
In the rt. △ *PAX*,	$\overline{AX}^2 = \overline{PX}^2 - \overline{PA}^2$.	§ 338
	$\therefore\ \overline{AX}^2 = 5^2 - 4^2$, or 3^2.	Ax. 9
	$\therefore\ AX = 3$.	Ax. 5

But $2\pi AX$ is the length of the ⊙ constructed. § 385

∴ 6π in., or 18.8496 in., is the length of the locus of *X*. Q. E. F.

2. *From a point P, 5 in. from a plane, a line PX is drawn meeting the plane at X. If PX is 12 in., what area is inclosed in the plane by the locus of X? Answer to two decimal places.*



Given PA the $\perp$, and PX an oblique line drawn to the plane MN from an external point P, so that $PA = 5$ in., and $PX = 12$ in.; also the locus of X, which is a $\odot$ in MN with A as a center and AX as a radius.

Required to find the area of the $\odot$ which is the locus of X.

Solution. PA is $\perp$ to AX. § 430

In the rt. $\triangle PAX$, $\overline{AX}^2 = \overline{PX}^2 - \overline{PA}^2$. § 338

$\therefore \overline{AX}^2 = 12^2 - 5^2$, or 119. Ax. 9

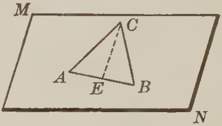
But $\pi \overline{AX}^2$ is the area of the given $\odot$. § 389

$\therefore 119\,\pi$ is the area of the $\odot$ which is the locus of X.

$\therefore$ 373.85 sq. in. is the required area. Q. E. F.

3. *The base AB of the isosceles triangle ABC, in the plane MN, is 6 in., and the perimeter of the triangle is 20 in. If the triangle revolves about its base as an axis, what is the greatest distance from the plane that is reached by C? Answer to three decimal places.*

Given in the plane MN the isosceles $\triangle ABC$, with its base $AB = 6$ in., and its perimeter = 20 in.; also the locus of C when the $\triangle ABC$ revolves in space about AB as an axis.



Required to find the greatest distance from MN reached by C.

Solution. Draw $CE \perp$ to AB.

The greatest distance from MN reached by C, when the $\triangle ABC$ revolves in space about AB as an axis, is the length of CE.

Now CE bisects AB. § 85

In the rt. $\triangle AEC$,

$AE = \frac{1}{2} AB = 3$, and $AC = \frac{1}{2}$ of $(20 - 6)$, or 7.

Since $\overline{CE}^2 = \overline{AC}^2 - \overline{AE}^2$, § 338

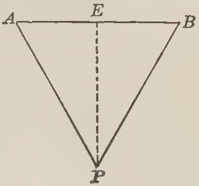
$\therefore \overline{CE}^2 = 7^2 - 3^2$, or 40. Ax. 9

$\therefore CE = 2\sqrt{10}$, or 6.324. Ax. 5

Therefore the length of CE is 6.324 in. Q. E. F.

4. *Two points A and B are 4 in. apart. A point P moves so as to be constantly 5 in. from each of these points. Find the length of the locus of P. Answer to three decimal places.*

Given the points A and B, 4 in. apart, and P a point moving in space so that PA and PB each equal 5 in.; also the ⊙ with the radius PE (a line from $P \perp$ to AB), which is the locus generated by the moving point P.



Required to find the length of the ⊙ which is the locus of P.

Solution. PE bisects AB. § 85

Then in the rt. $\triangle PEA$,

$AE = \frac{1}{2} AB = 2$ in., and $PA = 5$ in.

Since $\overline{PE}^2 = \overline{PA}^2 - \overline{AE}^2$, § 338

$\overline{PE}^2 = 5^2 - 2^2$, or 21. Ax. 9

$\therefore PE = \sqrt{21}$. Ax. 5

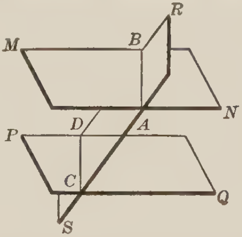
But $2\pi \times PE$ is the length of the ⊙ which is the locus of P. § 385

$\therefore 2\pi\sqrt{21}$, or 28.793, is the length of the locus, by Ax. 9.

That is, 28.793 in. is the length of the locus of P. Q.E.F.

5. *Two parallel planes MN and PQ are cut by a third plane RS so as to make one of the dihedral angles 27° 15′ 30″. Find the other dihedral angles.*

Given the ∥ planes MN and PQ cut by the plane RS in the lines AB and CD so that the dihedral $\angle R\text{-}BA\text{-}N$ is 27° 15′ 30″.



Required to find the values of the other dihedral $\angle$s.

Solution. The $\angle M\text{-}AB\text{-}R$ is a supplement of the $\angle R\text{-}BA\text{-}N$. § 470, 1

$\therefore \angle M\text{-}AB\text{-}R = 180° - 27°\ 15'\ 30''$, or $152°\ 44'\ 30''$.

Then since $\angle$s $M\text{-}BA\text{-}D$ and $N\text{-}BA\text{-}D$ are vertical $\angle$s of $R\text{-}BA\text{-}N$ and $M\text{-}AB\text{-}R$ respectively, the $\angle$s $M\text{-}BA\text{-}D$ and $N\text{-}BA\text{-}D$ are equal to $R\text{-}BA\text{-}N$ and $M\text{-}AB\text{-}R$ respectively. § 470, 3

$\therefore \angle M\text{-}BA\text{-}D = 27°\,15'\,30''$.

and $\angle N\text{-}BA\text{-}D = 152°\,44'\,30''$. Ax. 9

Now $\angle A\text{-}DC\text{-}Q = \angle R\text{-}BA\text{-}N$,

$\angle S\text{-}CD\text{-}Q = \angle N\text{-}BA\text{-}D$,

$\angle P\text{-}CD\text{-}A = \angle M\text{-}AB\text{-}R$,

and $\angle S\text{-}CD\text{-}P = \angle M\text{-}BA\text{-}D$. § 470, 4

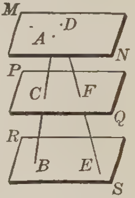
$\therefore \angle A\text{-}DC\text{-}Q = 27°\,15'\,30''$,

$\angle S\text{-}CD\text{-}Q = 152°\,44'\,30''$,

$\angle P\text{-}CD\text{-}A = 152°\,44'\,30''$,

and $\angle S\text{-}CD\text{-}P = 27°\,15'\,30''$, by Ax. 9. Q.E.F.

6. *Two lines are cut by three parallel planes. The segments cut from one line are 3 in. and $5\frac{1}{2}$ in., and those cut from the other line are $7\frac{3}{4}$ in. and x. Find the value of x.*



Given the $\parallel$ planes MN, PQ, and RS cutting the lines AB and DE into the segments AC and BC, and DF and EF, respectively, so that $AC = 3$ in., $BC = 5\frac{1}{2}$ in., $DF = 7\frac{3}{4}$ in., and $EF = x$.

Required to find the value of x.

Solution. $AC : BC = DF : EF$. § 462

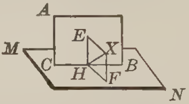
$\therefore 3 : 5\frac{1}{2} = 7\frac{3}{4} : x$. Ax. 9

$\therefore 3x = \frac{341}{8}$. § 261

$\therefore x = \frac{341}{24}$, or $14\frac{5}{24}$. Ax. 4

$\therefore$ the length of x is $14\frac{5}{24}$ in. Q.E.F.

7. *Two given planes are at right angles to each other. A point X is 8 in. from each plane. How far is X from the edge of the right dihedral angle?*



Given the planes MN and AB, intersecting at right ∡ in the line CB; also X, a point located between the planes such that XF and XE, the respective distances of X from MN and AB, each equal 8 in., and the line XH, the distance at which X is from the edge CB of the rt. dihedral $\angle N\text{-}CB\text{-}A$.

Required to find the length of XH.

Solution. Through XE and XF pass a plane intersecting the planes MN and AB in FH and EH respectively.

XF is $\perp$ to plane MN. Given

$\therefore$ plane $EHFX$ is $\perp$ to plane MN. § 477

$\therefore EH$ is $\perp$ to plane MN. § 478

$\therefore EH$ is $\parallel$ to XF. § 444

Also XE is $\perp$ to plane AB. Given

$\therefore$ plane $EHFX$ is $\perp$ to plane AB. § 477

$\therefore FH$ is $\perp$ to plane AB. § 478

$\therefore FH$ is $\parallel$ to XE. § 444

$\therefore EHFX$ is a ▱. § 118

But EH and FH are each $\perp$ to CB. § 430

$\therefore$ the $\angle FHE$ is the plane $\angle$ of the rt. dihedral $\angle N\text{-}CB\text{-}A$. § 468

$\therefore$ the $\angle FHE$ is a rt. $\angle$.

Since XF and XE are equal, Given

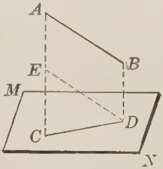
$\therefore EHFX$ is a square.

Now CH is $\perp$ to the plane $EHFX$. § 431
$\therefore XH$ lies in the plane $EHFX$. § 432
Hence $\overline{XH}^2 = 8^2 + 8^2$, or $2 \cdot 8^2$. § 337
$\therefore XH = 8\sqrt{2}$, or 11.3136. Ax. 5
$\therefore$ the length of XH is 11.3136 in. Q. E. F.

8. *What is the length of the projection on a plane of a line whose length is* $10\sqrt{2}$, *the inclination of the line to the plane being* 45°?

Given CD the projection of line AB, inclined at an $\angle$ of 45° to plane MN, on plane MN; also $AB = 10\sqrt{2}$.

Required to find the length of CD.



Solution. Draw AC and BD.
Then AC and BD are each $\perp$ to MN. § 481
$\therefore AC$ and BD are $\parallel$. § 444

Pass a plane through AC and BD. This plane will intersect MN in line CD, and will contain line AB.

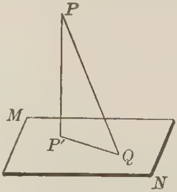
In the plane AD draw $DE \parallel$ to BA, meeting AC in E.

Then $ED = AB$. § 127
$\angle BDE = 45°$. § 104
$\therefore \angle EDC = 45°$.
$\therefore \angle CED = 45°$. § 107
$\therefore \triangle CED$ is a rt. isosceles $\triangle$. § 76
Now $\overline{CD}^2 + \overline{CE}^2 = \overline{ED}^2$. § 337
$\therefore 2\,\overline{CD}^2 = \overline{AB}^2$. Ax. 9
$\therefore 2\,\overline{CD}^2 = (10\sqrt{2})^2$, or 200. Ax. 9
$\therefore \overline{CD}^2 = 100$. Ax. 4
$\therefore CD = 10$, by Ax. 5. Q. E. F.

9. *From the external point* P *a perpendicular* PP', 9 *in. long, is drawn to a plane* MN. *From* P *the line* PQ *is drawn to the plane, making the angle* $P'PQ$ *equal to* 30°. *Find the length of the projection of* PQ *on the plane* MN.

Given the line PP', 9 in. long, drawn $\perp$ to the plane MN from an external point P, the line PQ meeting the plane MN in Q such that $\angle P'PQ = 30°$, and the line $P'Q$ the projection of PQ on the plane MN.

Required to find the length of $P'Q$.



Solution. Since PP' is $\perp$ to the plane MN, Given
$\therefore PP'$ is $\perp$ to $P'Q$. § 470

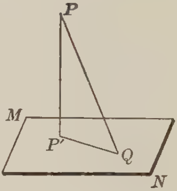
In the rt. $\triangle PP'Q$,

$\angle P'PQ = 30°.$ Given

$\therefore \angle PQP' = 60°.$ § 107

$\therefore P'Q = \frac{1}{2} PQ.$ Ex. 2, p. 80

$\therefore 2\,P'Q = PQ.$ Ax. 2



But in the rt. $\triangle PP'Q$,

$\overline{P'Q}^2 = \overline{PQ}^2 - \overline{PP'}^2.$ § 338

$\therefore \overline{P'Q}^2 = 4\,\overline{P'Q}^2 - \overline{PP'}^2.$ Ax. 9

$\therefore 3\,\overline{P'Q}^2 = \overline{PP'}^2.$ Axs. 1, 2

$\therefore \overline{P'Q}^2 = \frac{1}{3}\,\overline{PP'}^2.$ Ax. 4

$\therefore \overline{P'Q}^2 = \frac{1}{3} \times 9^2$, or 27. Ax. 9

$\therefore P'Q = \sqrt{27}$, or $3\sqrt{3}$. Ax. 5

$\therefore$ the length of $P'Q$ is $3\sqrt{3}$ in. Q.E.F.

BOOK VII

Exercise 84. Page 321

1. *Find the lateral area of the right prism whose altitude is* 18 *in. and perimeter of base* 29 *in.*

Solution. $l = ap.$ § 513

$\therefore l = 18 \times 29$ Ax. 9

$= 522.$

Therefore the lateral area is 522 sq. in. Q.E.F.

2. *Find the lateral area of the right prism whose altitude is* 22 *in. and perimeter of base* 37 *in.*

Solution. $l = ap.$ § 513

$\therefore l = 22 \times 37$ Ax. 9

$= 814.$

Therefore the lateral area is 814 sq. in. Q.E.F.

3. *Find the lateral area of the right prism whose altitude is* 4.25 *in. and perimeter of base* 6.75 *in.*

Solution. $l = ap.$ § 513

$\therefore l = 4.25 \times 6.75$ Ax. 9

$= 28.6875.$

Therefore the lateral area is 28.6875 sq. in. Q.E.F.

4. *Find the lateral area of the right prism whose altitude is* 1 *ft.* 7 *in. and perimeter of base* 2 *ft.* 9 *in.*

Solution. 1 ft. 7 in. = 19 in., and 2 ft. 9 in. = 33 in.

Now $l = ap.$ § 513

$\therefore l = 19 \times 33$ Ax. 9

$= 627.$

Therefore the lateral area is 627 sq. in. Q.E.F.

5. *Find the lateral area of the right prism whose altitude is* 3 *ft.* 8 *in. and perimeter of base* 5 *ft.* 7 *in.*

Solution. 3 ft. 8 in. = 44 in., and 5 ft. 7 in. = 67 in.

Now $l = ap.$ § 513

$\therefore l = 44 \times 67$ Ax. 9

$= 2948.$

Therefore the lateral area is 2948 sq. in. Q.E.F

6. *Find the lateral area of the right prism whose altitude is 12 ft. 2 in. and perimeter of base 27 ft. 9 in.*

Solution. 12 ft. 2 in. = 146 in., and 27 ft. 9 in. = 333 in.

Now $l = ap.$ § 513

$\therefore l = 146 \times 333$ Ax. 9

$= 48{,}618.$

Therefore the lateral area is 48,618 sq. in. Q.E.F.

7. *Find the lateral area of the prism whose lateral edge is 17 in. and perimeter of a right section 27 in.*

Solution. $l = ep.$ § 512

$\therefore l = 17 \times 27$ Ax. 9

$= 459.$

Therefore the lateral area is 459 sq. in. Q.E.F.

8. *Find the lateral area of the prism whose lateral edge is 23 in. and perimeter of a right section 35 in.*

Solution. $l = ep.$ § 512

$\therefore l = 23 \times 35$ Ax. 9

$= 805.$

Therefore the lateral area is 805 sq. in. Q.E.F.

9. *Find the lateral area of the prism whose lateral edge is $2\frac{3}{4}$ in. and perimeter of a right section $4\frac{7}{8}$ in.*

Solution. $2\frac{3}{4}$ in. $= \frac{11}{4}$ in., and $4\frac{7}{8}$ in. $= \frac{39}{8}$ in.

Now $l = ep.$ § 512

$\therefore l = \frac{11}{4} \times \frac{39}{8}$ Ax. 9

$= \frac{429}{32}$, or $13\frac{13}{32}$.

Therefore the lateral area is $13\frac{13}{32}$ sq. in. Q.E.F.

10. *Find the lateral area of the prism whose lateral edge is 1 ft. 3 in. and perimeter of a right section 2 ft. 3 in.*

Solution. 1 ft. 3 in. = 15 in., and 2 ft. 3 in. = 27 in.

Now $l = ep.$ § 512

$\therefore l = 15 \times 27$ Ax. 9

$= 405.$

Therefore the lateral area is 405 sq. in. Q.E.F.

11. *Find the lateral area of the prism whose lateral edge is 2 ft. 7 in. and perimeter of a right section 3 ft. 9 in.*

Solution. 2 ft. 7 in. = 31 in., and 3 ft. 9 in. = 45 in.

Now $l = ep$. § 512

$\therefore l = 31 \times 45$ Ax. 9

$= 1395$.

Therefore the lateral area is 1395 sq. in. Q.E.F.

12. *Find the lateral area of the prism whose lateral edge is* 6 *ft.* $1\frac{1}{4}$ *in. and perimeter of a right section* 8 *ft.* $9\frac{1}{8}$ *in.*

Solution. 6 ft. $1\frac{1}{4}$ in. $= \frac{293}{4}$ in., and 8 ft. $9\frac{1}{8}$ in. $= \frac{841}{8}$ in.

Now $l = ep$. § 512

$\therefore l = \frac{293}{4} \times \frac{841}{8}$ Ax. 9

$= \frac{246413}{32}$

$= 7700\frac{13}{32}$.

Therefore the lateral area is $7700\frac{13}{32}$ sq. in. Q.E.F.

13. *Find the lateral edge of the prism whose lateral area is* 187 *sq. in. and perimeter of a right section* 11 *in.*

Solution. $l = ep$. § 512

That is, $ep = l$.

$\therefore e = \frac{l}{p}$. Ax. 4

$\therefore e = \frac{187}{11}$ Ax. 9

$= 17$.

Therefore the lateral edge is 17 in. Q.E.F.

14. *Find the lateral edge of the prism whose lateral area is* 357 *sq. in. and perimeter of a right section* 21 *in.*

Solution. $l = ep$. § 512

$\therefore e = \frac{l}{p}$. Ax. 4

$\therefore e = \frac{357}{21}$ Ax. 9

$= 17$.

Therefore the lateral edge is 17 in. Q.E.F.

15. *Find the lateral edge of the prism whose lateral area is* 169 *sq. in. and perimeter of a right section* 1 *ft.* 1 *in.*

Solution. $l = ep$. § 512

$\therefore e = \frac{l}{p}$. Ax. 4

$\therefore e = \frac{169}{13}$ Ax. 9

$= 13$.

Therefore the lateral edge is 13 in. Q.E.F

16. *The lateral surface of an iron bar 5 ft. long is to be gilded. The right section is a square whose area is 2.89 sq. in. How many square inches of gilding are required?*

Solution. The length of a side of the right section of the iron bar is $\sqrt{2.89}$, or 1.7 in. § 320

$\therefore$ the perimeter of the right section is 4×1.7 in., or 6.8 in.

Now the length of the iron bar is 5 ft., or 60 in. Given

Since $l = ap$, § 513

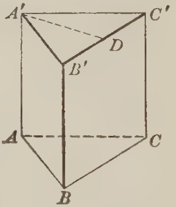
$\therefore l = 60 \times 6.8$ Ax. 9

$= 408.$

Therefore the lateral area of the given iron bar is 408 sq. in.

$\therefore$ 408 sq. in. of gilding are required. Q.E.F.

17. *A right prism of glass is $2\frac{1}{8}$ in. long. Its right section is an equilateral triangle whose altitude is 0.866 in. $(\frac{1}{2}\sqrt{3}$ in.$)$. Find the lateral surface.*



Given $ABC\text{-}B'$ a right triangular prism whose right section $A'B'C'$ is an equilateral $\triangle$. $AA' = 2\frac{1}{8}$ in., and $A'D$, the altitude of $\triangle A'B'C'$, $= \frac{1}{2}\sqrt{3}$ in.

Required to find the lateral surface of the prism $ABC\text{-}B'$.

Solution. $B'D = C'D.$ § 85

That is, $B'D = \frac{1}{2}B'C'.$

Since $\triangle A'B'C'$ is an equilateral $\triangle$, Given

$\therefore B'D = \frac{1}{2}A'B'.$ Ax. 9

In the rt. $\triangle A'DB'$,

$\overline{A'B'}^2 - \overline{B'D}^2 = \overline{A'D}^2.$ § 338

$\therefore \overline{A'B'}^2 - \frac{1}{4}\overline{A'B'}^2 = \overline{A'D}^2.$ Ax. 9

$\therefore \frac{3}{4}\overline{A'B'}^2 = \overline{A'D}^2.$

Since $A'D = \frac{1}{2}\sqrt{3},$

$\therefore \overline{A'D}^2 = \frac{3}{4}.$ Ax. 5

$\therefore \frac{3}{4}\overline{A'B'}^2 = \frac{3}{4}.$ Ax. 9

$\therefore \overline{A'B'}^2 = 1.$ Ax. 4

$\therefore A'B' = 1.$ Ax. 5

Now the perimeter of $\triangle A'B'C' = A'B' + A'C' + B'C'$ § 28

$= 3\,A'B'$ Ax. 9

$= 3.$ Ax. 9

Since $l = ap,$ § 513

$\therefore l = 2\frac{1}{8} \times 3$ Ax. 9

$= 6\frac{3}{8}.$

Therefore the lateral surface of the prism $ABC\text{-}B'$ is $6\frac{3}{8}$ sq. in. Q.E.F.

18. *Find the total area of a right prism whose base is a square with area* 5.29 *sq. in., and whose length is twice its thickness.*

Solution. One side of the base of the given right prism is $\sqrt{5.29}$, or 2.3 in. § 320

Then the perimeter of its base is 4×2.3, or 9.2 in. § 138

The altitude of the given prism is 2×2.3, or 4.6 in.

Then $a = 4.6$ in., $p = 9.2$ in.

Since $l = ap$, § 513

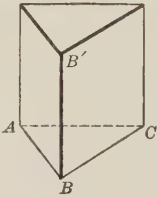
$\therefore l = 4.6 \times 9.2$ Ax. 9

$= 42.32.$

Therefore the lateral area of the given prism is 42.32 sq. in.

Therefore the total area of the given prism is 42.32 sq. in. + (2×5.29) sq. in. = 52.9 sq. in. Q.E.F.

19. *What is the total area of a right prism whose altitude is* 32 *in., and whose base is a right triangle with hypotenuse* 106 *in. and with one side* 84.8 *in.?*



Given the right prism $ABC\text{-}B'$ whose base ABC is a rt. △ with the rt. ∠ at B. $BB' = 32$ in., $AC = 106$ in., and $BC = 84.8$ in.

Required to find the total area of the right prism $ABC\text{-}B'$.

Solution. In the rt. △ ABC,

$\overline{AB}^2 = \overline{AC}^2 - \overline{BC}^2$. § 338

But $\overline{AC}^2 = 106^2 = 11{,}236$,

and $\overline{BC}^2 = 84.8^2 = 7191.04$. Ax. 5

$\therefore \overline{AB}^2 = 11{,}236 - 7191.04$ Ax. 9

$= 4044.96.$

$\therefore AB = 63.6.$ Ax. 5

Then $AC + AB + BC = 106 + 63.6 + 84.8$ § 28

$= 254.4.$

Since $l = ap$, § 513

$\therefore l = 32 \times 254.4$ Ax. 9

$= 8140.8.$

Therefore the lateral area of the right prism $ABC\text{-}B'$ is 8140.8 sq. in.

Now the area of △ $ABC = \frac{1}{2} AB \times BC$ § 325

$= \frac{1}{2}(63.6 \times 84.8)$ sq. in. Ax. 9

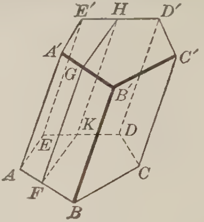
$= 2696.64$ sq. in.

Therefore the total area of the right prism $ABC\text{-}B'$

$= 8140.8$ sq. in. + (2×2696.64) sq. in.

$= 13{,}534.08$ sq. in. Q.E.F.

20. *Every section of a prism made by a plane parallel to the lateral edges is a parallelogram.*



Given $FGHK$ a section of any prism AD' ∥ to its lateral edges.

To prove that $FGHK$ is a ▱.

Proof. $ABCDE\cdots$ is ∥ to $A'B'C'D'E'\cdots$. § 503

$\therefore FK$ is ∥ to GH. § 453

Furthermore, AA' and EE' are each ∥ to $FGHK$. Given

$\therefore AA'$ is ∥ to FG, and EE' is ∥ to KH. Ex. 3, p. 285

Since AA' is ∥ to EE', § 118

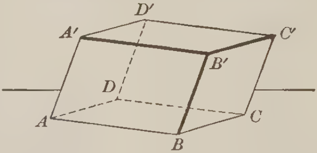
$\therefore FG$ is ∥ to KH. § 446

$\therefore FGHK$ is a ▱, by § 118. Q.E.D.

Exercise 85. Page 325

1. *If in the above figure the three plane angles at A are* 80°, 70°, 75°, *what are all the other angles in the faces?*

Given in the parallelepiped AC' the plane $\angle$s DAA', BAA', and BAD equal to 80°, 70°, and 75° respectively.



Required to find all the other angles in the faces of the parallelepiped AC'.

Solution. The opposite lateral faces of the parallelepiped AC' are congruent. § 526

$\therefore$ the corresponding $\angle$s of the opposite lateral faces are equal. § 67

In the ▱ $AA'D'D$, $\angle DAA' = 80°$. Given

$\therefore \angle AA'D' = 180° - 80° = 100°$,

$\angle A'D'D = 80°$,

and $\angle D'DA = 100°$. § 124

Then in the ▱ $BB'C'C$,

$\angle CBB' = 80°$,

$\angle BB'C' = 100°$,

$\angle B'C'C = 80°$,

and $\angle C'CB = 100°$. Ax. 9

Also in the ▱ $AA'B'B$,

$\angle BAA' = 70°$. Given

$\therefore \angle AA'B' = 180° - 70° = 110°$,

$\angle A'B'B = 70°$,

and $\angle B'BA = 110°$. § 124

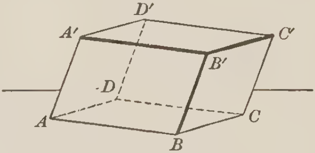
Then in the ▱ $DD'C'C$, $\angle CDD' = 70°$,
$\angle DD'C' = 110°$,
$\angle D'C'C = 70°$,
and $\angle C'CD = 110°$. Ax. 9

Furthermore, in the ▱ $ABCD$,
$\angle BAD = 75°$. Given
$\therefore \angle ADC = 180° - 75° = 105°$,
$\angle DCB = 75°$,
and $\angle CBA = 105°$. § 124

Then in the ▱ $A'B'C'D'$, $\angle B'A'D' = 75°$,
$\angle A'D'C' = 105°$,
$\angle D'C'B' = 75°$,
and $\angle C'B'A' = 105°$, by Ax. 9. Q.E.F.

2. *Given a parallelepiped with the three plane angles at one of the vertices* 85°, 75°, 60°, *to find all the other angles in the faces.*

Given in the parallelepiped AC' the plane $\angle$s DAA', BAA', and BAD equal to 85°, 75°, and 60° respectively.



Required to find all the other $\angle$s in the faces of the parallelepiped AC'.

Solution. The opposite lateral faces of the parallelepiped AC' are congruent. § 526

$\therefore$ the corresponding $\angle$s of the opposite lateral faces are equal. § 67

In the ▱ $AA'D'D$, $\angle DAA' = 85°$. Given
$\therefore \angle AA'D = 180° - 85° = 95°$,
$\angle A'D'D = 85°$,
and $\angle D'DA = 95°$. § 124

Then in the ▱ $BB'C'C$, $\angle CBB' = 85°$,
$\angle BB'C' = 95°$,
$\angle B'C'C = 85°$,
and $\angle C'CB = 95°$. Ax. 9

Also in the ▱ $AA'B'B$, $\angle BAA' = 75°$. Given
$\therefore \angle AA'B' = 180° - 75° = 105°$,
$\angle A'B'B = 75°$,
and $\angle B'BA = 105°$. § 124

Then in the ▱ $DD'C'C$, $\angle D'DC = 75°$,
$\angle DD'C' = 105°$,
$\angle D'C'C = 75°$,
and $\angle DCC' = 105°$. Ax. 9

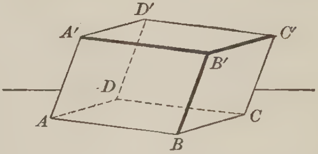
Furthermore, in the ▱ $ABCD$,

$\angle BAD = 60°$. Given

$\therefore \angle ADC = 180° - 60° = 120°$,

$\angle DCB = 60°$,

and $\angle CBA = 120°$. § 124



Then in the ▱ $A'B'C'D'$,

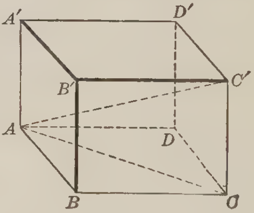
$\angle B'A'D' = 60°$,

$\angle A'D'C' = 120°$,

$\angle D'C'B' = 60°$,

and $\angle C'B'A' = 120°$, by Ax. 9. Q. E. F.

3. *Given a rectangular parallelepiped, lettered as in the above figure, and with $AB = 4$, $BC = 3$, and $CC' = 3\frac{3}{4}$, to find the length of the diagonal AC'.*



Given, in the rectangular parallelepiped AC', $AB = 4$, $BC = 3$, and $CC' = 3\frac{3}{4}$; also the diagonal AC'.

Required to find the length of the diagonal AC'.

Solution. Pass a plane through AA' and CC', cutting the rectangle $ABCD$ in AC.

Draw in the plane $AA'C'C$ the line AC'.

Now CC' is ⊥ to the plane $ABCD$. § 515

$\therefore CC'$ is ⊥ to AC. § 430

In the rt. △ ACC', $\overline{AC'}^2 = \overline{AC}^2 + \overline{CC'}^2$. § 337

But in the rt. △ ABC, $\overline{AC}^2 = \overline{AB}^2 + \overline{BC}^2$. § 337

$\therefore \overline{AC'}^2 = \overline{AB}^2 + \overline{BC}^2 + \overline{CC'}^2$. Ax. 9

$\therefore \overline{AC'}^2 = 4^2 + 3^2 + (3\frac{3}{4})^2$ Ax. 9

$= 16 + 9 + \frac{225}{16}$

$= \frac{625}{16}$.

$\therefore AC' = \frac{25}{4}$, or $6\frac{1}{4}$, by Ax. 5. Q. E. F.

4. *The four diagonals of a rectangular parallelepiped are equal.*

Given AG, BH, CE, and DF, the four diagonals of the rectangular parallelepiped AG.

To prove that AG, BH, CE, and DF are equal.

Proof. AE is ∥ to DH. § 503

Also DH is ∥ to CG. § 503

$\therefore AE$ is ∥ to CG. § 446

Similarly, BF is ∥ to DH.

Pass planes through the ∥ edges AE and CG, and BF and DH. § 428

Then the diagonals AG and CE lie in the section $AEGC$, and the diagonals BH and DF lie in the section $BFHD$. § 422

The sections $AEGC$ and $BFHD$ are ▱. § 130

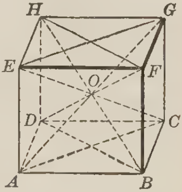
Furthermore, since AE, BF, CG, and DH are ⊥ to the bases AC and EG, § 515

therefore the $\angle$s of the sections $AEGC$ and $BFHD$ are rt. $\angle$s. § 430

$\therefore$ $AEGC$ and $BFHD$ are rectangles. § 119

$\therefore$ $AG = CE$, and $BH = DF$, § 67

being corresponding parts of the congruent $\triangle$s ACG and CAE, and BDH and DBF respectively.



Moreover, $AEGC$ and $BFHD$ are congruent, § 133

for their bases are the equal diagonals of the base of the given rectangular parallelepiped, § 526

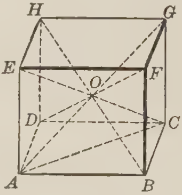
and their altitudes are its altitude.

$\therefore$ $AG = CE = BH = DF$, by § 67. Q.E.D.

5. *Compute the lengths of the diagonals of a rectangular parallelepiped whose edges from any vertex are a, b, c.*

Given, in the rectangular parallelepiped AG, $AE = a$, $AD = b$, and $AB = c$; also the diagonals AG, BH, CE, and DF.

Required to compute the lengths of AG, BH, CE, and DF in terms of a, b, and c.



Solution. Pass a plane through AG and GC, cutting the base AC in AC. § 429

Since GC is ⊥ to the base AC, § 515

$\therefore$ GC is ⊥ to the line AC. § 430

In the rt. $\triangle ACG$, $\overline{AG}^2 = \overline{AC}^2 + \overline{CG}^2$. § 337

But in the rt. $\triangle ABC$, $\overline{AC}^2 = \overline{AB}^2 + \overline{BC}^2$. § 337

$\therefore$ $\overline{AG}^2 = \overline{AB}^2 + \overline{BC}^2 + \overline{CG}^2$. Ax. 9

Now $BC = AD$, § 125

and $CG = AE$. § 503

$\therefore$ $\overline{AG}^2 = \overline{AE}^2 + \overline{AD}^2 + \overline{AB}^2$ Ax. 9

$= a^2 + b^2 + c^2$. Ax. 9

$\therefore$ $AG = \sqrt{a^2 + b^2 + c^2}$. Ax. 5

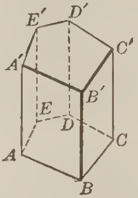
Since $AG = BH = CE = DF$, Ex. 4

$\therefore$ AG, BH, CE, and DF each equal $\sqrt{a^2 + b^2 + c^2}$, by Ax. 8. Q.E.F.

Exercise 86. Page 327

1. *The lateral faces of a right prism are rectangles.*

Given the right prism $ABCDE \cdots C'$ with the lateral faces AB', BC', CD', etc.



To prove that the lateral faces AB', BC', CD', etc. are rectangles.

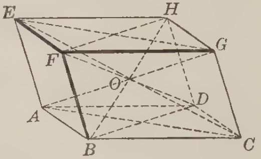
Proof. In the right prism $ABCDE \cdots C'$, AA', BB', CC', etc. are $\perp$ to the bases AD and $A'D'$. § 505

$\therefore$ AA' is $\perp$ to AB and $A'B'$,
BB' is $\perp$ to AB and $A'B'$,
BB' is $\perp$ to BC and $B'C'$,
CC' is $\perp$ to BC and $B'C'$,
and CC' is $\perp$ to CD and $C'D'$, etc. § 430

Therefore AB', BC', CD', etc. are rectangles, by § 119. Q.E.D.

2. *The diagonals of a parallelepiped bisect one another.*

Given AG, BH, CE, and DF, the four diagonals of the parallelepiped AG.



To prove that AG, BH, CE, and DF bisect each other.

Proof. Through the opposite and ∥ edges AE and CG pass a plane intersecting the ∥ bases in the ∥ lines AC and EG.

Then the section $ACGE$ is a ▱. § 118

$\therefore$ AG and CE bisect each other. § 131

That is, O is the mid-point of AG and CE.

Likewise, a plane passed through the opposite and ∥ edges FG and AD will form a ▱ $AFGD$.

Therefore O, the mid-point of the diagonal AG, is also the mid-point of FD. § 131

Also a plane passed through the opposite and ∥ edges EH and BC will form a ▱ $EBCH$.

Therefore O, the mid-point of the diagonal CE, is also the mid-point of BH. § 131

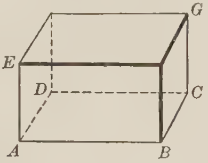
Therefore the four diagonals AG, BH, CE, and DF bisect each other at the point O. Q.E.D.

3. *The three edges of the trihedral angle at one of the vertices of a rectangular parallelepiped are* 5 *in.*, 6 *in.*, *and* 7 *in. respectively. Required the total area of the six faces of the parallelepiped.*

Given, in the rectangular parallelepiped AG, $AD = 5$ in., $AE = 6$ in., and $AB = 7$ in.

Required to find the total area of the rectangular parallelepiped AG.

Solution. Here $a = AE = 6$ in., and $p = 2(AD + AB) = 2(5 + 7)$ in., or 24 in.



Since $l = ap$, § 513

$\therefore l = 6 \times 24$ Ax. 9

$= 144.$

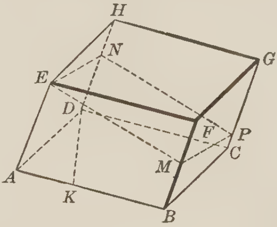
Furthermore, the area of base AC + base EG

$= 2 \times 5 \times 7 = 70.$ § 320

Therefore the total area of AG

$= 144$ sq. in. $+ 70$ sq. in. $= 214$ sq. in. Q.E.F.

4. *The three face angles at one vertex of a parallelepiped are each* 60°, *and the three edges of the trihedral angle with that vertex are* 3 *in.*, 2 *in.*, *and* 1 *in. respectively. Required the total area of the six faces. Answer to two decimal places.*



Given, in the parallelepiped AG, the face $\angle$s DAE, BAE, and BAD each equal to 60°; also $AB = 3$ in., $AD = 2$ in., and $AE = 1$ in.

Required to find the total area of the parallelepiped AG.

Solution. Through E pass the plane $EP \perp$ to AE, cutting FB, GC, and HD in the points M, P, and N respectively.

Then the plane EP is $\perp$ to FB, GC, and HD. § 445

$\therefore EP$ is a right section of the parallelepiped AG. § 508

Moreover, since EM is $\parallel$ to NP, and EN is $\parallel$ to MP, § 453

the figure EP is a ▱. § 118

$\therefore EM = NP$, and $EN = MP$. § 125

Now the faces AF and AH are ▱s. § 503

$\therefore \angle BAE = \angle EFB$,

and $\angle DAE = \angle EHD$. § 124

Since $\angle BAE = \angle DAE = 60°$, Given

$\therefore \angle EFB = \angle EHD = 60°$. Ax. 9

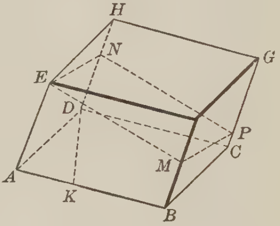
The $\triangle$s EMF and ENH are rt. $\triangle$s. § 430

$\therefore \angle MEF = \angle NEH = 30°$. § 107

Furthermore, in the ▱s AF and AH,

$EF = AB$, and $EH = AD$. § 125

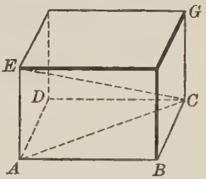
Since $AB = 3$ in. and $AD = 2$ in., Given
$\therefore EF = 3$ in. and $EH = 2$ in. Ax. 9
$\therefore FM = 1\frac{1}{2}$ in. and $HN = 1$ in. Ex. 2, p. 80
In the rt. $\triangle EMF$, $\overline{EM}^2 = \overline{EF}^2 - \overline{FM}^2$ § 338
$= 3^2 - (1\frac{1}{2})^2 = \frac{27}{4}$. Ax. 9
$\therefore EM = \frac{3}{2}\sqrt{3}$. Ax. 5
In the rt. $\triangle ENH$, $\overline{EN}^2 = \overline{EH}^2 - \overline{HN}^2$ § 338
$= 2^2 - 1^2$ Ax. 9
$= 3$.
$\therefore EN = \sqrt{3}$. Ax. 5
Since $EM = NP$, and $EN = MP$, Proved
$p = 2\,EM + 2\,EN$ Ax. 9
$= 3\sqrt{3} + 2\sqrt{3}$ Ax. 9
$= 5\sqrt{3}$. Ax. 11
Now $e = AE = 1$ in.
$\therefore$ since $l = ep$, § 512
$l = 1 \times 5\sqrt{3}$ Ax. 9
$= 5\sqrt{3}$
Next draw $DK \perp$ to AB. § 82



In the rt. $\triangle AKD$,
$\angle BAD$ or $\angle KAD = 60°$. Given
$\therefore \angle ADK = 30°$. § 107
Since $AD = 2$ in., Given
$\therefore AK = 1$ in. Ex. 2, p. 80
In the rt. $\triangle AKD$, $\overline{DK}^2 = \overline{AD}^2 - \overline{AK}^2$ § 338
$= 2^2 - 1^2$ Ax. 9
$= 3$.
$\therefore DK = \sqrt{3}$. Ax. 5
The area of the $\square AC = AB \times DK$ § 322
$= 3 \times \sqrt{3}$ Ax. 9
$= 3\sqrt{3}$.
But $\square EG = \square AC$. § 526
$\therefore$ area $\square EG$ + area $\square AC = 2 \times 3\sqrt{3}$, or $6\sqrt{3}$.
Therefore the total area of $AG = 5\sqrt{3} + 6\sqrt{3}$ Ax. 11
$= 11\sqrt{3}$, or 19.05.
Therefore the total area of the parallelepiped AG is 19.05 sq. in. Q.E.F.

5. *In a rectangular parallelepiped the square on any diagonal is equivalent to the sum of the squares on any three edges that meet at one of the vertices.*

Given CE, a diagonal of the rectangular parallelepiped AG.
To prove that $\overline{CE}^2 = \overline{AB}^2 + \overline{AD}^2 + \overline{AE}^2$.



Proof. Through the edge EA and diagonal CE pass a plane, cutting the base AC in AC. § 429

AE is $\perp$ to the base $ABCD$. § 515

Then AE is $\perp$ to AC.

In the rt. $\triangle EAC$,

$\overline{CE}^2 = \overline{AC}^2 + \overline{AE}^2$. § 337

But in the rt. $\triangle ABC$, § 516

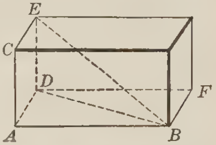
$\overline{AC}^2 = \overline{AB}^2 + \overline{BC}^2$. § 337

$\therefore \overline{CE}^2 = \overline{AB}^2 + \overline{BC}^2 + \overline{AE}^2$. Ax. 9

Now $AD = BC$. § 125

$\therefore \overline{CE}^2 = \overline{AB}^2 + \overline{AD}^2 + \overline{AE}^2$, by Ax. 9. Q.E.D.

6. *In a box 3 in. deep and 6 in. wide a wire 1 ft. long can be stretched to reach from one corner to the diagonally opposite corner. Required the length of the box. Answer to two decimal places.*



Given the rectangular parallelepiped BE representing a box; $AC = 3$ in., $AD = 6$ in., and $BE = 1$ ft. $= 12$ in.

Required to find the length of AB.

Solution. Through the edge ED and the diagonal BE pass a plane cutting the base of the parallelepiped BE in BD. § 429

ED is $\perp$ to the base $ABFD$. § 515

$\therefore ED$ is $\perp$ to DB. § 430

In the rt. $\triangle EDB$, $\overline{BE}^2 = \overline{ED}^2 + \overline{BD}^2$. § 337

In the rt. $\triangle DAB$, $\overline{BD}^2 = \overline{AD}^2 + \overline{AB}^2$. § 337

$\therefore \overline{BE}^2 = \overline{ED}^2 + \overline{AD}^2 + \overline{AB}^2$. Ax. 9

But $ED = AC = 3$ in. § 125

$\therefore 12^2 = 3^2 + 6^2 + \overline{AB}^2$. Ax. 9

$\therefore \overline{AB}^2 = 12^2 - 3^2 - 6^2$ Ax. 2

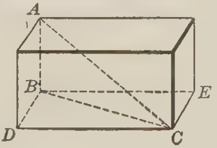
$= 99.$

$\therefore AB = 3\sqrt{11}$ Ax. 5

$= 9.95.$

Therefore 9.95 in. is the length of the given box. Q.E.F.

7. *The diagonal of the base of a rectangular parallelepiped is $31\frac{3}{5}$ in. and the height of the parallelepiped is 23.7 in. Required the length of the diagonal of the parallelepiped.*

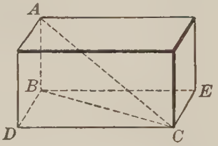


Given AC, a diagonal of the rectangular parallelepiped AC, and BC, a diagonal of its base DE. $BC = 31\frac{3}{5}$ in., and $AB = 23.7$ in.

Required to find the length of AC.

Solution. AB is $\perp$ to the base $BDCE$. § 515

$\therefore AB$ is $\perp$ to BC. § 430



In the rt. $\triangle ABC$,

$\overline{AC}^2 = \overline{AB}^2 + \overline{BC}^2$. § 337

$\therefore \overline{AC}^2 = 23.7^2 + (31\frac{3}{5})^2$ Ax. 9

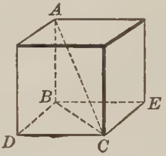
$= 561.69 + 998.56$

$= 1560.25$.

$\therefore AC = 39.5$. Ax. 5

Therefore 39.5 in. is the length of AC. Q.E.F.

8. *The total area of the six faces of a cube is* 18 *sq. in. Find the diagonal of the cube.*



Given the cube AC whose total area is 18 sq. in.

Required to find the length of AC, its diagonal.

Solution. Since a cube has six equal square faces, § 517

the area of the square $DE = \frac{1}{6} \times 18$ sq. in., or 3 sq. in.

$\therefore \overline{DC}^2 = 3$. § 320

Draw BC.

The edge AB is $\perp$ to the face DE. § 515

$\therefore$ the edge AB is $\perp$ to BC. § 430

In the rt. $\triangle ABC$, $\overline{AC}^2 = \overline{AB}^2 + \overline{BC}^2$. § 337

But in the rt. $\triangle BDC$, $\overline{BC}^2 = \overline{BD}^2 + \overline{DC}^2$. § 337

$\therefore \overline{AC}^2 = \overline{AB}^2 + \overline{BD}^2 + \overline{DC}^2$. Ax. 9

Now $AB = BD = DC$, § 65

$\therefore \overline{AC}^2 = 3\,\overline{DC}^2$. Ax. 9

Since $\overline{DC}^2 = 3$, Proved

$\therefore \overline{AC}^2 = 9$. Ax. 9

$\therefore AC = 3$. Ax. 5

Therefore 3 in. is the length of AC. Q.E.F.

9. *The diagonal of the face of a cube equals* $\sqrt{14}$. *Find the diagonal of the cube.*

Given the cube AC; also BC, a diagonal of its face DE, equal to $\sqrt{14}$.

Required to find the length of AC, its diagonal.

Solution. In the rt. $\triangle BDC$,

$\overline{BD}^2 + \overline{CD}^2 = \overline{BC}^2$. § 337

Now $BC = \sqrt{14}$, Given

$\therefore \overline{BD}^2 + \overline{CD}^2 = 14$. Ax. 9

But $BD = CD.$ § 65

$\therefore 2\,\overline{BD}^2 = 14.$ Ax. 9

$\therefore \overline{BD}^2 = 7.$ Ax. 4

Since $AB = BD,$ § 65

$\overline{AB}^2 = \overline{BD}^2$ Ax. 5

$= 7.$ Ax. 4

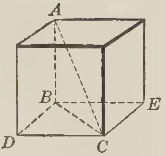
The edge AB is $\perp$ to the face $DE.$ § 515

$\therefore$ the edge AB is $\perp$ to $BC.$ § 430

In the rt. $\triangle ABC,$ $\overline{AC}^2 = \overline{AB}^2 + \overline{BC}^2.$ § 337

$\therefore \overline{AC}^2 = 7 + 14,$ or 21. Ax. 9

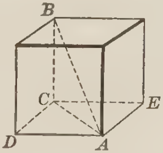
$\therefore AC = \sqrt{21},$ by Ax. 5. Q.E.F.



10. *The diagonal of a cube equals* $2.75\sqrt{3}$. *Find the diagonal of a face of the cube.*

Given AB, a diagonal of the cube AB, equal to $2.75\sqrt{3}$.

Required to find the length of AC, the diagonal of a face.



Solution. BC is $\perp$ to the face $CDAE.$ § 515

$\therefore BC$ is $\perp$ to $AC.$ § 430

In the rt. $\triangle BCA,$ $\overline{AB}^2 = \overline{BC}^2 + \overline{AC}^2.$ § 337

But in the rt. $\triangle CDA,$ $\overline{AC}^2 = \overline{CD}^2 + \overline{AD}^2.$ § 337

Now $CD = AD = BC,$ § 65

$\therefore \overline{AC}^2 = 2\,\overline{BC}^2.$ Ax. 9

$\therefore \overline{BC}^2 = \frac{1}{2}\overline{AC}^2.$ Ax. 4

$\therefore \overline{AB}^2 = \frac{1}{2}\overline{AC}^2 + \overline{AC}^2 = \frac{3}{2}\overline{AC}^2.$ Ax. 9

$\therefore \overline{AC}^2 = \frac{2}{3}\overline{AB}^2.$ Ax. 3

But $AB = 2.75\sqrt{3} = 2\frac{3}{4}\sqrt{3},$

whence $\overline{AB}^2 = \left(2\frac{3}{4}\sqrt{3}\right)^2 = \frac{363}{16}.$ Ax. 5

$$\therefore \overline{AC}^2 = \frac{2}{3} \times \frac{363}{16} = \frac{121 \times 2}{16}. \qquad \text{Ax. 9}$$

$\therefore AC = \frac{11}{4}\sqrt{2},$ or $2.75\sqrt{2},$ by Ax. 5. Q.E.F.

11. *A water tank is 3 ft. long, 2 ft. 6 in. wide, and 1 ft. 9 in. deep. How many square feet of zinc will be required to line the four sides and the base, allowing* $1\frac{7}{8}$ *sq. ft. for overlapping and for turning the top edge?*

Solution. The total area of the two sides $= 2 \times 3 \times 1\frac{3}{4} \times 1$ sq. ft. $= 10\frac{1}{2}$ sq. ft.; the total area of the two ends $= 2 \times 2\frac{1}{2} \times 1\frac{3}{4} \times 1$ sq. ft. $= 8\frac{3}{4}$ sq. ft.; and the area of the base $= 3 \times 2\frac{1}{2} \times 1$ sq. ft. $= 7\frac{1}{2}$ sq. ft.

Therefore the number of square feet of zinc required, an allowance of $1\frac{7}{8}$ sq. ft. being made for overlapping and for turning the top edge, is $(10\frac{1}{2} + 8\frac{3}{4} + 7\frac{1}{2} + 1\frac{7}{8})$ sq. ft., or $28\frac{5}{8}$ sq. ft. Q.E.F.

Exercise 87. Page 333

1. *Find the ratio of two rectangular parallelepipeds, if their dimensions are* 3, 4, 5, *and* 9, 8, 10 *respectively.*

Solution. Let P and P' denote the volumes of the given rectangular parallelepipeds, taken in order.

Then $$\frac{P}{P'} = \frac{3 \times 4 \times 5}{9 \times 8 \times 10} = \frac{1}{12}, \text{ by § 534.}$$ Q.E.F.

2. *Find the ratio of two rectangular parallelepipeds, if their altitudes are each* 6 *in., and their bases* 5 *in. by* 4 *in., and* 10 *in. by* 8 *in. respectively.*

Solution. Let P and P' denote the volumes of the given rectangular parallelepipeds, taken in order.

Then $$\frac{P}{P'} = \frac{5 \times 4}{10 \times 8} = \frac{1}{4}, \text{ by § 531.}$$ Q.E.F.

3. *Find the volume of a rectangular parallelepiped* 2 *ft.* 6 *in. long,* 1 *ft.* 8 *in. wide, and* 1 *ft.* 6 *in. high.*

Solution. The required volume is $2\frac{1}{2} \times 1\frac{2}{3} \times 1\frac{1}{2} \times 1$ cu. ft. $= 6\frac{1}{4}$ cu. ft., by § 534. Q.E.F.

4. *Find the volume of a rectangular parallelepiped whose base is* 27 *sq. in. and whose altitude is* $13\frac{1}{2}$ *in.*

Solution. The required volume is $27 \times 13\frac{1}{2} \times 1$ cu. in. $= 364\frac{1}{2}$ cu. in., by § 535. Q.E.F.

5. *The volume of a rectangular parallelepiped is* 1152 *cu. in. and the area of the base is half a square foot. Find the altitude.*

Solution. Let V, b, and a designate the volume, area of base, and altitude respectively of the given rectangular parallelepiped.

The number of square inches in 1 sq. ft. is 144.

$\therefore b = \frac{1}{2}$ of 144 sq. in., or 72 sq. in. Given

Then $V = a \times b.$ § 536

$\therefore 1152 = a \times 72.$ Ax. 9

$\therefore a = \frac{1152}{72}$ Ax. 4

$= 16.$

Therefore the altitude is 16 in. Q.E.F.

6. *The volume of a rectangular parallelepiped with a square base is* 273.8 *cu. in. and the altitude is* 5 *in. Find the dimensions.*

Solution. Let V, b^2, and a designate the volume, area of base, and altitude respectively of the given rectangular parallelepiped.

Then $V = a \times b^2.$ § 536

$\therefore 273.8 = 5b^2.$ Ax. 9

That is, $5b^2 = 273.8.$

$\therefore b^2 = \frac{273.8}{5} = 54.76.$ Ax. 4

$\therefore b = \sqrt{54.76} = 7.4.$ Ax. 5

Therefore the required dimensions of the given rectangular parallelepiped are 7.4 in., 7.4 in., and 5 in. Q.E.F.

7. *A rectangular tank full of water is* 7 *ft.* 3 *in. long by* 4 *ft.* 6 *in. wide. How many cubic feet of water must be drawn off in order that the surface may be lowered a foot?*

Solution. The required number of cubic feet of water drawn off is $7\frac{1}{4} \times 4\frac{1}{2} \times 1 = 32\frac{5}{8}$, by § 534. Q.E.F.

8. *Find to two decimal places the length of each side of a cubic reservoir that will contain exactly a gallon* (231 *cu. in.*).

Solution. Let s denote the length of each side of the given reservoir.

Then $s^3 = 231.$ § 534

$\therefore s = \sqrt[3]{231}$ Ax. 5

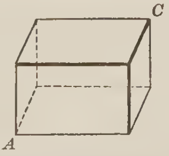
$= 6.133.$

Therefore the length of each side of the reservoir is 6.13 in. Q.E.F.

9. *A box has as its internal dimensions* 18 *in.*, $9\frac{1}{2}$ *in., and* $4\frac{1}{4}$ *in. The box and cover are made of steel* $\frac{1}{8}$ *in. thick. If steel weighs* 490 *lb. per cubic foot, what is the weight of the box?*

Given AC, a rectangular parallelepiped representing a box, with cover made of steel $\frac{1}{8}$ in. thick, whose internal dimensions are 18 in., $9\frac{1}{2}$ in., and $4\frac{1}{4}$ in.

Required to find the weight of the given box, the weight of steel being 490 lb. per cubic foot.



Solution. The external dimensions of AC are $18\frac{1}{4}$ in., $9\frac{3}{4}$ in., and $4\frac{1}{2}$ in.

Therefore the volume of a rectangular parallelepiped having these dimensions is $18\frac{1}{4} \times 9\frac{3}{4} \times 4\frac{1}{2} \times 1$ cu. in. $= (\frac{73}{4} \times \frac{39}{4} \times \frac{9}{2})$ cu. in. § 534

$= 800\frac{23}{32}$ cu. in.

But the volume of the space included by the internal dimensions of the box is $18 \times 9\frac{1}{2} \times 4\frac{1}{4} \times 1$ cu. in. $= 726\frac{3}{4}$ cu. in. § 534

Therefore $800\frac{23}{32} - 726\frac{3}{4}$, or $73\frac{31}{32}$ cu. in. of steel are used in the construction of the given box and cover.

Therefore $\frac{73\frac{31}{32}}{1728} \times 490 = 20\frac{2995}{3072}$, or 21, is the weight of the given box and cover expressed in pounds. Q.E.F.

10. *A steel rod 4 ft. 8 in. long is 2 in. wide and $1\frac{1}{2}$ in. thick. How much does it weigh, at 490 lb. per cubic foot?*

Solution. The volume of the given steel rod is $4\frac{2}{3} \times \frac{1}{6} \times \frac{1}{8} \times 1$ cu. ft. = $\frac{7}{72}$ cu. ft. § 534

Since 1 cu. ft. of steel weighs 490 lb., $\frac{7}{72}$ cu. ft. weighs $\frac{7}{72} \times 490$, or $47\frac{23}{36}$ lb.

Therefore the weight of the given steel rod is $47\frac{23}{36}$ lb. Q. E. F.

11. *If 3 cu. in. of gold beaten into gold leaf will cover 75,000 sq. in. of surface, find the thickness of the leaf.*

Solution. Let x be the thickness of the gold leaf expressed in inches.

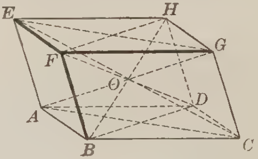
$\therefore$ 75,000 x is the volume of the gold leaf. § 536

Then $75{,}000x = 3.$ Ax. 8

$\therefore x = .00004.$ Ax. 4

Therefore .00004 in. is the thickness of the gold leaf. Q. E. F.

12. *The sum of the squares on the four diagonals of a parallelepiped is equivalent to the sum of the squares on the twelve edges.*



Given AG, CE, BH, and DF, the four diagonals of the parallelepiped AG.

To prove that $\overline{AG}^2 + \overline{BH}^2 + \overline{CE}^2 + \overline{DF}^2 = \overline{AB}^2 + \overline{BC}^2 + \overline{CD}^2 + \overline{DA}^2 + \overline{BF}^2 + \overline{CG}^2 + \overline{DH}^2 + \overline{AE}^2 + \overline{EF}^2 + \overline{FG}^2 + \overline{GH}^2 + \overline{HE}^2$.

Proof. Draw AC, EG, FH, and BD.

AE and CG are equal and ∥, and FB and HD are equal and ∥. § 503

The sections $ACGE$ and $BDHF$ are ▱s. § 118

In the $\triangle ACG$, $2\overline{OG}^2 + 2\overline{OC}^2 = \overline{AC}^2 + \overline{CG}^2$. § 343

In the $\triangle AGE$, $2\overline{OG}^2 + 2\overline{OE}^2 = \overline{AE}^2 + \overline{EG}^2$. § 343

$\therefore 4\overline{OG}^2 + 2\overline{OC}^2 + 2\overline{OE}^2 = \overline{AC}^2 + \overline{CG}^2 + \overline{AE}^2 + \overline{EG}^2$. Ax. 1

But $OG = \frac{1}{2}AG$, and $OC = OE = \frac{1}{2}EC$. § 131

$\therefore \overline{AG}^2 + \overline{EC}^2 = \overline{AC}^2 + \overline{CG}^2 + \overline{AE}^2 + \overline{EG}^2$. Ax. 9

Similarly, $\overline{BH}^2 + \overline{FD}^2 = \overline{BD}^2 + \overline{DH}^2 + \overline{BF}^2 + \overline{FH}^2$.

$\therefore \overline{AG}^2 + \overline{BH}^2 + \overline{CE}^2 + \overline{DF}^2$
$= \overline{AE}^2 + \overline{CG}^2 + \overline{BF}^2 + \overline{DH}^2 + \overline{EG}^2 + \overline{FH}^2 + \overline{AC}^2 + \overline{BD}^2$ (1). Ax. 1

Again, the bases EG and AC are ▱s. § 514

$\therefore \overline{EG}^2 + \overline{FH}^2 = \overline{EF}^2 + \overline{FG}^2 + \overline{GH}^2 + \overline{HE}^2$,

and $\overline{AC}^2 + \overline{BD}^2 = \overline{AB}^2 + \overline{BC}^2 + \overline{CD}^2 + \overline{DA}^2$.

Substitute these values for $\overline{EG}^2 + \overline{FH}^2$ and $\overline{AC}^2 + \overline{BD}^2$ in (1).

Then $\overline{AG}^2 + \overline{BH}^2 + \overline{CE}^2 + \overline{DF}^2$
$= \overline{AB}^2 + \overline{BC}^2 + \overline{CD}^2 + \overline{DA}^2 + \overline{BF}^2 + \overline{CG}^2 + \overline{DH}^2 + \overline{AE}^2 + \overline{EF}^2$
$+ \overline{FG}^2 + \overline{GH}^2 + \overline{HE}^2$, by Ax. 9. Q. E. D.

Exercise 88. Page 334

1. *Find the volume of the triangular prism whose base is 17 sq. in. and altitude 8 in.*

$v = ba.$ § 538

$\therefore v = 17 \times 8$ Ax. 9

$= 136.$

$\therefore$ the volume is 136 cu. in.

2. *Find the volume of the triangular prism whose base is 15.75 sq. ft. and altitude 3 ft.*

$v = ba.$ § 538

$\therefore v = 15.75 \times 3$ Ax. 9

$= 47.25.$

$\therefore$ the volume is 47.25 cu. ft.

3. *Find the volume of the triangular prism whose base is $3\frac{1}{4}$ sq. ft. and altitude 1 ft. 8 in.*

$v = ba.$ § 538

$\therefore v = 3\frac{1}{4} \times 1\frac{2}{3}$ Ax. 9

$= \frac{65}{12}$, or $5\frac{5}{12}$.

$\therefore$ the volume is $5\frac{5}{12}$ cu. ft.

4. *Find the volume of the triangular prism whose base is $5\frac{1}{2}$ sq. ft. and altitude 2 ft. 9 in.*

$v = ba.$ § 538

$\therefore v = 5\frac{1}{2} \times 2\frac{3}{4}$ Ax. 9

$= \frac{121}{8}$, or $15\frac{1}{8}$.

$\therefore$ the volume is $15\frac{1}{8}$ cu. ft.

5. *Find the volume of the triangular prism whose base is 15.84 sq. ft. and altitude 3 ft. 10 in.*

$v = ba.$ § 538

$\therefore v = 15.84 \times 3\frac{5}{6}$ Ax. 9

$= 60.72.$

$\therefore$ the volume is 60.72 cu. ft.

6. *Find the volume of the triangular prism whose base is $16\frac{2}{3}$ sq. in. and altitude $2\frac{2}{5}$ in.*

$v = ba.$ § 538

$\therefore v = 16\frac{2}{3} \times 2\frac{2}{5}$ Ax. 9

$= \frac{50}{3} \times \frac{12}{5}$

$= 40.$

$\therefore$ the volume is 40 cu. in.

7. *Find the volume of the triangular prism whose base is $22\frac{1}{2}$ sq. in. and altitude $4\frac{1}{2}$ in.*

$v = ba.$ § 538

$\therefore v = 22\frac{1}{2} \times 4\frac{1}{2}$ Ax. 9

$= \frac{45}{2} \times \frac{9}{2}$

$= \frac{405}{4}$, or $101\frac{1}{4}$.

$\therefore$ the volume is $101\frac{1}{4}$ cu. in.

8. *Find the volume of the triangular prism whose base is $33\frac{1}{3}$ sq. in. and altitude $7\frac{1}{5}$ in.*

$v = ba.$ § 538

$\therefore v = 33\frac{1}{3} \times 7\frac{1}{5}$ Ax. 9

$= \frac{100}{3} \times \frac{36}{5}$

$= 240.$

$\therefore$ the volume is 240 cu. in.

9. *Find the volume of the triangular prism whose base is $42\frac{7}{8}$ sq. in. and altitude $3\frac{3}{8}$ in.*

$v = ba.$ § 538

$\therefore v = 42\frac{7}{8} \times 3\frac{3}{8}$ Ax. 9

$= \frac{343}{8} \times \frac{27}{8}$

$$= \frac{343 \times 27}{64}$$

$= \frac{9261}{64}$, or $144\frac{45}{64}$.

$\therefore$ the volume is $144\frac{45}{64}$ cu. in.

10. *Find the volume of the triangular prism whose base is* $27\frac{3}{5}$ *sq. in. and altitude* $3\frac{3}{4}$ *in.*

$v = ba.$ § 538

$\therefore v = 27\frac{3}{5} \times 3\frac{3}{4}$ Ax. 9

$= \frac{138}{5} \times \frac{15}{4}$

$= 103\frac{1}{2}.$

Therefore the volume is $103\frac{1}{2}$ cu. in.

11. *Find the volume of the triangular prism whose base is* 12 *sq. ft.* 75 *sq. in. and altitude* 2 *ft.* 7 *in.*

12 sq. ft. 75 sq. in. = $12\frac{25}{48}$ sq. ft.,

and 2 ft. 7 in. = $2\frac{7}{12}$ ft.

Now $v = ba.$ § 538

$\therefore v = 12\frac{25}{48} \times 2\frac{7}{12}$ Ax. 9

$= \frac{601}{48} \times \frac{31}{12}$

$= 32\frac{199}{576}.$

Therefore the volume is $32\frac{199}{576}$ cu. ft.

Exercise 89. Page 336

1. *If the length of a rectangular parallelepiped is* 18 *in., the width* 9 *in., and the height* 8 *in., find the total area of the surface.*

Solution. The total area of the surface of the given rectangular parallelepiped is its lateral surface plus the sum of its bases.

Now $l = ep.$ § 512

Here $e = 8$ in., and $p = 2(18 + 9)$, or 54 in.

$\therefore l = 8 \times 54$, or 432. Ax. 9

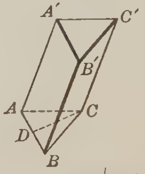
Therefore the lateral surface of the given rectangular parallelepiped is 432 sq. in.

The bases of the given rectangular parallelepiped are equal rectangles. § 526

$\therefore (2 \times 18 \times 9)$, or 324 sq. in., is the sum of the areas of the two bases.

Therefore (432 + 324) sq. in., or 756 sq. in., is the area required, by Ax. 11. Q. E. F.

2. *Find the volume of a triangular prism, if its height is* 15 *in. and the sides of the base are* 6 *in.,* 5 *in., and* 5 *in.*



Given, in the triangular prism ABC-B', $AB = 6$ in., $BC = 5$ in., $AC = 5$ in., and its height $a = 15$ in.

Required to find the volume of the prism ABC-B'.

Solution. Bisect AB in D and draw CD.

Then CD is $\perp$ to AB. § 151

In the rt. $\triangle CDB$, $\overline{CD}^2 = \overline{BC}^2 - \overline{BD}^2.$ § 338

Now $BC = 5$, and $BD = \frac{1}{2} AB = 3$. Given

$\therefore \overline{CD}^2 = 5^2 - 3^2$, or 4^2. Ax. 9

$\therefore CD = 4$. Ax. 5

The area of $\triangle ABC = \dfrac{AB \times CD}{2}$ § 325

$= \dfrac{6 \times 4}{2}$ Ax. 9

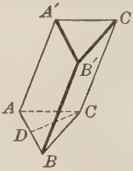
$= 12$.

Since $v = ba$, § 538

$\therefore v = 12 \times 15 = 180$. Ax. 9

Hence the volume of the prism is 180 cu. in. Q.E.F.

3. *Find the volume of a prism whose height is* 15 *ft., if each side of the triangular base is* 10 *in.*



Given the prism $ABC\text{-}B'$ whose base ABC is an equilateral $\triangle$, 10 in. on a side, and whose height is 15 ft.

Required to find the volume of $ABC\text{-}B'$.

Solution. Draw CD to the mid-point of AB.

Then CD is $\perp$ to AB. § 151

In the rt. $\triangle ADC$, $\overline{CD}^2 = \overline{AC}^2 - \overline{AD}^2$. § 338

Now $AC = 10$, and $AD = \frac{1}{2} AB = 5$.

$\therefore \overline{CD}^2 = 10^2 - 5^2$, or 75. Ax. 9

$\therefore CD = \sqrt{75}$ Ax. 5

$= 5\sqrt{3}$.

The area of $\triangle ABC = \dfrac{CD \times AB}{2}$ § 325

$= \dfrac{10 \times 5\sqrt{3}}{2}$, or $25\sqrt{3}$ sq. in. Ax. 9

Now $v = ba$. § 538

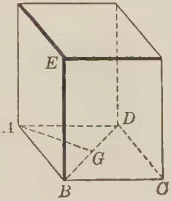
But $a = 15$ ft., or 15×12 in.

$\therefore v = 25\sqrt{3} \times 15 \times 12$ Ax. 9

$= 4500\sqrt{3}$.

Therefore $4500\sqrt{3}$ cu. in. is the volume of the prism. Q.E.F.

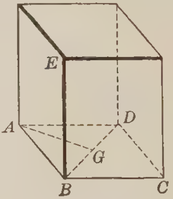
4. *The base of a right prism is a rhombus of which one side is* 20 *in., and the shorter diagonal* 24 *in. The height of the prism is* 30 *in. Find the entire surface and the volume.*



Given the right prism $ABCD\text{-}E$ whose base is the rhombus $ABCD$, with the shorter diagonal BD, and whose height is BE. $AB = 20$ in., $BD = 24$ in., and $BE = 30$ in.

Required to find (1) the entire surface of *ABCD-E*; (2) the volume of *ABCD-E*.

Solution. (1) The perimeter of $ABCD = 4 \times 20$, or 80 in. Ax. 11



Since $l = ap$, § 513

$\therefore l = 30 \times 80$ Ax. 9

$= 2400.$

Draw $AG \perp$ to BD. § 227

Then AG bisects BD. § 85

That is, $BG = \frac{1}{2} BD$.

In the rt. $\triangle AGB$, $\overline{AG}^2 = \overline{AB}^2 - \overline{BG}^2$ § 338

$= \overline{AB}^2 - (\frac{1}{2} BD)^2$ Ax. 9

$= \overline{AB}^2 - \frac{1}{4}\overline{BD}^2.$

$\therefore \overline{AG}^2 = 20^2 - \frac{1}{4} \times 24^2$ Ax. 9

$= 400 - \frac{1}{4} \times 576$

$= 400 - 144$

$= 256.$

$\therefore AG = 16.$ Ax. 5

Now the area of $\triangle ABD = \frac{1}{2} AG \times BD$ § 325

$= \frac{1}{2} \times 16 \times 24$ sq. in. Ax. 9

$= 192$ sq. in.

But the $\triangle$s ABD and CBD are congruent. § 80

Therefore the area of the rhombus $ABCD = 2 \times \triangle ABD$

$= 2 \times 192$ sq. in. Ax. 9

$= 384$ sq. in.

Therefore the area of the two bases of the right prism $ABCD\text{-}E = 2 \times 384$ sq. in. § 503

$= 768$ sq. in.

Therefore the total area of the right prism $ABCD\text{-}E = 2400 + 768$ sq. in. Ax. 11

$= 3168$ sq. in. Q.E.F.

(2) Since $v = ba$, § 539

$\therefore$ the volume of the right prism $ABCD\text{-}E = 384 \times 30$ cu. in. Ax. 9

$= 11{,}520$ cu. in. Q.E.F.

5. *How many square feet of lead will be required to line an open cistern which is* 4 *ft.* 6 *in. long,* 2 *ft.* 8 *in. wide, and contains* 42 *cu. ft.?*

Solution. $v = ba.$ § 539

But $b = 4\frac{1}{2} \times 2\frac{2}{3}.$ § 320

$\therefore 42 = 4\frac{1}{2} \times 2\frac{2}{3} \times a$ Ax. 9

$= 12\,a.$

That is, $12a = 42$.

$\therefore a = 3\frac{1}{2}$. Ax. 4

That is, $3\frac{1}{2}$ ft., or 3 ft. 6 in., is the depth of the given cistern.

Now $l = ap$. § 513

Here $a = 3\frac{1}{2}$ ft., and $p = 2\,(4\frac{1}{2} + 2\frac{2}{3})$, or $14\frac{1}{3}$ ft.

$\therefore l = 3\frac{1}{2} \times 14\frac{1}{3}$ Ax. 9

$= \frac{301}{6}$, or $50\frac{1}{6}$.

That is, the lateral surface of the cistern is $50\frac{1}{6}$ sq. ft.

The area of the bottom of the cistern $= 4\frac{1}{2} \times 2\frac{2}{3}$, or 12 sq. ft., as shown above.

Therefore the number of square feet of lead required to line the given cistern is $(50\frac{1}{6} + 12)$,or $62\frac{1}{6}$ sq. ft., by Ax. 11. Q.E.F.

6. *An open cistern* 6 *ft. long and* $4\frac{1}{2}$ *ft. wide holds* 108 *cu. ft. of water. How many square feet of lead will it take to line the sides and bottom?*

Solution. $v = ba$. § 539

But $b = 6 \times 4\frac{1}{2}$. § 320

$\therefore 108 = 6 \times 4\frac{1}{2} \times a$ Ax. 9

$= 27\,a$.

That is, $27\,a = 108$.

$\therefore a = 4$. Ax. 4

That is, 4 ft. is the depth of the given cistern.

Now $l = ap$. § 513

Here $a = 4$ ft., and $p = 2\,(6 + 4\frac{1}{2})$, or 21 ft.

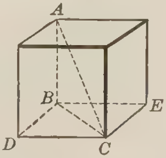
$\therefore l = 4 \times 21$, or 84 sq. ft. Ax. 9

That is, the lateral surface of the cistern is 84 sq. ft.

The area of the bottom of the cistern $= 6 \times 4\frac{1}{2}$, or 27 sq. ft., as shown above.

Therefore the number of square feet of lead required to line the given cistern is $(84 + 27)$, or 111 sq. ft., by Ax. 11. Q.E.F.

7. *One edge of a cube is e. Find in terms of e the surface, the volume, and the length of a diagonal of the cube.*



Given the cube AC, whose edge $AB = e$.

Required to find in terms of e (1) the surface of the cube AC; (2) volume of the cube AC; and (3) the length of the diagonal AC of the cube AC.

Solution. (1) The given cube AC has six equal square faces. § 517

The area of one of these squares is e^2. § 320

$\therefore$ the entire surface of the cube $AC = 6\,e^2$.

(2) The volume of the given cube AC is e^3. § 534

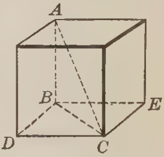
(3) Draw BC.

Then, in the rt. $\triangle BDC$,

$\overline{BC}^2 = 2e^2$. § 337

AB is $\perp$ to the face $DCEB$. § 431

$\therefore AB$ is $\perp$ to BC. § 430



Then in the rt. $\triangle ABC$,

$\therefore \overline{AC}^2 = \overline{AB}^2 + \overline{BC}^2$ § 337

$= e^2 + 2e^2$ Ax. 9

$= 3e^2$.

$\therefore AC = e\sqrt{3}$, by Ax. 5. Q.E.F.

8. *The diagonal of one of the faces of a cube is d. Find in terms of d the volume of the cube.*

Solution. Clearly two times the square of an edge of the given cube equals the square of a diagonal of one of its faces, for that diagonal is the hypotenuse of a rt. $\triangle$ having two edges of the cube as legs. § 337

That is, when x is the edge of the cube,

$2x^2 = d^2$.

$\therefore x^2 = \frac{1}{2}d^2$. Ax. 4

$\therefore x = \frac{1}{2}\sqrt{2}\,d$. Ax. 5

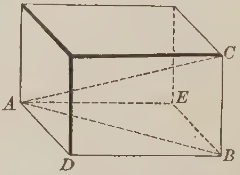
$\therefore x^3 = \frac{1}{4}\sqrt{2}\,d^3$. Ax. 5

That is, the volume of the given cube is $\frac{1}{4}d^3\sqrt{2}$, by § 534. Q.E.F.

9. *The three dimensions of a rectangular parallelepiped are a, b, c. Find in terms of a, b, and c the volume and the area of the surface.*

Given AC, a rectangular parallelepiped whose dimensions BC, BD, and BE are equal to a, b, and c respectively.

Required to find (1) the volume of the rectangular parallelepiped AC, and (2) the area of the surface of the rectangular parallelepiped AC.



Solution. (1) The volume of the given parallelepiped $= abc$. § 534

(2) The perimeter of the base DE of the rectangular parallelepiped AC is $2(b + c)$. § 125

Since $l = ap$, § 513

$\therefore l = 2a(b + c)$. Ax. 9

That is, the area of the lateral surface of AC is $2a(b + c)$.

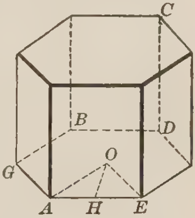
Moreover, the sum of the areas of the two bases is twice the area of the base DE. § 526

Now the area of the base DE is bc. § 320

$\therefore$ the sum of the areas of the two bases is $2bc$.

Hence $2a(b+c)+2bc$, that is, $2(ab+bc+ac)$, is the total surface of the given rectangular parallelepiped AC, by Ax. 11. Q.E.F.

10. *Find the volume of a prism with bases regular hexagons, if the height is* 10 *ft. and each side of the hexagons is* 10 *in.*



Given the prism AC, whose bases are regular hexagons 10 in. on a side, the height of AC being 10 ft.

Required to find the volume of AC.

Solution. Let O denote the center of the regular hexagon AD.

Draw the radii OA and OE. § 358

Also draw $OH \perp$ to AE.

Then in the rt. $\triangle AHO$, $AH = HE$ § 85

$= \frac{1}{2} AE.$

$\angle EAG = 120°.$ § 145

$\therefore \angle HAO = 60°.$ § 363

$\therefore \angle AOH = 30°.$ § 107

$\therefore OA = 2AH.$ Ex. 2, p. 80

Now $\overline{OH}^2 = \overline{OA}^2 - \overline{AH}^2.$ § 338

$\therefore \overline{OH}^2 = (2AH)^2 - \overline{AH}^2$ Ax. 9

$= 3\overline{AH}^2.$

$\therefore OH = \sqrt{3\overline{AH}^2}$, or $AH\sqrt{3}$. Ax. 5

But $AH = \frac{1}{2} \times 10$ in., or 5 in. Ax. 9

$\therefore OH = 5\sqrt{3}$ in. Ax. 9

$\therefore$ the area of the regular hexagon AD

$= \frac{1}{2} \times 6 \times 10 \times 5\sqrt{3}$ sq. in. § 386

$= 150\sqrt{3}$ sq. in.

Therefore the volume of $AC = \dfrac{10 \times 12 \times 150\sqrt{3}}{1728}$ cu. ft. § 539

$= 10\frac{5}{12}\sqrt{3}$ cu. ft. Q.E.F.

11. *An open cistern is made of iron* $\frac{1}{4}$ *in. thick. The inner dimensions are: length,* 4 *ft.* 6 *in.; breadth,* 3 *ft.; depth,* 2 *ft.* 6 *in. What will the cistern weigh when empty? when full of water? (A cubic foot of water weighs* $62\frac{1}{2}$ *lb. Iron is* 7.2 *times as heavy as water; that is, the specific gravity of iron is* 7.2.)

Solution. The outer dimensions of the given cistern are: length, 4 ft. 6 in. + $2 \times \frac{1}{4}$ in. = 4 ft. $6\frac{1}{2}$ in., or $54\frac{1}{2}$ in.; breadth, 3 ft. + $2 \times \frac{1}{4}$ in. = 3 ft. $\frac{1}{2}$ in., or $36\frac{1}{2}$ in., depth, 2 ft. 6 in. + $\frac{1}{4}$ in. = 2 ft. $6\frac{1}{4}$ in., or $30\frac{1}{4}$ in.

Therefore the volume of the iron forming the cistern plus its capacity

$$= 54\tfrac{1}{2} \times 36\tfrac{1}{2} \times 30\tfrac{1}{4} \text{ cu. in.} \qquad \S\ 534$$

$$= \tfrac{109}{2} \times \tfrac{73}{2} \times \tfrac{121}{4} \text{ cu. in.}$$

$$= \frac{109 \times 73 \times 121}{16 \times 1728} \text{ cu. ft.}$$

$$= \tfrac{962797}{27648} \text{ cu. ft., or } 34\tfrac{22765}{27648} \text{ cu. ft.}$$

But the capacity of the cistern

$$= 4\tfrac{1}{2} \times 3 \times 2\tfrac{1}{2} \text{ cu. ft.}$$

$$= \tfrac{9}{2} \times 3 \times \tfrac{5}{2} \text{ cu. ft.}$$

$$= \tfrac{135}{4} \text{ cu. ft., or } 33\tfrac{3}{4} \text{ cu. ft.} \qquad \S\ 534$$

Therefore the volume of iron forming the cistern

$$= 34\tfrac{22765}{27648} \text{ cu. ft.} - 33\tfrac{3}{4} \text{ cu. ft.}$$

$$= 1\tfrac{2029}{27648} \text{ cu. ft.}$$

Therefore the weight of iron forming the cistern

$$= 1\tfrac{2029}{27648} \times 62\tfrac{1}{2} \times 7.2 \text{ lb.}$$

$$= 483.019 \text{ lb.}$$

That is, the weight of the cistern when empty is 483.019 lb.

Since the capacity of the cistern is $33\frac{3}{4}$ cu. ft.,

∴ the weight of water the cistern will contain

$$= 33\tfrac{3}{4} \times 62\tfrac{1}{2} \text{ lb.}$$

$$= 33.75 \times 62.5 \text{ lb.}$$

$$= 2109.375 \text{ lb.}$$

Therefore the weight of the cistern full of water

$$= 483.0192 \text{ lb.} + 2109.375 \text{ lb.}$$

$$= 2592.394 \text{ lb.}$$

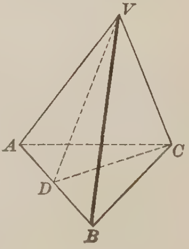
Q.E.F.

Exercise 90. Page 343

1. *The slant height of a regular pyramid is 6 in., and the base is an equilateral triangle of altitude $2\sqrt{3}$ in. Find the lateral area of the pyramid.*

Given $VD = 6$ in., the slant height of the pyramid $V\text{-}ABC$, whose base ABC is an equilateral △ with altitude $CD = 2\sqrt{3}$ in.

Required to find the lateral area of $V\text{-}ABC$.



Solution. In the rt. △ ADC,

$$\overline{AC}^2 - \overline{AD}^2 = \overline{CD}^2. \qquad \S\ 338$$

Now CD bisects AB. § 85

$$\therefore AD = \tfrac{1}{2} AB$$

$$= \tfrac{1}{2} AC. \qquad \text{Ax. } 9$$

$$\therefore \overline{AC}^2 - \tfrac{1}{4}\overline{AC}^2 = \overline{CD}^2. \qquad \text{Ax. 9}$$

That is, $$\tfrac{3}{4}\overline{AC}^2 = \overline{CD}^2.$$

$$\therefore \overline{AC}^2 = \tfrac{4}{3}\overline{CD}^2. \qquad \text{Ax. 3}$$

$$\therefore AC = \tfrac{2}{3}\sqrt{3}\,CD. \qquad \text{Ax. 5}$$

But $$CD = 2\sqrt{3}. \qquad \text{Given}$$

$$\therefore AC = \tfrac{2}{3}\sqrt{3} \times 2\sqrt{3}$$
$$= \tfrac{4}{3}(\sqrt{3})^2 = 4.$$

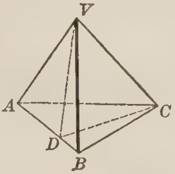
Then the perimeter of $\triangle ABC = 3 \times 4$, or 12. Ax. 11

Since $$A = \tfrac{1}{2}sp, \qquad \text{§ 553}$$

$$\therefore A = \tfrac{1}{2} \times 6 \times 12 \qquad \text{Ax. 9}$$
$$= 36.$$

Therefore the lateral area of $V\text{-}ABC$ is 36 sq. in. Q.E.F.

2. *The slant height of a regular triangular pyramid equals the altitude of the base. The area of the base is $\sqrt{3}$ sq. ft. Find the total area of the pyramid.*



Given, in the regular triangular pyramid $V\text{-}ABC$, VD, the slant height, equal to CD, the altitude of the base ABC whose area is $\sqrt{3}$ sq. ft.

Required to find the total area of $V\text{-}ABC$.

Solution. The triangular pyramid $V\text{-}ABC$ is a regular pyramid. § 547

Then the face △ VAB, VAC, and VBC are congruent. § 548, 2

Now the △ VAB and ABC have the common base AB, and the equal altitudes VD and CD respectively. Given

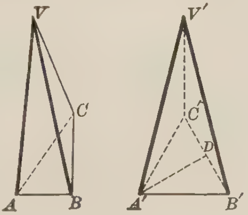
∴ the △ VAB and ABC are equivalent. § 326

∴ the △ VAB, VAC, VBC, and CAB are equivalent. Ax. 8

∴ the total area of the regular triangular pyramid

$$V\text{-}ABC = 4 \times \triangle ABC \qquad \text{Ax. 11}$$
$$= 4\sqrt{3} \text{ sq. ft., by Ax. 9.} \qquad \text{Q.E.F.}$$

3. *A pyramid has for its base a right triangle with hypotenuse 5 and shortest side 3. Another one of equal altitude has for its base an equilateral triangle with side $2\sqrt{2\sqrt{3}}$. Prove the pyramids equivalent.*



Given the pyramid $V\text{-}ABC$ whose base ABC is a rt. △ with hypotenuse $AC = 5$ and shorter leg $AB = 3$; also the pyramid $V'\text{-}A'B'C'$ whose altitude equals the altitude of $V\text{-}ABC$, and whose base $A'B'C'$ is an equilateral △ with side $2\sqrt{2\sqrt{3}}$.

To prove that $V\text{-}ABC$ and $V'\text{-}A'B'C'$ are equivalent.

Solution. In the $\triangle ABC$, $\overline{BC}^2 = \overline{AC}^2 - \overline{AB}^2$. § 338

$\therefore BC^2 = 5^2 - 3^2 = 4^2$. Ax. 9

$\therefore BC = 4$.

The area of the $\triangle ABC = \frac{1}{2} AB \times BC$ § 325

$= \frac{1}{2} \times 3 \times 4$ Ax. 9

$= 6$.

In the $\triangle A'B'C'$ draw $A'D \perp$ to $B'C'$.

Then $A'D$ bisects $B'C'$. § 85

That is, $B'D = \frac{1}{2} B'C'$.

$\therefore B'D = \frac{1}{2} A'B'$. Ax. 9

Since $\overline{A'D}^2 = \overline{A'B'}^2 - \overline{B'D}^2$, § 338

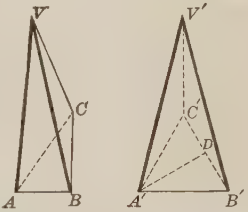
$\therefore \overline{A'D}^2 = \overline{A'B'}^2 - \frac{1}{4}\overline{A'B'}^2$ Ax. 9

$= \frac{3}{4}\overline{A'B'}^2$.

$\therefore A'D = \sqrt{\frac{3}{4}\overline{A'B'}^2}$ Ax. 5

$= \frac{1}{2} A'B'\sqrt{3}$.

$\therefore A'D = \frac{1}{2} B'C'\sqrt{3}$. Ax. 9



But the area of the $\triangle A'B'C' = \frac{1}{2} A'D \times B'C'$. § 325

$\therefore$ the area of the $\triangle A'B'C' = (\frac{1}{2})^2\sqrt{3} \times \overline{B'C'}^2$ Ax. 9

$= (\frac{1}{2})^2\sqrt{3} \times \left(2\sqrt{2\sqrt{3}}\right)^2$ Ax. 9

$= \frac{1}{4} \times \sqrt{3} \times 4 \times 2\sqrt{3} = 6$.

Then the given pyramids $V\text{-}ABC$ and $V'\text{-}A'B'C'$ have equivalent bases and equal altitudes.

Therefore $V\text{-}ABC$ and $V'\text{-}A'B'C'$ are equivalent, by § 558. Q.E.F.

Exercise 91. Page 346

1. *Find the lateral area of a regular pyramid whose slant height is 34 in. and perimeter of base 57 in.*

Solution. $a = \frac{1}{2} sp$. § 553

$\therefore a = \frac{1}{2} \times 34 \times 57$ Ax. 9

$= 969$.

Therefore the lateral area is 969 sq. in. Q.E.F.

2. *Find the lateral area of a regular pyramid whose slant height is $8\frac{3}{4}$ in. and perimeter of base $17\frac{1}{8}$ in.*

Solution. $a = \frac{1}{2} sp$. § 553

$\therefore a = \frac{1}{2} \times 8\frac{3}{4} \times 17\frac{1}{8}$ Ax. 9

$= 74\frac{59}{64}$.

Therefore the lateral area is $74\frac{59}{64}$ sq. in. Q.E.F

3. *Find the lateral area of a regular pyramid whose slant height is 2 ft. 7 in. and perimeter of base 4 ft. 6 in.*

Solution. $s = 2$ ft. 7 in. $= 2\frac{7}{12}$ ft.,

and $p = 4$ ft. 6 in. $= 4\frac{1}{2}$ ft.

Now $a = \frac{1}{2}sp.$ § 553

$\therefore a = \frac{1}{2} \times 2\frac{7}{12} \times 4\frac{1}{2}$ Ax. 9

$= \frac{1}{2} \times \frac{31}{12} \times \frac{9}{2}$

$= \frac{93}{16}$, or $5\frac{13}{16}$.

Therefore the lateral area is $5\frac{13}{16}$ sq. ft. Q.E.F.

4. *Find the lateral area of a regular pyramid whose slant height is 127 ft. 5 in. and perimeter of base 63 ft. 2 in.*

Solution. $s = 127$ ft. 5 in. $= 127\frac{5}{12}$ ft.,

and $p = 63$ ft. 2 in. $= 63\frac{1}{6}$ ft.

Now $a = \frac{1}{2}sp.$ § 553

$\therefore a = \frac{1}{2} \times 127\frac{5}{12} \times 63\frac{1}{6}$ Ax. 9

$= \frac{579491}{144}$, or $4024\frac{35}{144}$.

Therefore the lateral area is $4024\frac{35}{144}$ sq. ft. Q.E.F.

5. *Find the lateral area of the frustum of a regular pyramid whose slant height is 4 in. and perimeters of bases 8 in. and 6 in.*

Solution. $a = \frac{1}{2}s(p + p').$ § 554

$\therefore a = \frac{1}{2} \times 4(8 + 6)$ Ax. 9

$= 28.$

Therefore the lateral area is 28 sq. in. Q.E.F.

6. *Find the lateral area of the frustum of a regular pyramid whose slant height is $5\frac{1}{8}$ in. and perimeters of bases $9\frac{3}{4}$ in. and $7\frac{3}{8}$ in.*

Solution. $a = \frac{1}{2}s(p + p').$ § 554

$\therefore a = \frac{1}{2} \times 5\frac{1}{8}(9\frac{3}{4} + 7\frac{3}{8})$ Ax. 9

$= 43\frac{113}{128}.$

Therefore the lateral area is $43\frac{113}{128}$ sq. in. Q.E.F.

7. *Find the lateral area of the frustum of a regular pyramid whose slant height is 2 ft. 3 in. and perimeters of bases 4 ft. 8 in. and 3 ft. 9 in.*

Solution. $a = \frac{1}{2}s(p + p').$ § 554

Now $s = 2$ ft. 3 in. $= 2\frac{1}{4}$ ft.,

$p = 4$ ft. 8 in. $= 4\frac{2}{3}$ ft.,

and $p' = 3$ ft. 9 in. $= 3\frac{3}{4}$ ft.

$\therefore a = \frac{1}{2} \times 2\frac{1}{4}(4\frac{2}{3} + 3\frac{3}{4})$ Ax. 9

$= \frac{1}{2} \times \frac{9}{4} \times \frac{101}{12}$

$= \frac{303}{32}$, or $9\frac{15}{32}$.

Therefore the lateral area is $9\frac{15}{32}$ sq. ft. Q.E.F

8. *Find the volume of a pyramid whose altitude is 7 in. and area of base 9 sq. in.*

Solution. $v = \frac{1}{3}ba.$ § 561

$\therefore v = \frac{1}{3} \times 9 \times 7$ Ax. 9

$= 21.$

Therefore the volume is 21 cu. in. Q.E.F.

9. *Find the volume of a pyramid whose altitude is 6 in. and area of base 23 sq. in.*

Solution. $v = \frac{1}{3}ba.$ § 561

$\therefore v = \frac{1}{3} \times 23 \times 6$ Ax. 9

$= 46.$

Therefore the volume is 46 cu. in. Q.E.F.

10. *Find the volume of a pyramid whose altitude is 17 in. and area of base 51 sq. in.*

Solution. $v = \frac{1}{3}ba.$ § 561

$\therefore v = \frac{1}{3} \times 51 \times 17$ Ax. 9

$= 289.$

Therefore the volume is 289 cu. in. Q.E.F.

11. *Find the volume of a pyramid whose altitude is $3\frac{1}{8}$ in. and area of base $5\frac{1}{8}$ sq. in.*

Solution. $v = \frac{1}{3}ba.$ § 561

$\therefore v = \frac{1}{3} \times 5\frac{1}{8} \times 3\frac{1}{8}$ Ax. 9

$= \frac{1}{3} \times \frac{41}{8} \times \frac{25}{8}$

$= \frac{1025}{192}$, or $5\frac{65}{192}$.

Therefore the volume is $5\frac{65}{192}$ cu. in. Q.E.F.

12. *Find the volume of a pyramid whose altitude is $4\frac{3}{8}$ in. and area of base 19 sq. in.*

Solution. $v = \frac{1}{3}ba.$ § 561

$\therefore v = \frac{1}{3} \times 19 \times 4\frac{3}{8}$ Ax. 9

$= \frac{1}{3} \times 19 \times \frac{35}{8}$

$= \frac{665}{24}$, or $27\frac{17}{24}$.

Therefore the volume is $27\frac{17}{24}$ cu. in. Q.E.F.

13. *Find the volume of a pyramid whose altitude is 27.5 ft. and area of base 325 sq. ft.*

Solution. $v = \frac{1}{3}ba.$ § 561

$\therefore v = \frac{1}{3} \times 325 \times 27.5$ Ax. 9

$= 2979.16.$

Therefore the volume is 2979.16 cu. ft. Q.E.F.

14. *Find the lateral area of a regular pyramid, given the slant height* 2.3 *in., the number of sides of the base* 4, *and the length of each side* 2.1 *in.*

Solution. $a = \frac{1}{2} sp.$ § 553
But $p = nl$ Ax. 11
$= 4 \times 2.1$, or 8.4. Ax. 9
$\therefore a = \frac{1}{2} \times 2.3 \times 8.4$ Ax. 9
$= 9.66.$

Therefore the lateral area is 9.66 sq. in. Q.E.F.

15. *Find the lateral area of a regular pyramid, given the slant height* 3.7 *in., the number of sides of the base* 6, *and the length of each side* 2.9 *in.*

Solution. $a = \frac{1}{2} sp.$ § 553
But $p = nl$ Ax. 11
$= 6 \times 2.9$, or 17.4. Ax. 9
$\therefore a = \frac{1}{2} \times 3.7 \times 17.4$ Ax. 9
$= 32.19.$

Therefore the lateral area is 32.19 sq. in. Q.E.F.

16. *Find the lateral area of a regular pyramid, given the slant height* 5.33 *in., the number of sides of the base* 8, *and the length of each side* 3 *in.*

Solution. $a = \frac{1}{2} sp.$ § 553
But $p = nl$ Ax. 11
$= 8 \times 3$, or 24. Ax. 9
$\therefore a = \frac{1}{2} \times 5.33 \times 24$ Ax. 9
$= 63.96.$

Therefore the lateral area is 63.96 sq. in. Q.E.F.

17. *Find the volume of a pyramid, given the altitude* 7 *in. and the base a square with side* 2 *in.*

Solution. $v = \frac{1}{3} ba.$ § 561
But $b = 2^2$, or 4. § 320
$\therefore v = \frac{1}{3} \times 4 \times 7$ Ax. 9
$= \frac{28}{3}$, or $9\frac{1}{3}$.

Therefore the volume is $9\frac{1}{3}$ cu. in. Q.E.F.

18. *Find the volume of a pyramid, given the altitude* $6\frac{3}{4}$ *in. and the base a square with diagonal* $3\sqrt{2}$ *in.*

Solution. $v = \frac{1}{3} ba.$ § 561

Since b is the area of the square which is the base of the given pyramid, $\sqrt{b}$ is one side of that square. § 320

Therefore the square on the diagonal of the base $= 2\left(\sqrt{b}\right)^2$. § 337

Then $2(\sqrt{b})^2 = (3\sqrt{2})^2.$

$\therefore\ 2b = 18,$

whence $b = 9.$ Ax. 4

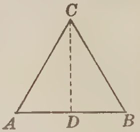
$\therefore\ v = \frac{1}{3} \times 9 \times 6\frac{3}{4}$ Ax. 9

$= \frac{1}{3} \times 9 \times \frac{27}{4}$

$= \frac{81}{4}$, or $20\frac{1}{4}$.

Therefore the volume is $20\frac{1}{4}$ cu. in. Q.E.F.

19. *Find the volume of a pyramid, given the altitude* 8.9 *in. and the base a triangle with each side* 3.7 *in.*



Solution. Let the equilateral $\triangle ABC$ represent the base of the given pyramid.

Draw $CD \perp$ to AB.

Then CD bisects AB. § 85

That is, $AD = \frac{1}{2}AB = \frac{1}{2}AC.$ Ax. 8

In the rt. $\triangle ADC$, $\overline{CD}^2 = \overline{AC}^2 - \overline{AD}^2.$ § 338

$\therefore\ \overline{CD}^2 = \overline{AC}^2 - \frac{1}{4}\overline{AC}^2$ Ax. 9

$= \frac{3}{4}\overline{AC}^2.$

$\therefore\ CD = \frac{1}{2}AC\sqrt{3}.$ Ax. 5

$\therefore$ the area of $\triangle ABC = \frac{1}{2}CD \times AB$ § 325

$= \frac{1}{2}CD \times AC$ Ax. 9

$= (\frac{1}{2})^2\sqrt{3}\cdot(3.7)^2$ Ax. 9

$= \frac{1}{4} \times \sqrt{3} \times 13.69$

$= 3.4225\sqrt{3}.$

That is, $b = 3.4225\sqrt{3}.$

Now $v = \frac{1}{3}ba.$ § 561

$\therefore\ v = \frac{1}{3} \times 3.4225\sqrt{3} \times 8.9$ Ax. 9

$= 17.586.$

Therefore the volume is 17.586 cu. in. Q.E.F.

20. *Find the lateral area of a regular pyramid, if the slant height is* 16 *ft. and the base is a hexagon with side* 12 *ft.*

Solution. $a = \frac{1}{2}sp.$ § 553

$p = 6 \times 12$, or 72 ft.

$\therefore\ a = \frac{1}{2} \times 16 \times 72$ Ax. 9

$= 576.$

Therefore the lateral area is 576 sq. ft. Q.E.F.

21. *Find the lateral area of a regular pyramid, if the slant height is* 8 *ft. and the base is a pentagon with side* 5 *ft.*

Solution. $a = \frac{1}{2}sp.$ § 553

$p = 5 \times 5$, or 25 ft.

$\therefore\ a = \frac{1}{2} \times 8 \times 25$ Ax. 9

$= 100.$

Therefore the lateral area is 100 sq. ft. Q. E. F.

22. *Find the total surface of a regular pyramid if the slant height is 6 ft. and the base is a square with side 4 ft.*

Solution. $a = \frac{1}{2} sp.$ § 553

$p = 4 \times 4$ ft., or 16 ft.

$\therefore\ a = \frac{1}{2} \times 6 \times 16$ Ax. 9

$= 48.$

Therefore the lateral area is 48 sq. ft.

Since the base of the given pyramid is a square with side 4 ft., its area is 4^2, or 16, sq. ft. § 320

Therefore the total area of the given pyramid is 48 sq. ft. + 16 sq. ft., or 64 sq. ft., by Ax. 11. Q. E. F.

23. *Find the total surface of a regular pyramid if the slant height is 18 ft. and the base is a square with side 8 ft.*

Solution. $a = \frac{1}{2} sp.$ § 553

$p = 4 \times 8$ ft., or 32 ft.

$\therefore\ a = \frac{1}{2} \times 18 \times 32$ Ax. 9

$= 288.$

Therefore the lateral area is 288 sq. ft.

Since the base of the given pyramid is a square with side 8 ft., its area is 8^2, or 64, sq. ft. § 320

Therefore the total area of the given pyramid is 288 sq. ft. + 64 sq. ft., or 352 sq. ft., by Ax. 11. Q. E. F.

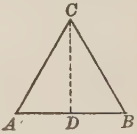
24. *Find the total surface of a regular pyramid if the slant height is 16 ft. and the base is a triangle with side 8 ft.*

Solution. Let $\triangle ABC$ represent the base of the given regular pyramid.

Then $\triangle ABC$ is an equilateral $\triangle$. § 546

Draw $CD \perp$ to AB.

Then CD bisects AB. § 85



That is, $AD = \frac{1}{2} AB = \frac{1}{2} AC.$ Ax. 8

Now in the rt. $\triangle ADC$, $\overline{CD}^2 = \overline{AC}^2 - \overline{AD}^2.$ § 338

$\therefore\ \overline{CD}^2 = \overline{AC}^2 - \frac{1}{4}\overline{AC}^2$ Ax. 9

$= \frac{3}{4}\overline{AC}^2.$

$\therefore\ CD = \frac{1}{2} AC\sqrt{3}$ Ax. 5

$= \frac{1}{2} AB\sqrt{3}.$ Ax. 9

$\therefore$ the area of $\triangle ABC = \frac{1}{2} CD \times AB$ § 325

$= (\frac{1}{2})^2 \sqrt{3} \cdot \overline{AB}^2$ Ax. 9

$= \frac{1}{4} \times 8^2 \sqrt{3}$ Ax. 9

$= 16\sqrt{3}$, or 27.713.

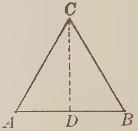
That is, the area of the base of the given pyramid is 27.713 sq. ft.

Furthermore $a = \frac{1}{2} sp.$ § 553

$p = 3 \times 8$, or 24 ft.

$\therefore a = \frac{1}{2} \times 16 \times 24$ Ax. 9

$= 192.$



Therefore the lateral area is 192 sq. ft.

Therefore the total area of the given pyramid is 27.713 sq. ft. + 192 sq. ft., or 219.713 sq. ft., by Ax. 11. Q. E. F.

25. *The volume of a pyramid is 26 cu. ft. 936 cu. in., and each side of its square base is 3 ft. 6 in. Find the height.*

Solution. $v = \frac{1}{3} ba.$ § 561

$\therefore a = \frac{3v}{b}.$ Ax. 4

Now $v = 26$ cu. ft. 936 cu. in. Given

$= 26 \times 1728$ cu. in. $+ 936$ cu. in.

$= 45{,}864$ cu. in.

Since the base of the given pyramid is a square with side 3 ft. 6 in., or 42 in.,

$\therefore b = 42^2$, or 1764 sq. in. § 320

$\therefore a = \frac{3 \times 45{,}864}{1764}$ Ax. 9

$= 78.$

Therefore the height of the given pyramid is 78 in., or 6 ft. 6 in. Q. E. F.

26. *The volume of a pyramid is 20 cu. ft. and the sides of its triangular base are 5 ft., 4 ft., and 3 ft. respectively. Find the height.*

Solution. Since $5^2 = 4^2 + 3^2$, the sides of the base $\triangle$ of the given pyramid whose lengths are 4 ft. and 3 ft. respectively include a rt. $\angle$. § 337

Therefore the area of the base $\triangle$ of the given pyramid is $\frac{1}{2} \times 4 \times 3$, or 6 sq. ft. § 325

Now $v = \frac{1}{3} ba.$ § 561

$\therefore a = \frac{3v}{b}.$ Ax. 4

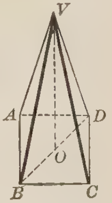
$\therefore a = \frac{3 \times 20}{6} = 10.$ Ax. 9

Therefore the height of the given pyramid is 10 ft. Q. E. F.

27. *Find the volume of a regular pyramid with a square base whose side is* 40 *ft., the lateral edge being* 101 *ft.*

Given the regular pyramid *V-ABCD*, having a square base *ABCD* with side 40 ft., and its lateral edge 101 ft.

Required to find the volume of *V-ABCD*.



Solution. Draw VO to the mid-point O of the diagonal BD.

O is the center of the base $ABCD$,

and VO is $\perp$ to it. § 546

In the rt. $\triangle VOB$, $\overline{VO}^2 = \overline{VB}^2 - \overline{BO}^2$. § 338

Since $BO = \frac{1}{2} BD$,

$\therefore\ \overline{VO}^2 = \overline{VB}^2 - \frac{1}{4}\overline{BD}^2$. Ax. 9

In the rt. $\triangle BCD$, $\overline{BD}^2 = \overline{BC}^2 + \overline{CD}^2$ § 337

$= 2\,\overline{BC}^2$.

$\therefore\ \overline{VO}^2 = \overline{VB}^2 - \frac{1}{2}\overline{BC}^2$ Ax. 9

$= 101^2 - \frac{1}{2} \times 40^2$ Ax. 9

$= 9401$.

$\therefore\ VO = \sqrt{9401}$. Ax. 5

That is, $a = \sqrt{9401}$.

Moreover, $b = 40^2$, or 1600. § 320

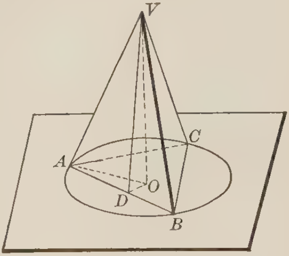
Since $v = \frac{1}{3} ba$, § 561

$\therefore\ v = \frac{1}{3} \times 1600 \times \sqrt{9401}$ Ax. 9

$= 51{,}712$.

Therefore the volume of the given pyramid is 51,712 cu. ft. Q. E. F.

28. *Find the volume of a regular pyramid whose slant height is* 12 *ft. and whose base is an equilateral triangle inscribed in a circle of radius* 10 *ft.*



Given the regular pyramid *V-ABC*, having for its slant height $VD = 12$ ft., and for its base the equilateral $\triangle ABC$ inscribed in a $\odot$ whose radius is 10 ft.

Required to find the volume of *V-ABC*.

Solution. Draw $VO \perp$ to the plane ABC.

Draw AO and OD.

Then O is the center of the equilateral $\triangle ABC$. § 546

That is, AO is the radius of the $\odot$ circumscribed about $\triangle ABC$. § 358

Since $\triangle AVB$ is isosceles, § 548

$\therefore\ VD$ bisects AB in D. § 85

$\therefore\ OD$ is $\perp$ to AB. § 151

Now AO bisects $\angle BAC$. § 363

$\therefore \angle DAO = \frac{1}{2}$ of 60°, or 30°.

$\therefore \angle AOD = 60°$. § 107

$\therefore OD = \frac{1}{2} AO = \frac{1}{2}$ of 10 ft., or 5 ft. Ex. 2, p. 80

VO is $\perp$ to OD. § 430

In the rt. $\triangle VOD$, $\overline{VO}^2 = \overline{VD}^2 - \overline{OD}^2$. § 338

$\therefore \overline{VO}^2 = 12^2 - 5^2$ Ax. 9

$= 119$.

$\therefore VO = \sqrt{119}$. Ax. 5

In the rt. $\triangle ODA$,

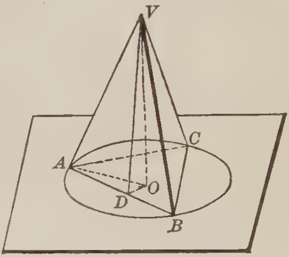
$\overline{AD}^2 = \overline{AO}^2 - \overline{OD}^2$ § 338

$= 10^2 - 5^2$, or 75. Ax. 9

$\therefore AD = 5\sqrt{3}$. Ax. 5

$\therefore AB = 10\sqrt{3}$. Ax. 3

Now the line OD produced will pass through C. § 150



$\therefore DC = OD + OC$ Ax. 11

$= 5 + 10$, or 15. Ax. 9

$\therefore$ the area of $\triangle ABC = \frac{1}{2} DC \times AB$ § 325

$= \frac{1}{2} \times 15 \times 10\sqrt{3}$ Ax. 9

$= 75\sqrt{3}$.

Since $v = \frac{1}{3} ba$, § 561

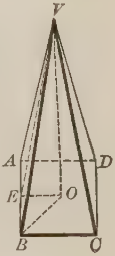
$\therefore v = \frac{1}{3} \times 75\sqrt{3} \times \sqrt{119}$ Ax. 9

$= 25\sqrt{357}$

$= 472.36$.

Therefore the volume of $V\text{-}ABC$ is 472.36 cu. ft. Q.E.F.

29. *Having given the base edge a and the total surface t of a regular pyramid with a square base, find the height h.*



Given the regular pyramid $V\text{-}ABCD$, with the square base $ABCD$ whose side is a, and with the total surface t.

Required to find the height h in terms of a and t.

Solution. Draw $VO \perp$ to the base $ABCD$ of the pyramid $V\text{-}ABCD$, draw $VE \perp$ to AB, and draw OE.

Now VE is the slant height of $V\text{-}ABCD$. § 547

$\therefore \frac{1}{2}(VE \times 4\,AB) + \overline{AB}^2 = t$. Ax. 11

$\therefore 2\,AB \times VE + \overline{AB}^2 = t$.

Since $AB = a$, Given

$\therefore 2\,a \times VE + a^2 = t$. Ax. 9

$\therefore 2\,a \times VE = t - a^2$. Ax. 2

$$\therefore VE = \frac{t - a^2}{2\,a}.$$ Ax. 4

Moreover, VE bisects AB; that is, E is the mid-point of AB. § 85

Also point O is equidistant from A and B. § 358

$\therefore OE$ is $\perp$ to AB. § 151

Draw BO.

In the rt. $\triangle BEO$, $\angle EBO = 45°$. § 363

$\therefore \angle EOB = 45°$. § 107

$\therefore OE = BE$. § 76

But $BE = \frac{1}{2}AB = \frac{1}{2}a$. Ax. 9

$\therefore OE = \frac{1}{2}a$. Ax. 8

In the rt. $\triangle VOE$, $\overline{VO}^2 = \overline{VE}^2 - \overline{OE}^2$. § 338

$$\therefore \overline{VO}^2 = \left(\frac{t-a^2}{2a}\right)^2 - \left(\frac{1}{2}a\right)^2 \qquad \text{Ax. 9}$$

$$= \frac{t^2 - 2a^2t + a^4}{4a^2} - \frac{a^2}{4}$$

$$= \frac{t^2 - 2a^2t + a^4}{4a^2} - \frac{a^4}{4a^2}$$

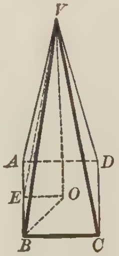
$$= \frac{t^2 - 2a^2t}{4a^2}.$$

$$\therefore VO = \sqrt{\frac{t^2 - 2a^2t}{4a^2}} \qquad \text{Ax. 5}$$

$$= \frac{1}{2a}\sqrt{t(t-2a^2)}.$$

$$\therefore h = \frac{1}{2a}\sqrt{t(t-2a^2)}, \text{ by Ax. 9.} \qquad \text{Q.E.F.}$$

30. *Having given the base edge a and the total surface t of a regular pyramid with a square base, find the volume v.*



Given the regular pyramid V-$ABCD$, with the square base $ABCD$ whose side is a, and with the total surface t.

Required to find the volume v in terms of a and t.

Solution. Draw $VO \perp$ to the base $ABCD$ of the regular pyramid V-$ABCD$.

Also draw $VE \perp$ to AB, and draw OE.

Now VE is the slant height of V-$ABCD$. § 547

$\therefore \frac{1}{2}(VE \times 4AB) + \overline{AB}^2 = t$. Ax. 11

$\therefore 2AB \times VE + \overline{AB}^2 = t$.

Since $AB = a$, Given

$\therefore 2a \times VE + a^2 = t$. Ax. 9

$\therefore 2a \times VE = t - a^2$. Ax. 2

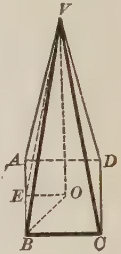
$$\therefore VE = \frac{t-a^2}{2a}. \qquad \text{Ax. 4}$$

Moreover, VE bisects AB; that is, E is the mid-point of AB. § 85

Also point O is equidistant from A and B. § 358

$\therefore OE$ is $\perp$ to AB. § 151

Draw BO.



In the rt. $\triangle BEO$, $\angle EBO = 45°$. § 363

$\therefore \angle EOB = 45°$. § 107

$\therefore OE = BE$. § 76

But $BE = \frac{1}{2} AB = \frac{1}{2} a$. Ax. 9

$\therefore OE = \frac{1}{2} a$. Ax. 8

In the rt. $\triangle VOE$, $\overline{VO}^2 = \overline{VE}^2 - \overline{OE}^2$. § 338

$$\therefore \overline{VO}^2 = \left(\frac{t - a^2}{2a}\right)^2 - \left(\frac{1}{2}a\right)^2 \quad \text{Ax. 9}$$

$$= \frac{t^2 - 2a^2t + a^4}{4a^2} - \frac{a^2}{4}$$

$$= \frac{t^2 - 2a^2t + a^4}{4a^2} - \frac{a^4}{4a^2}$$

$$= \frac{t^2 - 2a^2t}{4a^2}.$$

$$\therefore VO = \sqrt{\frac{t^2 - 2a^2t}{4a^2}} \quad \text{Ax. 5}$$

$$= \frac{1}{2a}\sqrt{t(t - 2a^2)}.$$

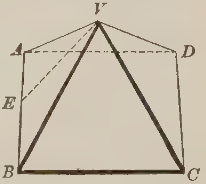
Since $v = \frac{1}{3}\overline{AB}^2 \times VO$, § 561

$$\therefore v = \frac{1}{3}a^2 \times \frac{1}{2a}\sqrt{t(t - 2a^2)} \quad \text{Ax. 9}$$

$$= \frac{1}{6}a\sqrt{t(t - 2a^2)}.$$

Q.E.F.

31. *The eight edges of a regular pyramid with a square base are equal and the total surface is t. Find the edge.*



Given the regular pyramid $V\text{-}ABCD$ whose base $ABCD$ is a square, having its eight edges all equal and with the total surface t.

Required to find the edge of $V\text{-}ABCD$ in terms of t.

Solution. Draw VE, the slant height of $V\text{-}ABCD$, to AB.

Then VE bisects AB. § 85

Now $\frac{1}{2}(VE \times 4\,AB) + \overline{AB}^2 = t$. Ax. 11

$$\therefore 2\,AB \times VE + \overline{AB}^2 = t.$$

But in the rt. $\triangle VEB$, $\overline{VE}^2 = \overline{VB}^2 - \overline{BE}^2$. § 338

Since $BE = \frac{1}{2} AB$ Proved

and $VB = AB,$ Given

$\therefore \overline{VE}^2 = \overline{AB}^2 - \frac{1}{4} \overline{AB}^2$ Ax. 9

$= \frac{3}{4} \overline{AB}^2.$

$\therefore VE = \frac{1}{2} AB \sqrt{3}.$ Ax. 5

Substituting this value of VE, we have

$2 AB \times \frac{1}{2} AB \sqrt{3} + \overline{AB}^2 = t.$ Ax. 9

$\therefore \overline{AB}^2 \cdot \sqrt{3} + \overline{AB}^2 = t.$

$\therefore \overline{AB}^2 (\sqrt{3} + 1) = t.$

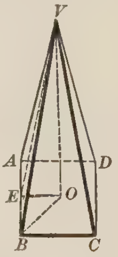
$\therefore \overline{AB}^2 = \frac{t}{\sqrt{3} + 1}.$ Ax. 4

$\therefore AB = \sqrt{\frac{t}{\sqrt{3} + 1}}.$ Ax. 5

Therefore the edge of V-$ABCD$ is $\sqrt{\frac{t}{\sqrt{3} + 1}}.$ Q.E.F.

32. *Find the base edge a of a regular pyramid with a square base, having given the height h and the total surface t.*

Given the regular pyramid V-$ABCD$ whose base $ABCD$ is a square, having the height VO equal to h and with the total surface t.



Required to find the base edge a of V-$ABCD$ in terms of h and t.

Solution. Draw VE, the slant height of V-$ABCD$, to AB; and draw OE.

Then OE is ⊥ to AB. § 151

Now $\frac{1}{2}(VE \times 4 AB) + \overline{AB}^2 = t.$ Ax. 11

$\therefore 2 AB \times VE + \overline{AB}^2 = t.$ (1)

In the rt. △ VOE, $\overline{VE}^2 = \overline{VO}^2 + \overline{OE}^2.$ (2) § 337

Draw BO.

In the rt. △ BEO, $\angle EBO = 45°.$ § 363

$\therefore \angle EOB = 45°.$ § 107

$\therefore OE = BE.$ § 76

But $BE = \frac{1}{2} AB.$ § 85

$\therefore OE = \frac{1}{2} AB.$ Ax. 8

Substituting in (2), we have

$\overline{VE}^2 = h^2 + \frac{1}{4} \overline{AB}^2.$ Ax. 9

$\therefore VE = \sqrt{h^2 + \frac{1}{4} \overline{AB}^2}.$ Ax. 5

Substituting this value of VE in (1), we have

$2 AB \times \sqrt{h^2 + \frac{1}{4} \overline{AB}^2} + \overline{AB}^2 = t.$ Ax. 9

$\therefore 2 AB \times \sqrt{h^2 + \frac{1}{4} \overline{AB}^2} = t - \overline{AB}^2.$ Ax. 2

$$\therefore\ 4\overline{AB}^2\left(h^2 + \tfrac{1}{4}\overline{AB}^2\right) = t^2 - 2t\cdot\overline{AB}^2 + \overline{AB}^4. \qquad \text{Ax. 5}$$

$$\therefore\ 4h^2\cdot\overline{AB}^2 + \overline{AB}^4 = t^2 - 2t\cdot\overline{AB}^2 + \overline{AB}^4.$$

$$\therefore\ 4h^2\cdot\overline{AB}^2 + 2t\cdot\overline{AB}^2 = t^2. \qquad \text{Ax. 2}$$

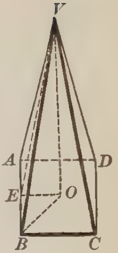
$$\therefore\ 2\overline{AB}^2(2h^2 + t) = t^2.$$

$$\therefore\ \overline{AB}^2 = \frac{t^2}{2(2h^2+t)}. \qquad \text{Ax. 4}$$

$$\therefore\ AB = \sqrt{\frac{t^2}{2(2h^2+t)}} \qquad \text{Ax. 5}$$

$$= \frac{t}{\sqrt{2(2h^2+t)}}.$$

$$\therefore\ a = \frac{t}{\sqrt{2(2h^2+t)}}, \text{ by Ax. 8.} \qquad \text{Q.E.F.}$$



33. *Show how to find the volume of any polyhedron by dividing the polyhedron into pyramids.*

The volume of any polyhedron may be found by dividing it into pyramids, computing their volumes separately and finding the sum of their volumes.

Exercise 92. Page 352

1. *Find the volume of a frustum of a pyramid, the altitude being* 3 *in. and the bases* 8 *sq. in and* 2 *sq. in.*

Solution.

$$v = \tfrac{1}{3}a\left(b + b' + \sqrt{bb'}\right). \qquad \S\ 565$$

$$\therefore\ v = \tfrac{1}{3} \times 3\left(8 + 2 + \sqrt{8 \times 2}\right) \qquad \text{Ax. 9}$$

$$= 14.$$

∴ the volume of the given frustum of a pyramid is 14 cu. in. Q.E.F.

2. *Find the volume of a frustum of a pyramid, the altitude being* 4½ *in. and the bases* 8⅓ *sq. in. and* 3 *sq. in.*

Solution.

$$v = \tfrac{1}{3}a\left(b + b' + \sqrt{bb'}\right). \qquad \S\ 565$$

$$\therefore\ v = \tfrac{1}{3} \times 4\tfrac{1}{2}\left(8\tfrac{1}{3} + 3 + \sqrt{8\tfrac{1}{3} \times 3}\right) \qquad \text{Ax. 9}$$

$$= 1\tfrac{1}{2} \times 16\tfrac{1}{3}$$

$$= 24\tfrac{1}{2}.$$

∴ the volume of the given frustum of a pyramid is 24½ cu. in. Q.E.F.

3. *Find the volume of a frustum of a pyramid, the altitude being* 3.2 *in. and the bases* 2 *sq. in. and* 0.18 *sq. in.*

Solution.

$$v = \tfrac{1}{3}a\left(b + b' + \sqrt{bb'}\right). \qquad \S\ 565$$

$$\therefore\ v = \tfrac{1}{3} \times 3{,}2\left(2 + 0.18 + \sqrt{2 \times 0.18}\right) \qquad \text{Ax. 9}$$

$$= \tfrac{1}{3} \times 3.2 \times 2.78$$

$$= 2.965.$$

∴ the volume of the given frustum of a pyramid is 2.965 cu. in. Q.E.F.

4. *Find the volume of a frustum of a pyramid, the altitude being 2 ft. 6 in. and the bases 10 sq. ft. and 2 sq. ft. 72 sq. in.*

Solution. $a = 2 \text{ ft. } 6 \text{ in.} = 2\frac{1}{2} \text{ ft.},$

and $b' = 2 \text{ sq. ft. } 72 \text{ sq. in.} = 2\frac{1}{2} \text{ sq. ft.}$

Now $v = \frac{1}{3} a\left(b + b' + \sqrt{bb'}\right).$ § 565

$\therefore v = \frac{1}{3} \times 2\frac{1}{2}\left(10 + 2\frac{1}{2} + \sqrt{10 \times 2\frac{1}{2}}\right)$ Ax. 9

$= \frac{1}{3} \times \frac{5}{2} \times \frac{35}{2}$

$= \frac{175}{12}$, or $14\frac{7}{12}$.

∴ the volume of the given frustum of a pyramid is $14\frac{7}{12}$ cu. ft., or 14 cu. ft. 1008 cu. in. Q.E.F.

5. *Find the volume of a frustum of a pyramid, the altitude being 3 ft. 7 in. and the bases 24 sq. ft. 72 sq. in. and 2 sq. ft.*

Solution. $a = 3 \text{ ft. } 7 \text{ in.} = 3\frac{7}{12} \text{ ft.},$

and $b = 24 \text{ sq. ft. } 72 \text{ sq. in.} = 24\frac{1}{2} \text{ sq. ft.}$

Now $v = \frac{1}{3} a\left(b + b' + \sqrt{bb'}\right).$ § 565

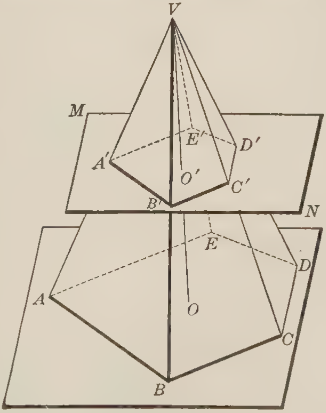
$\therefore v = \frac{1}{3} \times 3\frac{7}{12}\left(24\frac{1}{2} + 2 + \sqrt{24\frac{1}{2} \times 2}\right)$ Ax. 9

$= \frac{1}{3} \times \frac{43}{12} \times \frac{67}{2}$

$= \frac{2881}{72}$, or $40\frac{1}{72}$.

∴ the volume of the given frustum of a pyramid is $40\frac{1}{72}$ cu. ft., or 40 cu. ft. 24 cu. in. Q.E.F.

6. *A pyramid 2 in. high, with a base whose area is 8 sq. in., is cut by a plane parallel to the base 1 in. from the vertex. Find the volume of the frustum.*



Given the pyramid $V\text{-}ABCDE$ cut by the plane $MN \parallel$ to the base, intersecting the lateral edges in A', B', C', D', E', and the altitude VO in O'. $VO = 2$ in., $VO' = 1$ in., and the area of $ABCDE = 8$ sq. in.

Required to find the volume of the frustum $A'B'C'D'E'\text{-}ABCDE$.

Solution. The polygons $ABCDE$ and $A'B'C'D'E'$ are similar. § 555, 2

$\therefore \frac{A'B'C'D'E'}{ABCDE} = \frac{\overline{VO'}^2}{\overline{VO}^2}.$ § 335

But $\frac{VO'}{VO} = \frac{1}{2}.$ Given

$\therefore \frac{A'B'C'D'E'}{ABCDE} = \frac{1}{2^2} = \frac{1}{4}.$ Ax. 9

But the area of $ABCDE$ is 8 sq. in.

$$\therefore \tfrac{1}{8} A'B'C'D'E' = \tfrac{1}{4}. \qquad \text{Ax. 9}$$

$$\therefore A'B'C'D'E' = 2. \qquad \text{Ax. 3}$$

That is, the area of $A'B'C'D'E'$ is 2 sq. in.

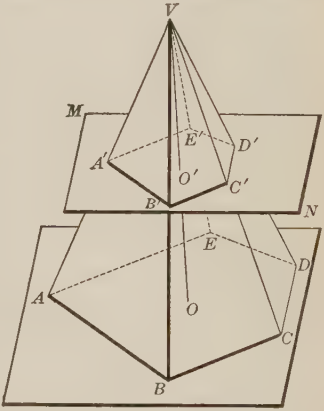
Now $$v = \tfrac{1}{3} a \left(b + b' + \sqrt{bb'}\right). \qquad \text{§ 565}$$

$$\therefore v = \tfrac{1}{3}(2-1)\left(8+2+\sqrt{8 \times 2}\right) \qquad \text{Ax. 9}$$

$$= \tfrac{14}{3}, \text{ or } 4\tfrac{2}{3}.$$

∴ the volume of the frustum $A'B'C'D'E'$-$ABCDE$ is $4\frac{2}{3}$ cu. in. Q. E. F.

7. *A pyramid 3 in. high, with a base whose area is 81 sq. in., is cut by a plane parallel to the base 2 in. from the base. Find the volume of the frustum.*



Given the pyramid V-$ABCDE$ cut by the plane $MN \parallel$ to the base, intersecting the lateral edges in A', B', C', D', E', and the altitude VO in O'. $VO = 3$ in., $OO' = 2$ in., and the area of $ABCDE = 81$ sq. in.

Required to find the volume of the frustum $A'B'C'D'E'$-$ABCDE$.

Solution. The polygons $ABCDE$ and $A'B'C'D'E'$ are similar. § 555, 2

$$\therefore \frac{A'B'C'D'E'}{ABCDE} = \frac{\overline{VO'}^2}{\overline{VO}^2} \qquad \text{§ 335}$$

$$= \frac{(VO - OO')^2}{\overline{VO}^2}. \qquad \text{Ax. 9}$$

$$\therefore \frac{A'B'C'D'E'}{ABCDE} = \frac{(3-2)^2}{3^2} = \frac{1}{9}. \qquad \text{Ax. 9}$$

But the area of $ABCDE$ is 81 sq. in.

$$\therefore \tfrac{1}{81} A'B'C'D'E' = \tfrac{1}{9}. \qquad \text{Ax. 9}$$

$$\therefore A'B'C'D'E' = 9. \qquad \text{Ax. 3}$$

That is, the area of $A'B'C'D'E'$ is 9 sq. in.

Now $$v = \tfrac{1}{3} a \left(b + b' + \sqrt{bb'}\right). \qquad \text{§ 565}$$

$$\therefore v = \tfrac{1}{3} \times 2\left(81 + 9 + \sqrt{81 \times 9}\right) \qquad \text{Ax. 9}$$

$$= \tfrac{2}{3} \times 117$$

$$= 78.$$

∴ the volume of the frustum $A'B'C'D'E'$-$ABCDE$ is 78 cu. in. Q. E. F.

8. *The lower base of a frustum of a pyramid is a square 4 in. on a side. The side of the upper base is half that of the lower base, and the altitude of the frustum is the same as the side of the upper base. Find the volume of the frustum.*

Solution. The area of the lower base of the given frustum of a pyramid is 4^2, or 16, sq. in. § 320

Since the side of the upper base of the given frustum is $\frac{1}{2}$ of 4, or 2, in., the area of the upper base is 2^2, or 4, sq. in. § 320

Moreover, the altitude of the frustum is 2 in.

Hence $a = 2$ in., $b = 16$ sq. in., and $b' = 4$ sq. in.

Now $v = \frac{1}{3} a \left(b + b' + \sqrt{bb'}\right).$ § 565

$\therefore\ v = \frac{1}{3} \times 2 \left(16 + 4 + \sqrt{16 \times 4}\right)$ Ax. 9

$= \frac{2}{3} \times 28$

$= \frac{56}{3}$, or $18\frac{2}{3}$.

$\therefore$ the volume of the given frustum of a pyramid is $18\frac{2}{3}$ cu. in. Q. E. F.

9. *The lower base of a frustum of a pyramid is a square* 3 *in. on a side. The area of the upper base is half that of the lower base, and the altitude of the frustum is* 2 *in. Find to two decimal places the volume of the frustum.*

Solution. The area of the lower base of the given frustum of a pyramid is 3^2, or 9, sq. in. § 320

Then the area of the upper base of the given frustum is $\frac{1}{2}$ of 9, or $4\frac{1}{2}$, sq. in.

Hence $a = 2$ in., $b = 9$ sq. in., and $b' = 4\frac{1}{2}$ sq. in.

Now $v = \frac{1}{3} a \left(b + b' + \sqrt{bb'}\right).$ § 565

$\therefore\ v = \frac{1}{3} \times 2 \left(9 + 4\frac{1}{2} + \sqrt{9 \times 4\frac{1}{2}}\right)$ Ax. 9

$= \frac{2}{3} \times 4\frac{1}{2} \left(3 + \sqrt{2}\right)$

$= 3 \left(3 + \sqrt{2}\right)$

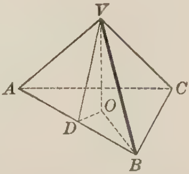
$= 9 + 3\sqrt{2}.$

But $3\sqrt{2} = 3 \times 1.414 = 4.242.$

$\therefore\ v = 9 + 4.24$, or 13.24. Ax. 11

$\therefore$ the volume of the given frustum of a pyramid is 13.24 cu. in. Q. E. F.

10. *A pyramid has six edges, each* 1 *in. long. Find to two decimal places the volume of the pyramid.*



Given the regular tetrahedron *V-ABC* whose edge is 1 in. long.

Required to find the volume of *V-ABC*.

Solution. Draw the altitude *VO* and the slant height *VD* to *AB*, in *V-ABC*.

Also draw *BO* and *DO*.

Then *O* is the center of the equilateral $\triangle ABC$. § 546

$\therefore$ *BO* bisects $\angle ABC$. § 363

But $\angle ABC = 60°$, for $\triangle ABC$ is equilateral and therefore equiangular. § 107

$\therefore \angle DBO = 30°$.

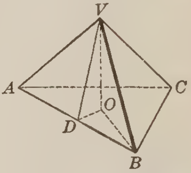
Since VD is $\perp$ to AB, § 547

$\therefore VD$ bisects AB. § 85

$\therefore DO$ is $\perp$ to AB. § 151

Then in the rt. $\triangle ODB$, $\angle BOD = 60°$. § 107

$\therefore DO = \frac{1}{2} BO$. Ex. 2, p. 80



Since $AB = 1$ in.,

$BD = \frac{1}{2}$ in. Ax. 4

Now $\overline{BO}^2 = \overline{BD}^2 + \overline{DO}^2$. § 337

$\therefore \overline{BO}^2 = (\frac{1}{2})^2 + (\frac{1}{2} BO)^2$ Ax. 9

$= \frac{1}{4} + \frac{1}{4} \overline{BO}^2$.

$\therefore \overline{BO}^2 - \frac{1}{4} \overline{BO}^2 = \frac{1}{4}$. Ax. 2

$\therefore \frac{3}{4} \overline{BO}^2 = \frac{1}{4}$.

$\therefore \overline{BO}^2 = \frac{1}{3}$. Ax. 3

In the rt. $\triangle VOB$, $\overline{VO}^2 = \overline{VB}^2 - \overline{BO}^2$. § 338

$\therefore \overline{VO}^2 = 1 - \frac{1}{3}$, or $\frac{2}{3}$. Ax. 9

$\therefore VO = \frac{1}{3}\sqrt{6}$. Ax. 5

Furthermore, in the rt. $\triangle VDB$,

$\overline{VD}^2 = \overline{VB}^2 - \overline{BD}^2$. § 338

$\therefore \overline{VD}^2 = 1 - (\frac{1}{2})^2$, or $\frac{3}{4}$. Ax. 9

$\therefore VD = \frac{1}{2}\sqrt{3}$. Ax. 5

Since the area of $\triangle AVB = \frac{1}{2} AB \times VD$, § 325

$\therefore$ the area of $\triangle AVB = \frac{1}{2} \times \frac{1}{2}\sqrt{3}$, or $\frac{1}{4}\sqrt{3}$. Ax. 9

But $\triangle$s ABC and AVB are congruent. § 80

$\therefore$ the area of $\triangle ABC = \frac{1}{4}\sqrt{3}$. Ax. 8

Now the volume of $V\text{-}ABC = \frac{1}{3} VO \times$ the area of $\triangle ABC$. § 559

$\therefore$ the volume of $V\text{-}ABC = \frac{1}{3} \times \frac{1}{3}\sqrt{6} \times \frac{1}{4}\sqrt{3}$ Ax. 9

$= \frac{1}{36}\sqrt{6}\sqrt{3}$

$= \frac{1}{12}\sqrt{2}$

$= 0.12$.

Therefore the volume of $V\text{-}ABC$ is 0.12 cu. in. Q.E.F.

11. *A regular tetrahedron has a volume $2\sqrt{2}$ cu. in. Find to two decimal places the length of an edge.*

Given the regular tetrahedron $V\text{-}ABC$ whose volume is $2\sqrt{2}$ cu. in.
Required to find the length of an edge of $V\text{-}ABC$.

Solution. Draw the altitude VO, and the slant height VD to AB in $V\text{-}ABC$.

Also draw BO and DO.

Then O is the center of the equilateral $\triangle ABC$. § 546

$\therefore$ BO bisects $\angle ABC$. § 363

But the $\angle ABC = 60°$, for $\triangle ABC$ is equilateral and therefore equiangular. § 107

$\therefore \angle DBO = 30°$.

Since VD is $\perp$ to AB, § 547

$\therefore$ VD bisects AB. § 85

$\therefore$ OD is $\perp$ to AB. § 151

Then in the rt. $\triangle ODB$, $\angle BOD = 60°$. § 107

$\therefore DO = \frac{1}{2} BO$. Ex. 2, p. 80

Now $\overline{BO}^2 = \overline{BD}^2 + \overline{DO}^2$. § 337

$\therefore \overline{BO}^2 = \overline{BD}^2 + (\frac{1}{2} BO)^2$ Ax 9

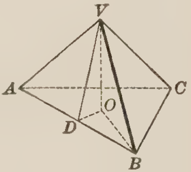
$= \overline{BD}^2 + \frac{1}{4}\overline{BO}^2$.

$\therefore \frac{3}{4}\overline{BO}^2 = \overline{BD}^2$, Ax. 2

whence $\overline{BO}^2 = \frac{4}{3}\overline{BD}$. Ax. 3

In the rt. $\triangle VOB$, $\overline{VO}^2 = \overline{VB}^2 - \overline{BO}^2$. § 338

$\therefore \overline{VO}^2 = \overline{VB}^2 - \frac{4}{3}\overline{BD}^2$. Ax. 9



But $AB = VB$, § 567

and $BD = \frac{1}{2} AB$.

$\therefore BD = \frac{1}{2} VB$. Ax. 9

$\therefore \overline{VO}^2 = \overline{VB}^2 - \frac{1}{3}\overline{VB}^2$ Ax. 9

$= \frac{2}{3}\overline{VB}^2$.

$\therefore VO = \frac{1}{3}\sqrt{6}\, VB$. Ax. 5

Furthermore, in the rt. $\triangle VDB$,

$\overline{VD}^2 = \overline{VB}^2 - \overline{BD}^2$. § 338

$\therefore \overline{VD}^2 = \overline{VB}^2 - \frac{1}{4}\overline{VB}^2$ Ax. 9

$= \frac{3}{4}\overline{VB}^2$.

$\therefore VD = \frac{1}{2}\sqrt{3}\, VB$. Ax. 5

But the area of $\triangle AVB = \frac{1}{2} AB \times VD$. § 325

$\therefore$ the area of $\triangle AVB = (\frac{1}{2})^2 AB \times VB\sqrt{3}$. Ax. 9

Since $AB = VB$,

$\therefore$ the area of $\triangle AVB = \frac{1}{4}\sqrt{3}\,\overline{VB}^2$. Ax. 9

But $\triangle$s ABC and AVB are congruent. § 80

$\therefore$ the area of $\triangle ABC = \frac{1}{4}\sqrt{3}\,\overline{VB}^2$. Ax. 8

Now the volume of $V\text{-}ABC = \frac{1}{3} VO \times$ the area of $\triangle ABC$. § 559

$\therefore$ the volume of $V\text{-}ABC = \frac{1}{3} \times \frac{1}{3}\sqrt{6}\, VB \times \frac{1}{4}\sqrt{3}\,\overline{VB}^2$ Ax. 9

$= \frac{1}{36}\sqrt{6}\sqrt{3} \times \overline{VB}^3$

$= \frac{1}{12}\sqrt{2} \times \overline{VB}^3$.

But the volume of $V\text{-}ABC = 2\sqrt{2}$ cu. in.

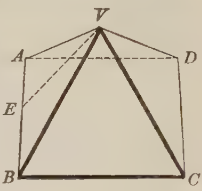
$\therefore \frac{1}{12}\sqrt{2} \times \overline{VB}^3 = 2\sqrt{2}$. Ax. 8

$\therefore \overline{VB}^3 = 24$, Ax. 4

whence $VB = 2\sqrt[3]{3} = 2.88$. Ax. 5

Therefore the length of an edge of $V\text{-}ABC$ is 2.88 in. Q.E.F.

12. *The base of a regular pyramid is a square l ft. on a side. The slant height is s ft. Find the area of the entire surface.*



Given the regular pyramid V-$ABCD$, with its base $ABCD$ a square l ft. on a side, and with its slant height VE equal to s ft.

Required to find the entire surface of V-$ABCD$ in terms of l and s.

Solution. The lateral surface of V-$ABCD$

$= \frac{1}{2}(VE \times 4\,AB)$ § 553

$= \frac{1}{2}(s \times 4\,l)$ Ax. 9

$= 2\,ls.$

Furthermore, the area of the base $ABCD = l^2$. § 320

Therefore the area of the entire surface of V-$ABCD$

$= 2\,ls + l^2 = l(2\,s + l)$. Ax. 11

Q.E.F.

13. *Consider the formula* $v = \frac{1}{3}a\left(b + b' + \sqrt{bb'}\right)$ *of* § 564, *when* $b' = 0$. *Discuss the meaning of the result. Also discuss the case in which* $b = b'$.

When $b' = 0$ the formula $v = \frac{1}{3}a\left(b + b' + \sqrt{bb'}\right)$ becomes $v = \frac{1}{3}ba$. In this case the given frustum of a pyramid ceases to be a frustum and becomes a pyramid instead.

Hence, by putting 0 for b' in the formula $v = \frac{1}{3}a\left(b + b' + \sqrt{bb'}\right)$, we simply deduce the formula for the volume of a pyramid.

When $b = b'$ the formula $v = \frac{1}{3}a\left(b + b' + \sqrt{bb'}\right)$ becomes $v = \frac{1}{3}a(2\,b + b)$, or $v = ab$. In this case the given frustum of a pyramid ceases to be a frustum and becomes a prism instead.

Hence, by putting b for b' in the formula $v = \frac{1}{3}a\left(b + b' + \sqrt{bb'}\right)$, we simply deduce the formula for the volume of a prism.

Exercise 93. Page 361

1. *The diameter of a well is 6 ft. and the water is 7 ft. deep. How many gallons of water are there in the well, reckoning $7\frac{1}{2}$ gal. to the cubic foot?*

Solution. The volume of water in the well is the volume of a right circular cylinder whose diameter is 6 ft. and altitude 7 ft.

Now the volume of any right circular cylinder is $\pi r^2 a$. § 590

Therefore the volume of water in the given well $= 3^2 \times 7\,\pi$ cu. ft. $= 63\,\pi$ cu. ft. Ax. 9

Therefore the number of gallons of water in the given well is $7\frac{1}{2} \times 63\,\pi$, or 1484.406. Q.E.F.

2. *When a body is placed under water in a right circular cylinder* 60 *centimeters in diameter, the level of the water rises* 40 *centimeters. Find the volume of the body.*

Solution. Since the water rises 40 centimeters in the given right circular cylinder when the given body is placed under water in it, and the increase in volume is obviously due to the size of the body immersed, it follows that the volume of a right circular cylinder 60 centimeters in diameter and 40 centimeters in height is the volume of the body immersed.

But the volume of a right circular cylinder with these measurements

$$= (\tfrac{1}{2} \times 60)^2 \times 40 \times \pi \qquad \S\ 590$$

$$= 36{,}000\,\pi, \text{ or } 113{,}097.6.$$

Therefore the volume of the body immersed is 113,097.6 cu. cm. Q. E. F.

3. *How many cubic yards of earth must be removed in constructing a tunnel* 100 *yd. long, the section being a semicircle with a radius of* 18 *ft.*

Solution. The volume of earth excavated is half the volume of a right circular cylinder 100 yd. long, having a radius of 18 ft., or 6 yd.

Therefore the number of cubic yards of earth excavated

$$= \tfrac{1}{2} \times 100 \times 6^2\,\pi \qquad \S\ 590$$

$$= 1800\,\pi, \text{ or } 5654.88. \qquad \text{Q. E. F.}$$

4. *How many square feet of sheet iron are required to make a pipe* 18 *in. in diameter and* 40 *ft. long?*

Solution. The circumference of the given pipe

$$= 18\,\pi \text{ in., or } 1.5\,\pi \text{ ft.} \qquad \S\ 384$$

Therefore the required number of square feet of sheet iron

$$= 40 \times 1.5\,\pi \qquad \S\ 588$$

$$= 60\,\pi, \text{ or } 188.496. \qquad \text{Q. E. F.}$$

5. *Find the radius of a cylindric pail* 14 *in. high that will hold exactly* 2 *cu. ft.*

Solution. Let r be the number of inches in the radius of the given cylindric pail.

Then, since $\quad 2 \text{ cu. ft.} = 2 \times 1728 \text{ cu. in., or } 3456 \text{ cu. in.,}$

$$14\,\pi r^2 = 3456. \qquad \S\ 590$$

$$\therefore\ r^2 = \frac{3456}{14\,\pi} = \frac{1728}{7\,\pi}. \qquad \text{Ax. 4}$$

$$\therefore\ r = \frac{24}{7\,\pi}\sqrt{21\,\pi} \qquad \text{Ax. 5}$$

$$= 8.864.$$

Therefore the required radius of the given cylindric pail is 8.864 in. Q. E. F.

6. *The height of a cylindric vessel that will hold 20 liters is equal to the diameter. Find the altitude and the radius.*

Solution. Let r be the number of centimeters in the radius of the given cylindric vessel.

Then, since the diameter of the vessel is $2r$ centimeters, the height of the vessel will be $2r$ centimeters.

$\therefore$ the volume of the vessel $= 2r \times \pi r^2$. § 590

Then, since 20 liters $= 20 \times 1000$ cu. cm., or 20,000 cu. cm.,

$$2r \times \pi r^2 = 20{,}000.$$

$$\therefore 2\pi r^3 = 20{,}000.$$

$$\therefore r^3 = \frac{20000}{2\pi}, \text{ or } \frac{10000}{\pi}. \qquad \text{Ax. 4}$$

$$\therefore r = \frac{10}{\pi}\sqrt[3]{10\pi^2}, \text{ or } 14.7. \qquad \text{Ax. 5}$$

$$\therefore 2r = 29.4. \qquad \text{Ax. 3}$$

Therefore the altitude and radius of the given vessel are 29.4 and 14.7 centimeters respectively. Q.E.F.

7. *If the total surface of a right circular cylinder is t and the radius of the base is r, find the altitude a.*

Solution. $t = 2\pi r(a + r)$. § 588

That is, $2\pi r(a + r) = t.$

$$\therefore 2\pi ra + 2\pi r^2 = t.$$

$$\therefore 2\pi ra = t - 2\pi r^2. \qquad \text{Ax. 2}$$

$$\therefore a = \frac{t - 2\pi r^2}{2\pi r}, \text{ by Ax. 4.} \qquad \text{Q.E.F.}$$

8. *If the lateral surface of a right circular cylinder is l and the volume is v, find the radius r and the altitude a.*

Solution. Let r be the radius of the base, and a the altitude.

Then $2\pi r \times a = l$, (1) § 588

and $\pi r^2 \times a = v$. (2) § 590

Multiply (2) by 2, and then divide by (1), $r = \frac{2v}{l}$. Axs. 3, 4

Substitute this value of r in (1).

Then $$2\pi\left(\frac{2v}{l}\right)a = l. \qquad \text{Ax. 9}$$

$$\therefore 4\pi va = l^2. \qquad \text{Ax. 3}$$

$$\therefore a = \frac{l^2}{4\pi v}. \qquad \text{Ax. 4}$$

Therefore the radius of the base and the altitude of the given right circular cylinder in terms of v and l are $\frac{2v}{l}$ and $\frac{l^2}{4\pi v}$ respectively. Q.E.F.

9. *If the circumference of the base of a right circular cylinder is c and the altitude is a, find the volume v.*

Solution. The radius of the base $= \dfrac{c}{2\pi}$. § 384

$\therefore$ the area of each base $= \left(\dfrac{c}{2\pi}\right)^2\pi$, or $\dfrac{c^2}{4\pi}$. § 389

$\therefore\ v = \dfrac{c^2}{4\pi} \times a$ § 589

$= \dfrac{ac^2}{4\pi}$. Q.E.F.

10. *If the circumference of the base of a right circular cylinder is c and the total surface is t, find the volume v.*

Solution. The radius of the base $= \dfrac{c}{2\pi}$. § 384

$\therefore$ the area of each base $= \left(\dfrac{c}{2\pi}\right)^2\pi$, or $\dfrac{c^2}{4\pi}$. § 389

$\therefore$ the lateral surface $= t - 2\left(\dfrac{c^2}{4\pi}\right)$, or $t - \dfrac{c^2}{2\pi}$.

$\therefore$ the altitude $= \left(t - \dfrac{c^2}{2\pi}\right) \div c$, or $\dfrac{2\pi t - c^2}{2\pi c}$.

$\therefore\ v = \dfrac{c^2}{4\pi} \times \dfrac{2\pi t - c^2}{2\pi c}$ § 589

$= \dfrac{c(2\pi t - c^2)}{8\pi^2}$. Q.E.F.

11. *If the volume of a right circular cylinder is v and the altitude is a, find the total surface t.*

Solution. The area of each base $= \dfrac{v}{a}$. § 589

Then the area of the two bases $= \dfrac{2v}{a}$, Ax. 3

and the radius of each base $= \sqrt{\dfrac{v}{a} \div \pi}$, or $\sqrt{\dfrac{v}{\pi a}}$. § 389

$\therefore$ the circumference of each base $= 2\pi\sqrt{\dfrac{v}{\pi a}}$. § 385

$\therefore$ the lateral surface $= 2\pi a\sqrt{\dfrac{v}{\pi a}}$, or $2\sqrt{\pi a v}$. § 588

$\therefore\ t = 2\sqrt{\pi a v} + \dfrac{2v}{a}$ Ax. 11

$= \dfrac{2(a\sqrt{\pi a v} + v)}{a}$. Q.E.F

12. *If v is the volume of a right circular cylinder in which the altitude equals the diameter, find the altitude a and the total surface t.*

Solution. Since the altitude of the given right circular cylinder is a, the diameter of the base is also a. Given

Therefore the radius of the base $= \frac{1}{2}a$.

$$\therefore v = (\tfrac{1}{2}a)^2\pi \times a \qquad \text{§ 590}$$

$$= \frac{\pi a^3}{4}.$$

$$\therefore a^3 = \frac{4v}{\pi}. \qquad \text{Ax. 4}$$

$$\therefore a = \sqrt[3]{\frac{4v}{\pi}} \qquad \text{Ax. 5}$$

$$= \frac{1}{\pi}\sqrt[3]{4\pi^2 v}.$$

Since the diameter of the base is a, the circumference is therefore πa. § 384

$$\therefore \text{the lateral surface} = a \cdot \pi a = \pi a^2. \qquad \text{§ 588}$$

Moreover, the area of each base $= (\frac{1}{2}a)^2\pi = \frac{1}{4}\pi a^2$. § 389

$$\therefore \text{the area of the two bases} = 2 \times \tfrac{1}{4}\pi a^2 = \tfrac{1}{2}\pi a^2. \qquad \text{Ax. 3}$$

$$\therefore t = \pi a^2 + \tfrac{1}{2}\pi a^2 = \tfrac{3}{2}\pi a^2 \qquad \text{Ax. 11}$$

$$= \tfrac{3}{2}\pi\left(\frac{1}{\pi}\sqrt[3]{4\pi^2 v}\right)^2$$

$$= \tfrac{3}{2}\pi \cdot \frac{1}{\pi^2} \cdot \sqrt[3]{16\pi^4 v^2}$$

$$= \tfrac{3}{2}\pi \cdot \frac{1}{\pi^2} \cdot 2\pi\sqrt[3]{2\pi v^2}$$

$$= 3\sqrt[3]{2\pi v^2}. \qquad \text{Q. E. F.}$$

13. *From the formula $t = 2\pi r(a + r)$ (§ 588) find the value of r.*

Solution. $t = 2\pi r(a + r)$. Given

That is, $2\pi r(a + r) = t$.

$$\therefore 2\pi ar + 2\pi r^2 = t,$$

or $2\pi r^2 + 2\pi ar = t$.

$$\therefore r^2 + ar = \frac{t}{2\pi}. \qquad \text{Ax. 4}$$

$$\therefore r^2 + ar + \frac{a^2}{4} = \frac{t}{2\pi} + \frac{a^2}{4} \qquad \text{Ax. 1}$$

$$= \frac{2t + \pi a^2}{4\pi}.$$

$$\therefore r+\frac{a}{2}=\pm\sqrt{\frac{2t+\pi a^2}{4\pi}} \qquad \text{Ax. 5}$$

$$=\pm\frac{1}{2}\sqrt{\frac{2t+\pi a^2}{\pi}}.$$

$$\therefore r=-\frac{a}{2}\pm\frac{1}{2}\sqrt{\frac{2t+\pi a^2}{\pi}} \qquad \text{Ax. 2}$$

$$=\frac{1}{2}\left(-a\pm\sqrt{\frac{2t+\pi a^2}{\pi}}\right). \qquad \text{Q.E.F.}$$

Exercise 94. Page 369

1. *Find the lateral area of a cone of revolution, given the slant height $2\frac{7}{8}$ in. and the circumference of the base $5\frac{3}{8}$ in.*

$$l=\tfrac{1}{2}sc. \qquad \S\ 609$$

$$\therefore l=\tfrac{1}{2}\times 2\tfrac{7}{8}\times 5\tfrac{3}{8} \qquad \text{Ax. 9}$$

$$=\tfrac{1}{2}\times\tfrac{23}{8}\times\tfrac{43}{8}$$

$$=\tfrac{989}{128}, \text{ or } 7\tfrac{93}{128}.$$

$\therefore$ the lateral area is $7\frac{93}{128}$ sq. in.

2. *Find the lateral area of a cone of revolution, given the slant height $4\frac{3}{8}$ in. and the circumference of the base $8\frac{1}{4}$ in.*

$$l=\tfrac{1}{2}sc. \qquad \S\ 609$$

$$\therefore l=\tfrac{1}{2}\times 4\tfrac{3}{8}\times 8\tfrac{1}{4} \qquad \text{Ax. 9}$$

$$=\tfrac{1}{2}\times\tfrac{35}{8}\times\tfrac{33}{4}$$

$$=\tfrac{1155}{64}, \text{ or } 18\tfrac{3}{64}.$$

$\therefore$ the lateral area is $18\frac{3}{64}$ sq. in.

3. *Find the lateral area of a cone of revolution, given the slant height $6\frac{5}{16}$ in. and the circumference of the base $10\frac{1}{2}$ in.*

$$l=\tfrac{1}{2}sc. \qquad \S\ 609$$

$$\therefore l=\tfrac{1}{2}\times 6\tfrac{5}{16}\times 10\tfrac{1}{2} \qquad \text{Ax. 9}$$

$$=\tfrac{1}{2}\times\tfrac{101}{16}\times\tfrac{21}{2}$$

$$=\tfrac{2121}{64}, \text{ or } 33\tfrac{9}{64}.$$

$\therefore$ the lateral area is $33\frac{9}{64}$ sq. in.

4. *Find the lateral area of a cone of revolution, given the slant height 3.7 in. and the circumference of the base 5.8 in.*

$$l=\tfrac{1}{2}sc. \qquad \S\ 609$$

$$\therefore l=\tfrac{1}{2}\times 3.7\times 5.8 \qquad \text{Ax. 9}$$

$$=3.7\times 2.9$$

$$=10.73.$$

$\therefore$ the lateral area is 10.73 sq. in.

5. *Find the lateral area of a cone of revolution, given the slant height 5.3 in. and the circumference of the base 9.7 in.*

$$l=\tfrac{1}{2}sc. \qquad \S\ 609$$

$$\therefore l=\tfrac{1}{2}\times 5.3\times 9.7 \qquad \text{Ax. 9}$$

$$=25.705.$$

$\therefore$ the lateral area is 25.705 sq. in.

6. *Find the lateral area of a cone of revolution, given the slant height 6.5 in. and the circumference of the base 11.6 in.*

$$l=\tfrac{1}{2}sc. \qquad \S\ 609$$

$$\therefore l=\tfrac{1}{2}\times 6.5\times 11.6 \qquad \text{Ax. 9}$$

$$=6.5\times 5.8$$

$$=37.7.$$

$\therefore$ the lateral area is 37.7 sq. in.

7. *Find the lateral area of a cone of revolution, given the slant height 2 ft. 6 in. and the circumference of the base 4 ft. 8 in.*

2 ft. 6 in. = $2\frac{1}{2}$ ft.,
and 4 ft. 8 in. = $4\frac{2}{3}$ ft.
Now $l = \frac{1}{2}sc$. § 609
$\therefore l = \frac{1}{2} \times 2\frac{1}{2} \times 4\frac{2}{3}$ Ax. 9
$= \frac{1}{2} \times \frac{5}{2} \times \frac{14}{3}$
$= \frac{35}{6}$, or $5\frac{5}{6}$.

∴ the lateral area is $5\frac{5}{6}$ sq. ft., or 5 sq. ft. 120 sq. in.

8. *Find the lateral area of a cone of revolution, given the slant height 3 ft. 7 in. and the circumference of the base 8 ft. 6 in.*

3 ft. 7 in. = $3\frac{7}{12}$ ft.,
and 8 ft. 6 in. = $8\frac{1}{2}$ ft.
Now $l = \frac{1}{2}sc$. § 609
$\therefore l = \frac{1}{2} \times 3\frac{7}{12} \times 8\frac{1}{2}$ Ax. 9
$= \frac{1}{2} \times \frac{43}{12} \times \frac{17}{2}$
$= \frac{731}{48}$, or $15\frac{11}{48}$.

∴ the lateral area is $15\frac{11}{48}$ sq. ft., or 15 sq. ft. 33 sq. in.

9. *Find the lateral area of a cone of revolution, given the slant height 5 ft. 8 in. and the circumference of the base 12 ft. 4 in.*

5 ft. 8 in. = $5\frac{2}{3}$ ft.,
and 12 ft. 4 in. = $12\frac{1}{3}$ ft.
Now $l = \frac{1}{2}sc$. § 609
$\therefore l = \frac{1}{2} \times 5\frac{2}{3} \times 12\frac{1}{3}$ Ax. 9
$= \frac{1}{2} \times \frac{17}{3} \times \frac{37}{3}$
$= \frac{629}{18}$, or $34\frac{17}{18}$.

∴ the lateral area is $34\frac{17}{18}$ sq. ft., or 34 sq. ft. 136 sq. in.

10. *Find the lateral area of a cone of revolution, given the slant height $3\frac{3}{4}$ in. and the radius of the base $2\frac{1}{2}$ in.*

$l = \pi rs$. § 610
$\therefore l = 2\frac{1}{2} \times 3\frac{3}{4}\pi$ Ax. 9
$= \frac{5}{2} \times \frac{15}{4}\pi$
$= \frac{75\pi}{8}$, or 29.45.

∴ the lateral area is 29.45 sq. in.

11. *Find the lateral area of a cone of revolution, given the slant height $2\frac{1}{2}$ in. and the radius of the base $1\frac{3}{4}$ in.*

$l = \pi rs$. § 610
$\therefore l = 1\frac{3}{4} \times 2\frac{1}{2}\pi$ Ax. 9
$= \frac{7}{4} \times \frac{5}{2}\pi$
$= \frac{35\pi}{8}$, or 13.74.

∴ the lateral area is 13.74 sq. in.

12. *Find the lateral area of a cone of revolution, given the slant height $4\frac{7}{8}$ in. and the radius of the base $3\frac{1}{4}$ in.*

$l = \pi rs$. § 610
$\therefore l = 3\frac{1}{4} \times 4\frac{7}{8}\pi$ Ax. 9
$= \frac{13}{4} \times \frac{39}{8}\pi$
$= \frac{507\pi}{32}$, or 49.77.

∴ the lateral area is 49.77 sq. in.

13. *Find the lateral area of a cone of revolution, given the slant height 6.4 in. and the radius of the base 4.8 in.*

$l = \pi rs$. § 610
$\therefore l = 4.8 \times 6.4\pi$ Ax. 9
$= 30.72\pi$, or 96.51.

∴ the lateral area is 96.51 sq. in.

14. *Find the lateral area of a cone of revolution, given the slant height 7.2 in. and the radius of the base 5.3 in.*

$l = \pi rs.$ § 610

$\therefore l = 5.3 \times 7.2\,\pi$ Ax. 9

$= 38.16\,\pi$, or 119.88.

$\therefore$ the lateral area is 119.88 sq. in.

15. *Find the lateral area of a cone of revolution, given the slant height 8.9 in. and the radius of the base 5.6 in.*

$l = \pi rs.$ § 610

$\therefore l = 5.6 \times 8.9\,\pi$ Ax. 9

$= 49.84\,\pi$, or 156.58.

$\therefore$ the lateral area is 156.58 sq. in.

16. *Find the lateral area of a cone of revolution, given the slant height 2 ft. 3 in. and the radius of the base 8 in.*

2 ft. 3 in. $= 2\frac{1}{4}$ ft.,

and 8 in. $= \frac{2}{3}$ ft.

Now $l = \pi rs.$ § 610

$\therefore l = \frac{2}{3} \times 2\frac{1}{4}\,\pi$ Ax. 9

$= \frac{2}{3} \times \frac{9}{4}\,\pi$

$= \frac{3}{2}\,\pi$, or 4.71.

$\therefore$ the lateral area is 4.71 sq. ft.

17. *Find the lateral area of a cone of revolution, given the slant height 4 ft. 6 in. and the radius of the base 2 ft.*

4 ft. 6 in. $= 4\frac{1}{2}$ ft.

Now $l = \pi rs.$ § 610

$\therefore l = 2 \times 4\frac{1}{2}\,\pi$ Ax. 9

$= 9\,\pi$, or 28.27.

$\therefore$ the lateral area is 28.27 sq. ft.

18. *Find the lateral area of a cone of revolution, given the slant height 6 ft. 9 in. and the radius of the base 3 ft. 2 in.*

6 ft. 9 in. $= 6\frac{3}{4}$ ft.,

and 3 ft. 2 in. $= 3\frac{1}{6}$ ft.

Now $l = \pi rs.$ § 610

$\therefore l = 3\frac{1}{6} \times 6\frac{3}{4}\,\pi$ Ax. 9

$= \frac{19}{6} \times \frac{27}{4}\,\pi$

$= \frac{171\,\pi}{8}$, or 67.15.

$\therefore$ the lateral area is 67.15 sq. ft.

19. *Find the total area of a cone of revolution, given the slant height 3 in. and the radius of the base 2 in.*

$t = \pi r(s + r).$ § 610

$\therefore t = 2\,\pi(3 + 2)$ Ax. 9

$= 10\,\pi$, or 31.416.

$\therefore$ the total area is 31.416 sq. in.

20. *Find the total area of a cone of revolution, given the slant height 5 in. and the radius of the base 3 in.*

$t = \pi r(s + r).$ § 610

$\therefore t = 3\,\pi(5 + 3)$ Ax. 9

$= 24\,\pi$, or 75.4.

$\therefore$ the total area is 75.4 sq. in.

21. *Find the total area of a cone of revolution, given the slant height 7 in. and the radius of the base 4 in.*

$t = \pi r(s + r).$ § 610

$\therefore t = 4\,\pi(7 + 4)$ Ax. 9

$= 44\,\pi$, or 138.23.

$\therefore$ the total area is 138.23 sq. in.

22. *Find the total area of a cone of revolution, given the slant height 9 in. and the radius of the base 5 in.*

$t = \pi r(s + r).$ § 610

$\therefore t = 5\,\pi(9 + 5)$ Ax. 9

$= 70\,\pi$, or 219.91.

$\therefore$ the total area is 219.91 sq. in.

23. *Find the total area of a cone of revolution, given the slant height 6 ft. and the radius of the base 4 ft.*

Solution. $t = \pi r(s + r).$ § 610

$\therefore t = 4\pi(6 + 4)$ Ax. 9

$= 40\pi$, or 125.66.

Therefore the total area is 125.66 sq. ft.

24. *Find the total area of a cone of revolution, given the slant height 12 ft. and the radius of the base 5 ft.*

Solution. $t = \pi r(s + r).$ § 610

$\therefore t = 5\pi(12 + 5)$ Ax. 9

$= 85\pi$, or 267.04.

Therefore the total area is 267.04 sq. ft. Q.E.F.

25. *Deduce a formula for finding the lateral area of a cone of revolution in terms of the radius of the base and the altitude.*

Solution. Denote the lateral area of the given cone of revolution by l, the radius of the base by r, the altitude by a, and the slant height by s.

Then $l = \pi rs.$ § 610

But $s^2 = a^2 + r^2,$ § 337

whence $s = \sqrt{a^2 + r^2}.$ Ax. 5

$\therefore l = \pi r\sqrt{a^2 + r^2}$, by Ax. 9. Q.E.F.

26. *Deduce a formula for finding the slant height in terms of the lateral area and the circumference of the base.*

Solution. Denote the circumference of the base of the given cone of revolution by c, the lateral area by l, and the slant height by s.

Then $l = \frac{1}{2}sc.$ § 609

$\therefore sc = 2l.$ Ax. 3

$\therefore s = \frac{2l}{c}$, by Ax. 4. Q.E.F.

27. *Deduce a formula for finding the slant height in terms of the lateral area and the radius of the base.*

Solution. Denote the slant height of the given cone of revolution by s, the lateral area by l, and the radius of the base by r.

Then $l = \pi rs.$ § 610

$\therefore s = \frac{l}{\pi r}$, by Ax. 4. Q.E.F.

28. *Deduce a formula for finding the radius of the base in terms of the lateral area and the slant height.*

Solution. Denote the slant height of the given cone of revolution by s, the lateral area by l, and the radius of the base by r.

Then $l = \pi r s.$ § 610

$\therefore r = \dfrac{l}{\pi s}$, by Ax. 4. Q. E. F.

29. *Deduce a formula for finding the slant height in terms of the total area and the radius of the base.*

Solution. Denote the slant height of the given cone of revolution by s, the total area by t, and the radius of the base by r.

Then $t = \pi r (s + r)$ § 610

$= \pi r s + \pi r^2.$

$\therefore \pi r s = t - \pi r^2.$ Ax. 2

$\therefore s = \dfrac{t - \pi r^2}{\pi r}$, by Ax. 4. Q. E. F.

30. *Deduce a formula for finding the circumference of the base in terms of the lateral area and the slant height.*

Solution. Denote the circumference of the given cone of revolution by c, the lateral area by l, and the slant height by s.

Then $l = \frac{1}{2} sc.$ § 609

$\therefore sc = 2l.$ Ax. 3

$\therefore c = \dfrac{2l}{s}$, by Ax. 4. Q. E. F.

Exercise 95. Page 371

1. *Find the volume of a circular cone, given the altitude 4 in. and the area of the base 8 sq. in.*

Solution. $v = \frac{1}{3} ba.$ § 611

$\therefore v = \frac{1}{3} \times 8 \times 4$ Ax. 9

$= \frac{32}{3}$, or $10\frac{2}{3}$.

Therefore the volume of the given cone is $10\frac{2}{3}$ cu. in. Q. E. F.

2. *Find the volume of a circular cone, given the altitude $3\frac{1}{4}$ in. and the area of the base $9\frac{3}{8}$ sq. in.*

Solution. $v = \frac{1}{3} ba.$ § 611

$\therefore v = \frac{1}{3} \times 9\frac{3}{8} \times 3\frac{1}{4}$ Ax. 9

$= \frac{1}{3} \times \frac{75}{8} \times \frac{13}{4}$

$= \frac{325}{32}$, or $10\frac{5}{32}$.

Therefore the volume of the given cone is $10\frac{5}{32}$ cu. in. Q. E. F.

3. *Find the volume of a circular cone, given the altitude* $5\frac{3}{8}$ *in. and the area of the base* $10\frac{1}{2}$ *sq. in.*

Solution. $v = \frac{1}{3}ba.$ § 611

$\therefore v = \frac{1}{3} \times 10\frac{1}{2} \times 5\frac{3}{8}$ Ax. 9

$= \frac{1}{3} \times \frac{21}{2} \times \frac{43}{8}$

$= \frac{301}{16}$, or $18\frac{13}{16}$.

Therefore the volume of the given cone is $18\frac{13}{16}$ cu. in. Q.E.F.

4. *Find the volume of a circular cone, given the altitude* 6.3 *in. and the area of the base* 3.8 *sq. in.*

Solution. $v = \frac{1}{3}ba.$ § 611

$\therefore v = \frac{1}{3} \times 3.8 \times 6.3$ Ax. 9

$= 2.1 \times 3.8$, or 7.98.

Therefore the volume of the given cone is 7.98 cu. in. Q.E.F.

5. *Find the volume of a circular cone, given the altitude* 7.8 *in. and the area of the base* 6.9 *sq. in.*

Solution. $v = \frac{1}{3}ba.$ § 611

$\therefore v = \frac{1}{3} \times 6.9 \times 7.8$ Ax. 9

$= 2.3 \times 7.8$, or 17.94.

Therefore the volume of the given cone is 17.94 cu. in. Q.E.F.

6. *Find the volume of a circular cone, given the altitude* 9.3 *in. and the area of the base* 16.8 *sq. in.*

Solution. $v = \frac{1}{3}ba.$ § 611

$\therefore v = \frac{1}{3} \times 16.8 \times 9.3$ Ax. 9

$= 5.6 \times 9.3$, or 52.08.

Therefore the volume of the given cone is 52.08 cu. in. Q.E.F.

7. *Find the volume of a circular cone, given the altitude* 4 *in. and the radius of the base* 3 *in.*

Solution. $v = \frac{1}{3}\pi r^2 a.$ § 612

$\therefore v = \frac{1}{3} \times 3^2 \times 4\pi$ Ax. 9

$= 12\pi$, or 37.7.

Therefore the volume of the given cone is 37.7 cu. in. Q.E.F.

8. *Find the volume of a circular cone, given the altitude* 6 *in. and the radius of the base* 4 *in.*

Solution. $v = \frac{1}{3}\pi r^2 a.$ § 612

$\therefore v = \frac{1}{3} \times 4^2 \times 6\pi$ Ax. 9

$= 32\pi$, or 100.53.

Therefore the volume of the given cone is 100.53 cu. in. Q.E.F.

9. *Find the volume of a circular cone, given the altitude* 8 *in. and the radius of the base* 5 *in.*

Solution. $v = \frac{1}{3}\pi r^2 a.$ § 612

$\therefore\ v = \frac{1}{3} \times 5^2 \times 8\pi$ Ax. 9

$= \dfrac{200\pi}{3}$, or 209.44.

Therefore the volume of the given cone is 209.44 cu. in. Q.E.F.

10. *Find the volume of a circular cone, given the altitude* 9.8 *in. and the radius of the base* 4.3 *in.*

Solution. $v = \frac{1}{3}\pi r^2 a.$ § 612

$\therefore\ v = \frac{1}{3} \times 4.3^2 \times 9.8\pi$ Ax. 9

$= 189.75.$

Therefore the volume of the given cone is 189.75 cu. in. Q.E.F.

11. *Find the volume of a circular cone, given the altitude* 10.5 *in. and the radius of the base* 6.2 *in.*

Solution. $v = \frac{1}{3}\pi r^2 a.$ § 612

$\therefore\ v = \frac{1}{3} \times 6.2^2 \times 10.5\pi$ Ax. 9

$= 3.5 \times 38.44\pi$

$= 422.67.$

Therefore the volume of the given cone is 422.67 cu. in. Q.E.F.

12. *Find the volume of a circular cone, given the altitude* 14.9 *in. and the radius of the base* 9.6 *in.*

Solution. $v = \frac{1}{3}\pi r^2 a.$ § 612

$\therefore\ v = \frac{1}{3} \times 9.6^2 \times 14.9\pi$ Ax. 9

$= \frac{1}{3} \times 92.16 \times 14.9\pi$

$= 30.72 \times 14.9\pi$

$= 1437.998.$

Therefore the volume of the given cone is 1437.998 cu. in. Q.E.F.

13. *How many cubic feet in a conical tent* 10 *ft. in diameter and* 7 *ft. high?*

Solution. $v = \frac{1}{3}\pi r^2 a.$ § 612

$\therefore\ v = \frac{1}{3} \times (\frac{10}{2}) \times 7\pi$ Ax. 9

$= \dfrac{175\pi}{3}$, or 183.26.

Therefore the number of cubic feet in the tent is 183.26. Q.E.F.

14. *How many cubic feet in a conical pile of earth* 15 *ft. in diameter and* 8 *ft. high?*

Solution. $v = \frac{1}{3}\pi r^2 a.$ § 612

$\therefore\ v = \frac{1}{3} \times (\frac{15}{2})^2 \times 8\pi$ Ax. 9

$= \frac{1}{3} \times \frac{225}{4} \times 8\pi$

$= 150\pi$, or 471.24.

Therefore the number of cubic feet in the pile of earth is 471.24. Q.E.F.

15. *Deduce a formula for finding the altitude of a circular cone in terms of the volume and the area of the base.*

Solution. Denote the altitude of the given circular cone by a, the volume by v, and the area of the base by b.

Then $v = \frac{1}{3} ba.$ § 611

$\therefore ba = 3v.$ Ax. 3

$\therefore a = \frac{3v}{b}$, by Ax. 4. Q. E. F.

16. *Deduce a formula for finding the area of the base of a circular cone in terms of the volume and the altitude.*

Solution. Denote the area of the base of the given circular cone by b, the volume by v, and the altitude by a.

Then $v = \frac{1}{3} ba.$ § 611

$\therefore ba = 3v.$ Ax. 3

$\therefore b = \frac{3v}{a}$, by Ax. 4. Q. E. F.

17. *Deduce a formula for finding the altitude of a circular cone in terms of the volume and the radius of the base.*

Solution. Denote the altitude of the given circular cone by a, the volume by v, and the radius of the base by r.

Then $v = \frac{1}{3} \pi r^2 a.$ § 612

$\therefore \pi r^2 a = 3v.$ Ax. 3

$\therefore a = \frac{3v}{\pi r^2}$, by Ax. 4. Q. E. F.

18. *Deduce a formula for finding the radius of the base of a circular cone in terms of the volume and the altitude.*

Solution. Denote the radius of the base of the given circular cone by r, the volume by v, and the altitude by a.

Then $v = \frac{1}{3} \pi r^2 a.$ § 612

$\therefore \pi r^2 a = 3v.$ Ax. 3

$\therefore r^2 = \frac{3v}{\pi a}.$ Ax. 4

$$\therefore r = \sqrt{\frac{3v}{\pi a}}$$

$= \frac{1}{\pi a}\sqrt{3\pi a v}$, by Ax. 5. Q. E. F.

19. *Deduce a formula for finding the volume of a cone of revolution in terms of the slant height and the radius of the base.*

Solution. Denote the volume of the given cone of revolution by v, the slant height by s, and the radius of the base by r.

Now $v = \frac{1}{3}\pi r^2 a.$ § 612

But $a^2 = s^2 - r^2,$ § 338

whence $a = \sqrt{s^2 - r^2}.$ Ax. 5

$\therefore\ v = \frac{1}{3}\pi r^2 \sqrt{s^2 - r^2}$, by Ax. 9. Q.E.F.

20. *Deduce formulas for finding the slant height and the altitude of a cone of revolution in terms of the volume and the radius of the base.*

Solution. Denote the slant height of the given cone of revolution by s, the altitude by a, the volume by v, and the radius of the base by r.

(1) $v = \frac{1}{3}\pi r^2 a.$ § 612

But $a^2 = s^2 - r^2,$ § 338

whence $a = \sqrt{s^2 - r^2}.$ Ax. 5

$\therefore\ v = \frac{1}{3}\pi r^2 \sqrt{s^2 - r^2}.$ Ax. 9

$\therefore\ \sqrt{s^2 - r^2} = \frac{3v}{\pi r^2}.$ Axs. 3, 4

$\therefore\ s^2 - r^2 = \frac{9v^2}{\pi^2 r^4}.$ Ax. 5

$\therefore\ s^2 = r^2 + \frac{9v^2}{\pi^2 r^4}$ Ax. 1

$= \frac{\pi^2 r^6 + 9v^2}{\pi^2 r^4}.$ Ax .11

$\therefore\ s = \frac{1}{\pi r^2}\sqrt{\pi^2 r^6 + 9v^2}$, by Ax. 5. Q.E.F.

(2) $v = \frac{1}{3}\pi r^2 a.$ § 612

$\therefore\ \pi r^2 a = 3v.$ Ax. 3

$\therefore\ a = \frac{3v}{\pi r^2}$, by Ax. 4. Q.E.F.

Exercise 96. Page 375

1. *Find the lateral area of a frustum of a cone, given the circumferences of the bases* 4 *in. and* 3 *in. and the slant height* 0.5 *in.*

Solution. $l = \frac{1}{2}(c + c')s.$ § 615

$\therefore\ l = \frac{1}{2}(4 + 3) \times 0.5$ Ax. 9

$= 3.5 \times 0.5$, or 1.75.

$\therefore$ the lateral area of the given frustum of a cone is 1.75 sq. in. Q.E.F.

2. *Find the lateral area of a frustum of a cone, given the circumferences of the bases* 6 *in. and* 5 *in. and the slant height* 1.4 *in.*

Solution. $l = \frac{1}{2}(c + c')s.$ § 615

$\therefore\ l = \frac{1}{2}(6 + 5) \times 1.4$ Ax. 9

$= 5.5 \times 1.4$, or 7.7.

$\therefore$ the lateral area of the given frustum of a cone is 7.7 sq. in. Q.E.F.

3. *Find the lateral area of a frustum of a cone, given the circumferences of the bases* $7\frac{1}{2}$ *in. and* $5\frac{3}{4}$ *in. and the slant height* $2\frac{1}{8}$ *in.*

Solution.

$$\begin{aligned} l &= \tfrac{1}{2}(c + c')\,s. && \text{§ 615} \\ \therefore\ l &= \tfrac{1}{2}\left(7\tfrac{1}{2} + 5\tfrac{3}{4}\right) \times 2\tfrac{1}{8} && \text{Ax. 9} \\ &= \tfrac{53}{8} \times \tfrac{17}{8} \\ &= \tfrac{901}{64},\ \text{or } 14\tfrac{5}{64}. \end{aligned}$$

$\therefore$ the lateral area of the given frustum of a cone is $14\frac{5}{64}$ sq. in. Q.E.F.

4. *Find the lateral area of a frustum of a cone, given the circumferences of the bases* 23 *in. and* 18 *in. and the slant height* 16 *in.*

Solution.

$$\begin{aligned} l &= \tfrac{1}{2}(c + c')\,s. && \text{§ 615} \\ \therefore\ l &= \tfrac{1}{2}(23 + 18) \times 16 && \text{Ax. 9} \\ &= \tfrac{1}{2} \times 41 \times 16 \\ &= 328. \end{aligned}$$

$\therefore$ the lateral area of the given frustum of a cone is 328 sq. in. Q.E.F.

5. *Find to two decimal places the volume of a frustum of a cone, given the altitude* 3 *in. and the areas of the bases* $4\frac{1}{2}$ *sq. in. and* 2 *sq. in.*

Solution.

$$\begin{aligned} v &= \tfrac{1}{3}a\left(b + b' + \sqrt{bb'}\right). && \text{§ 617} \\ \therefore\ v &= \tfrac{1}{3} \times 3\left(4\tfrac{1}{2} + 2 + \sqrt{4\tfrac{1}{2} \times 2}\right) && \text{Ax. 9} \\ &= 9.5. \end{aligned}$$

$\therefore$ the volume of the given frustum of a cone is 9.5 cu. in. Q.E.F.

6. *Find to two decimal places the volume of a frustum of a cone, given the altitude* 4 *in. and the areas of the bases* $8\frac{1}{3}$ *sq. in. and* 3 *sq. in.*

Solution.

$$\begin{aligned} v &= \tfrac{1}{3}a\left(b + b' + \sqrt{bb'}\right). && \text{§ 617} \\ \therefore\ v &= \tfrac{1}{3} \times 4\left(8\tfrac{1}{3} + 3 + \sqrt{8\tfrac{1}{3} \times 3}\right) && \text{Ax. 9} \\ &= \tfrac{4}{3} \times 16\tfrac{1}{3} \\ &= \tfrac{4}{3} \times \tfrac{49}{3} \\ &= \tfrac{196}{9},\ \text{or } 21.78. \end{aligned}$$

$\therefore$ the volume of the given frustum of a cone is 21.78 cu. in. Q.E.F.

7. *Find to two decimal places the volume of a frustum of a cone, given the altitude* $5\frac{1}{2}$ *in. and the areas of the bases* 16 *sq. in. and* 9 *sq. in.*

Solution.

$$\begin{aligned} v &= \tfrac{1}{3}a\left(b + b' + \sqrt{bb'}\right). && \text{§ 617} \\ \therefore\ v &= \tfrac{1}{3} \times 5\tfrac{1}{2}\left(16 + 9 + \sqrt{16 \times 9}\right) && \text{Ax. 9} \\ &= \tfrac{11}{6} \times 37 \\ &= 67.83. \end{aligned}$$

$\therefore$ the volume of the given frustum of a cone is 67.83 cu. in. Q.E.F.

8. *Find to two decimal places the volume of a frustum of a cone, given the altitude* 6 *in. and the areas of the bases* 17 *sq. in. and* 11 *sq. in.*

Solution. $v = \frac{1}{3} a \left(b + b' + \sqrt{bb'}\right).$ § 617

$\therefore v = \frac{1}{3} \times 6 \left(17 + 11 + \sqrt{17 \times 11}\right)$ Ax. 9

$$= 2\left(28 + \sqrt{187}\right)$$
$$= 56 + 2\sqrt{187}$$
$$= 56 + 27.35$$
$$= 83.35.$$

$\therefore$ the volume of the given frustum of a cone is 83.35 cu. in. Q.E.F.

9. *Find to two decimal places the volume of a frustum of a cone of revolution, given the altitude* 4 *in. and the radii of the bases* 3 *in. and* 2 *in.*

Solution. $v = \frac{1}{3} \pi a (r^2 + r'^2 + rr').$ § 618

$\therefore v = \frac{1}{3} \pi \times 4 (3^2 + 2^2 + 3 \times 2)$ Ax. 9

$$= \tfrac{1}{3} \pi \times 4 (9 + 4 + 6)$$
$$= \tfrac{1}{3} \pi \times 76$$
$$= \frac{76 \pi}{3}, \text{ or } 79.59.$$

$\therefore$ the volume of the given frustum of a cone of revolution is 79.59 cu. in. Q.E.F.

10. *Find to two decimal places the volume of a frustum of a cone of revolution, given the altitude* 5 *in. and the radii of the bases* $3\frac{1}{2}$ *in. and* $2\frac{1}{4}$ *in.*

Solution. $v = \frac{1}{3} \pi a (r^2 + r'^2 + rr').$ § 618

$\therefore v = \frac{1}{3} \pi \times 5 \left[(3\frac{1}{2})^2 + (2\frac{1}{4})^2 + 3\frac{1}{2} \times 2\frac{1}{4}\right]$ Ax. 9

$$= \tfrac{5}{3} \pi \left(\tfrac{49}{4} + \tfrac{81}{16} + \tfrac{63}{8}\right)$$
$$= \tfrac{5}{3} \pi \times \tfrac{403}{16}$$
$$= \frac{2015 \pi}{48}, \text{ or } 131.88.$$

$\therefore$ the volume of the given frustum of a cone of revolution is 131.88 cu. in. Q.E.F.

11. *Find to two decimal places the volume of a frustum of a cone of revolution, given the altitude* 6 *in. and the radii of the bases* 3.7 *in. and* 3.1 *in.*

Solution. $v = \frac{1}{3} \pi a (r^2 + r'^2 + rr').$ § 618

$\therefore v = \frac{1}{3} \pi \times 6 (3.7^2 + 3.1^2 + 3.7 \times 3.1)$ Ax. 9

$$= 2 \pi (13.69 + 9.61 + 11.47)$$
$$= 2 \pi \times 34.77$$
$$= 69.54 \pi, \text{ or } 218.47.$$

$\therefore$ the volume of the given frustum of a cone of revolution is 218.47 cu. in. Q.E.F.

12. *Find to two decimal places the volume of a frustum of a cone of revolution, given the altitude $7\frac{1}{2}$ in. and the radii of the bases $4\frac{3}{4}$ in. and $3\frac{1}{8}$ in.*

Solution.

$$v = \tfrac{1}{3}\pi a(r^2 + r'^2 + rr'). \qquad \S\ 618$$

$$\therefore\ v = \tfrac{1}{3}\pi \times 7\tfrac{1}{2}\left[(4\tfrac{3}{4})^2 + (3\tfrac{1}{8})^2 + 4\tfrac{3}{4} \times 3\tfrac{1}{8}\right] \qquad \text{Ax. } 9$$

$$= \tfrac{5}{2}\pi\left(\tfrac{361}{16} + \tfrac{625}{64} + \tfrac{475}{32}\right)$$

$$= \tfrac{5}{2}\pi \times \tfrac{3019}{64}$$

$$= \frac{15095\pi}{128}, \text{ or } 370.49.$$

∴ the volume of the given frustum of a cone of revolution is 370.49 cu. in. Q.E.F.

13. *Deduce a formula for finding the altitude of a frustum of a circular cone in terms of the volume and the areas of the bases.*

Solution. Denote the altitude of the given frustum of a circular cone by a, the volume by v, and the areas of the bases by b and b' respectively.

Then
$$v = \tfrac{1}{3}a\left(b + b' + \sqrt{bb'}\right). \qquad \S\ 617$$

$$\therefore\ a\left(b + b' + \sqrt{bb'}\right) = 3v. \qquad \text{Ax. } 3$$

$$\therefore\ a = \frac{3v}{b + b' + \sqrt{bb'}}, \text{ by Ax. 4.} \qquad \text{Q.E.F.}$$

14. *Deduce a formula for finding the altitude of a frustum of a cone of revolution in terms of the volume and the radii of the bases.*

Solution. Denote the altitude of the given frustum of a cone of revolution by a, the volume by v, and the radii of the bases by r and r' respectively.

Then
$$v = \tfrac{1}{3}\pi a(r^2 + r'^2 + rr'). \qquad \S\ 618$$

$$\therefore\ \pi a(r^2 + r'^2 + rr') = 3v. \qquad \text{Ax. } 3$$

$$\therefore\ a = \frac{3v}{\pi(r^2 + r'^2 + rr')}, \text{ by Ax. 4.} \qquad \text{Q.E.F.}$$

Exercise 97. Page 376

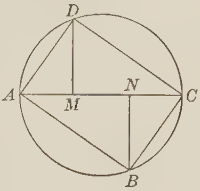
1. *There is a rule for calculating the strongest beam that can be cut from a cylindric log, as follows:*

Erect perpendiculars MD and NB on opposite sides of a diameter AC, at the trisection points M and N, meeting the circle in D and B. Then ABCD is a section of the beam.

Calculate the dimensions, the log being 16 *in. in diameter.*

Given the ⊙ *ADC*, a right section of the given cylindric log; *MD* and *NB* ⊥s on opposite sides to *AC*, the diameter of the ⊙ *ADC*, at its points of trisection M and N, meeting the ⊙ in D and B; and the lines DA, DC, BA, and BC. The length of $AC = 16$ in.

Required to find the dimensions of $ABCD$.



Solution. Since MD and NB are each $\perp$ to AC, Given

$\therefore \overline{DA}^2 = AC \times AM,$

and $\overline{BC}^2 = AC \times CN.$ § 298

But $AM = CN.$ Given

$\therefore AC \times AM = AC \times CN.$ Ax. 3

$\therefore \overline{DA}^2 = \overline{BC}^2.$ Ax. 8

Whence $DA = BC.$ Ax. 5

Similarly, $DC = BA.$

$\therefore ABCD$ is a ▱. § 129

Now ∠s ADC and CBA are rt. ∠s. § 215

Since ∠s BAD and DCB are equal, § 124

$\therefore$ ∠s BAD and DCB are rt. ∠s. § 144

$\therefore ABCD$ is a ▭. § 119

Since $\overline{DA}^2 = AC \times AM,$ Proved

and $AM = \frac{1}{3} AC,$ Given

$\therefore \overline{DA}^2 = \frac{1}{3} \overline{AC}^2.$ Ax. 9

$\therefore \overline{DA}^2 = \frac{1}{3}(16)^2$ Ax. 9

$= \frac{256}{3}.$

$\therefore DA = \frac{16}{3}\sqrt{3}$, or 9.238. Ax. 5

Also since $\overline{DC}^2 = AC \times CM,$ Proved

and $CM = \frac{2}{3} AC,$ Given

$\therefore \overline{DC}^2 = \frac{2}{3} \overline{AC}^2.$ Ax. 9

$\therefore \overline{DC}^2 = \frac{2}{3}(16)^2 = \frac{512}{3}.$ Ax. 9

$\therefore DC = \frac{16}{3}\sqrt{6}$, or 13.064. Ax. 5

Therefore the dimensions of $ABCD$ are 9.238 in. and 13.064 in. Q. E. F.

2. *A cylindric funnel for a steamboat is 4 ft. 3 in. in diameter. It is built up of four plates in girth, and the lap of each joint is $1\frac{3}{8}$ in. Find one dimension of each plate.*

Solution. 4 ft. 3 in. = 51 in.

The circumference of the funnel $= \pi d$ § 384

$= 51\,\pi$ in. Ax. 9

Therefore, making no allowance for a joint, we have for a dimension of one plate $\frac{51\,\pi}{4}$ in.

Therefore the required dimension of one of the plates

$= \frac{51}{4}\pi$ in. $+ 1\frac{3}{8}$ in. Ax. 11

$= 41.4304$ in.

$= 3$ ft. 5.4304 in. Q. E. F.

3. *A tubular boiler has 124 tubes, each $3\frac{7}{8}$ in. in diameter and 18 ft. long. Required the total tube surface. Answer to the nearest square foot.*

Solution. The surface of one of the given tubes is the lateral surface of a cylinder having the same dimensions.

Now $3\frac{7}{8}$ in. $= \frac{31}{8}$ in. $= \frac{31}{96}$ ft.

Since $l = 2\pi ra$, where $2r =$ the diameter, § 588

$\therefore\ l = \frac{31}{96} \times 18\pi$ Ax. 9

$$= \frac{93\pi}{16}.$$

That is, the lateral surface of one tube is $\frac{93\pi}{16}$ sq. ft.

Therefore the total tube surface of 124 such tubes

$= 124 \times \frac{93\pi}{16}$ sq. ft., or 2264 sq. ft., by Ax. 11. Q. E. F.

4. *A room in a factory is heated by steam pipes. There are 235 ft. of 2-inch pipe and 26 ft. 3 in. of 3-inch pipe, besides 2 ft. 8 in. of $4\frac{1}{4}$-inch feed pipe. Required the total heating surface. Answer to the nearest square foot.*

Solution. 2 in. $= \frac{1}{6}$ ft.,
3 in. $= \frac{1}{4}$ ft.,
26 ft. 3 in. $= 26\frac{1}{4}$ ft.,
2 ft. 8 in. $= 2\frac{2}{3}$ ft.,
and $4\frac{1}{4}$ in. $= \frac{17}{48}$ ft.

Since $l = 2\pi ra$, where $2r =$ the diameter, § 588

$\therefore\ l = \frac{1}{6} \times 235\pi$ Ax. 9

$$= \frac{235\pi}{6}.$$

That is, the heating surface of 235 ft. of 2-inch pipe is $\frac{235\pi}{6}$ sq. ft.

For like reason the heating surface of 26 ft. 3 in. of 3-inch pipe is $\frac{1}{4} \times 26\frac{1}{4}\pi$ sq. ft., or $\frac{105\pi}{16}$ sq. ft.; and the heating surface of 2 ft. 8 in. of $4\frac{1}{4}$-inch pipe is $\frac{17}{48} \times 2\frac{2}{3}\pi$ sq. ft., or $\frac{17\pi}{18}$ sq. ft.

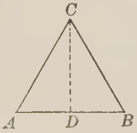
Therefore the total heating surface

$= \left(\frac{235\pi}{6} + \frac{105\pi}{16} + \frac{17\pi}{18}\right)$ sq. ft. Ax. 11

$= \pi\left(\frac{235}{6} + \frac{105}{16} + \frac{17}{18}\right)$ sq. ft.

$= \frac{6721\pi}{144}$ sq. ft., or 147 sq. ft. Q. E. F.

5. *A triangular plate of wrought iron $\frac{5}{8}$ in. thick is 2 ft. 7 in. on each side. If the weight of a plate 1 ft. square and $\frac{1}{8}$ in. thick is 5 lb., find to the nearest pound the weight of the given triangular plate.*



Solution. Let the equilateral $\triangle ABC$ represent the face of the given triangular plate of wrought iron whose side $= 2$ ft. 7 in. $= 2\frac{7}{12}$ ft.

Draw $CD \perp$ to AB.

Then CD bisects AB. § 85

That is, $AD = \frac{1}{2}AB = \frac{1}{2}AC.$ Ax. 9

In the rt. $\triangle ADC$, $\overline{CD}^2 = \overline{AC}^2 - \overline{AD}^2.$ § 338

$\therefore \overline{CD}^2 = \overline{AC}^2 - \frac{1}{4}\overline{AC}^2$ Ax. 9

$= \frac{3}{4}\overline{AC}^2.$

$\therefore CD = \frac{1}{2}\sqrt{3}\,AC.$ Ax. 5

Now the area of $\triangle ABC = \frac{1}{2}AB \times CD$ § 325

$= \frac{1}{2}AC \times CD.$ Ax. 9

$\therefore$ the area of $\triangle ABC = (\frac{1}{2})^2\sqrt{3}\,\overline{AC}^2$ Ax. 9

$= \frac{1}{4}\sqrt{3}\,(2\frac{7}{12})^2$ sq. ft. Ax. 9

$= \frac{1}{4}\sqrt{3}\,(\frac{31}{12})^2$ sq. ft.

$$= \frac{961\sqrt{3}}{576} \text{ sq. ft.}$$

Now $v = ba$, where $v =$ the volume of the given triangular plate, $b =$ the area of $\triangle ABC$, its base; and $a = \frac{5}{8}$ in. $= \frac{5}{96}$ ft., its altitude. § 538

$$\therefore v = \frac{961\sqrt{3}}{576} \times \frac{5}{96} \text{ cu. ft.} \qquad \text{Ax. 9}$$

$$= \frac{4805\sqrt{3}}{55296} \text{ cu. ft., or } 0.1505 \text{ cu. ft.}$$

Furthermore, the volume of a square plate 1 ft. on a side and $\frac{1}{8}$ in., or $\frac{1}{96}$ ft., thick is $\frac{1}{96}$ cu. ft. § 534

But this square plate weighs 5 lb.

Therefore $\frac{1}{96}$ cu. ft. of the given wrought iron weighs 5 lb.

Whence 1 cu. ft. of it weighs 96 × 5 lb., or 480 lb.

Therefore the weight of the given triangular plate is 0.1505 × 480 lb., or 72 lb. Q.E.F.

6. *The water surface of an upright cylindric boiler is 2 ft. 8 in. below the top of the boiler, and is 12.57 sq. ft. in area. What is the volume of the steam space?*

Solution. The volume of the steam space is the volume of a right circular cylinder whose base is 12.57 sq. ft. and whose altitude is 2 ft. 8 in. $= 2\frac{2}{3}$ ft.

Since $v = ba,$ § 589
$\therefore\ v = 12.57 \times 2\frac{2}{3}$ Ax. 9
$= 33.52.$

That is, the volume of the given steam space is 33.52 cu. ft. Q.E.F.

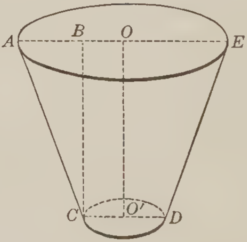
7. *A cylinder* 16 *in. in diameter is required to hold* 50 *gal. of water. What must be its height, to the nearest tenth of an inch, allowing* 231 *cu. in. to the gallon?*

Solution. 50 gal. = 50 × 231 cu. in., or 11,550 cu. in.
Now $v = \pi r^2 a.$ § 590
Here $v = 11{,}550$ cu. in.,
and $r = \frac{1}{2}$ of 16 in., or 8 in.
$\therefore\ 11{,}550 = 8^2\,\pi a$ Ax. 9
$= 64\,\pi a.$
Whence $a = \dfrac{11550}{64\,\pi}$ Ax. 4
$= 57.4.$

Therefore the height of the cylinder required is 57.4 in., or 4 ft. 9.4 in. Q.E.F.

8. *How many square feet of tin are required to make a funnel, if the diameters of the top and bottom are* 30 *in. and* 15 *in. respectively, and the height is* 25 *in.?*

Given AD, a frustum of a right circular cone representing a conical funnel, in which AE and CD are the diameters of the upper and lower bases respectively, AE being 30 in. and CD 15 in., and the height being 25 in.



Required to find the number of square feet of tin necessary to make the given funnel.

Solution. Draw OO' joining the centers O and O' of the bases.
Through OO' and AO pass a plane cutting the base CD in CO'. § 425
Then draw CB in the plane $AO' \perp$ to AO.
The lines BO and CO' are $\parallel$. § 453
Also the lines BC and OO' are $\parallel$, § 95
since the given frustum is a frustum of a right circular cone.
$\therefore\ BCO'O$ is a ▱. § 118
$\therefore\ BC = OO'$, and $BO = CO'$. § 125

$\therefore BC = 25$ in., Ax. 8

and $AB = AO - BO$

$= AO - CO'$ Ax. 9

$= \frac{30}{2}$ in. $- \frac{15}{2}$ in., or $\frac{15}{2}$ in. Ax. 9

Now $\triangle ABC$ is right angled at B. Const.

In the rt. $\triangle ABC$, $\overline{AC}^2 = \overline{AB}^2 + \overline{BC}^2$. § 337

$\therefore \overline{AC}^2 = (\frac{15}{2})^2 + 25^2$ Ax. 9

$= \frac{2725}{4}$.

$\therefore AC = \frac{5}{2}\sqrt{109}$. Ax. 5

But $l = \frac{1}{2}(c + c')s$. § 615

Since $c = 30\pi$, and $c' = 15\pi$, § 384

$\therefore l = \frac{1}{2}(30\pi + 15\pi) \times \frac{5}{2}\sqrt{109}$ Ax. 9

$= \frac{5}{4} \times 45\pi \times \sqrt{109}$

$= \dfrac{225\pi\sqrt{109}}{4}$.

Therefore the number of square inches of tin required to make the given funnel $= \dfrac{225\pi\sqrt{109}}{4}$.

Therefore the number of square feet of tin required to make the funnel

$= \dfrac{225\pi\sqrt{109}}{4 \times 144}$

$= \dfrac{225\pi\sqrt{109}}{576}$, or 12.81. Q.E.F.

9. *Find to two decimal places the weight of a steel plate* 4 *ft. by* 3 *ft.* 2 *in. by* $1\frac{3}{8}$ *in., allowing* 490 *lb. per cubic foot.*

Solution. 3 ft. 2 in. $= 3\frac{1}{6}$ ft.,

and $1\frac{3}{8}$ in. $= \frac{11}{96}$ ft.

Since the volume of any rectangular parallelepiped is equal to the product of its three dimensions, § 534

$\therefore$ the volume of the given steel plate

$= 4 \times 3\frac{1}{6} \times \frac{11}{96}$ cu. ft. Ax. 9

$= 4 \times \frac{19}{6} \times \frac{11}{96}$ cu. ft.

$= \frac{209}{144}$ cu. ft.

Therefore the weight of the steel plate

$= \frac{209}{144} \times 490$ lb., or 711.18 lb. Q.E.F.

10. *A steel plate for a steamship is* 5 *ft. long,* 3 *ft.* 6 *in. wide, and* $\frac{1}{2}$ *in. thick. A porthole* 10 *in. in diameter is cut through the plate. Required the weight of the finished plate, allowing* 0.29 *lb. per cubic inch. Answer to two decimal places.*

Solution. 5 ft. = 60 in.,
and 3 ft. 6 in. = 42 in.

Since the volume of any rectangular parallelepiped is equal to the product of its three dimensions, § 534

∴ the volume of the entire steel plate including the circular plate cut out for the porthole $= 60 \times 42 \times \frac{1}{2}$ cu. in. **Ax. 9**
$= 1260$ cu. in.

But since the volume of any right circular cylinder is $\pi r^2 a$, where πr^2 is the area of the base and a the altitude, § 590

∴ the volume of the circular plate cut from the given plate
$= 5^2 \times \frac{1}{2}\pi$ cu. in. **Ax. 9**
$= 39.27$ cu. in.

∴ the number of cubic inches in the finished steel plate
$= 1260 - 39.27$, or 1220.73.

Since the weight of the steel is 0.29 lb. per cubic inch, Given
∴ the finished plate weighs 1220.73×0.29 lb., or 354.01 lb. Q.E.F.

11. *A cast-iron base for a column is in the form of a frustum of a pyramid, the lower base being a square 2 ft. on a side, and the upper base having a fourth of the area of the lower base. The altitude of the frustum is 9 in. Required the weight to the nearest pound, allowing* 460 *lb. per cubic foot.*

Solution. The area of the lower base of the given frustum of a pyramid is 2^2 sq. ft., or 4 sq. ft. § 320

Then the area of the upper base is $\frac{1}{4}$ of 4 sq. ft., or 1 sq. ft.

Now $v = \frac{1}{3}a\left(b + b' + \sqrt{bb'}\right)$. § 565

$\therefore\ v = \frac{1}{3} \times \frac{9}{12}\left(4 + 1 + \sqrt{4}\right)$ cu. ft. **Ax. 9**
$= \frac{7}{4}$ cu. ft., or $1\frac{3}{4}$ cu. ft.

That is, the volume of the given cast iron base is $1\frac{3}{4}$ cu. ft.

But the weight of 1 cu. ft. of the cast iron used in the base is 460 lb.

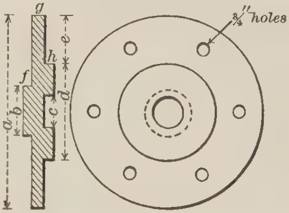
Therefore the weight of the base
$= 1\frac{3}{4} \times 460$ lb., or 805 lb. Q.E.F.

12. *A cylinder head for a steam engine has the shape shown in the figure, where the dimensions in inches are:* $a = 12$, $b = 3$, $c = 2$, $d = 6$, $e = 3$, $f = \frac{1}{2}$, $g = \frac{7}{8}$, and $h = \frac{3}{4}$. *There are six* $\frac{3}{4}$*-inch holes for bolts. Compute the weight of the plate, allowing* 41 *lb. for the weight of a steel plate* 1 *ft. square and* 1 *in. thick. Answer to the nearest tenth of a pound.*

Solution. It is clear that the volume of the given cylinder head with no holes in it equals the total volume of the first three right circular cylinders given below, *minus* the volume of the fourth which follows:

(1) A right circular cylinder with diameter $a = 12$ in. and height $g = \frac{7}{8}$ in.

(2) A right circular cylinder with diameter $b = 3$ in. and height $f = \frac{1}{2}$ in.



(3) A right circular cylinder with diameter $d = 6$ in. and height $h = \frac{3}{4}$ in.

(4) A right circular cylinder with diameter $c = 2$ in. and height $h = \frac{3}{4}$ in.

Now $v = \pi r^2 a$. § 590

Therefore the volume of a right circular cylinder 12 in. in diameter with height $\frac{7}{8}$ in. $= (\frac{12}{2})^2 \times \frac{7}{8}\pi$ cu. in. Ax. 9
$= \frac{63}{2}\pi$ cu. in.;

the volume of a right circular cylinder 3 in. in diameter with height $\frac{1}{2}$ in. $= (\frac{3}{2})^2 \times \frac{1}{2}\pi$ cu. in. Ax. 9
$= \frac{9}{8}\pi$ cu. in.;

the volume of a right circular cylinder 6 in. in diameter with height $\frac{3}{4}$ in. $= (\frac{6}{2})^2 \times \frac{3}{4}\pi$ cu. in. Ax. 9
$= \frac{27}{4}\pi$ cu. in.;

and the volume of a right circular cylinder 2 in. in diameter with height $\frac{3}{4}$ in. $= (\frac{2}{2})^2 \times \frac{3}{4}\pi$ cu. in. Ax. 9
$= \frac{3}{4}\pi$ cu. in.

Therefore the total volume of the given cylinder head with no holes in it $= (\frac{63}{2}\pi + \frac{9}{8}\pi + \frac{27}{4}\pi - \frac{3}{4}\pi)$ cu. in. Ax. 11
$= \pi(\frac{63}{2} + \frac{9}{8} + \frac{27}{4} - \frac{3}{4})$ cu. in.
$= \frac{309}{8}\pi$ cu. in., or $38\frac{5}{8}\pi$ cu. in.

The six $\frac{3}{4}$-inch holes each have a depth of $g = \frac{7}{8}$ in.

Then the volume of a right circular cylinder with dimensions equal to those of one of the holes $= (\frac{1}{2} \times \frac{3}{4})^2 \times \frac{7}{8}\pi$ cu. in. Ax. 9
$= \frac{63}{512}\pi$ cu. in.

Therefore the total volume of six such cylinders, each with dimensions equal to those of one of the holes,
$= 6 \times \frac{63}{512}\pi$ cu. in. Ax. 11
$= \frac{189}{256}\pi$ cu. in.

Therefore the volume of the given cylinder head with the six $\frac{3}{4}$-inch holes in it $= (38\frac{5}{8}\pi - \frac{189}{256}\pi)$ cu. in.
$= 37\frac{227}{256}\pi$ cu. in., or 119.34 cu. in.

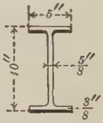
The volume of a steel plate 1 ft., or 12 in., square and 1 in. thick is 144 cu. in. § 534

Since this plate weighs 41 lb.,

$\therefore$ 1 cu. in. of the steel weighs $\frac{41}{144}$ lb.

Therefore the given cylinder head with the six $\frac{3}{4}$-inch holes in it weighs $119.34 \times \frac{41}{144}$ lb., or 34 lb. Q.E.F.

13. *A steel beam* 10 *in. by* 5 *in., in the form here shown, is* 18 *ft. long. The thickness of the beam is* $\frac{5}{8}$ *in. and the average thickness of the flanges is* $\frac{3}{8}$ *in. Find the weight of the beam to the nearest pound, allowing* 0.29 *lb. per cubic inch.*



Solution. From the adjoining figure it is obvious that the volume of the given steel beam equals the volume of a solid steel beam of the same length, 10 in. by 5 in. in right section, *minus* the total volume of two equal solid steel beams of the same length as the given beam, each of which is $(10 - 2 \times \frac{3}{8})$ in., or $9\frac{1}{4}$ in., by $\frac{1}{2}(5 - \frac{5}{8})$ in., or $\frac{35}{16}$ in., in right section.

Now the volume of a solid steel beam 18 ft. long and 10 in. by 5 in. in right section $= 18 \times 12 \times 10 \times 5$ cu. in., or 10,800 cu. in. § 534

And the total volume of two equal solid steel beams, each 18 ft. long and $9\frac{1}{4}$ in. by $\frac{35}{16}$ in. in right section, equals the volume of a single solid steel beam 18 ft. long and $9\frac{1}{4}$ in. by $2 \times \frac{35}{16}$ in. $= \frac{35}{8}$ in. in right section.

The volume of a solid steel beam 18 ft. long and $9\frac{1}{4}$ in. by $\frac{35}{8}$ in. in right section $= 18 \times 12 \times 9\frac{1}{4} \times \frac{35}{8}$ cu. in. § 534

$= 18 \times 12 \times \frac{37}{4} \times \frac{35}{8}$ cu. in.

$= 8741\frac{1}{4}$ cu. in.

Therefore the volume of the given steel beam

$= 10{,}800$ cu. in. $- 8741\frac{1}{4}$ cu. in.

$= 2058\frac{3}{4}$ cu. in.

But the weight per cubic inch of the given steel beam is 0.29 lb.

Therefore the weight of the given steel beam

$= 2058\frac{3}{4} \times 0.29$ lb., or 597 lb. Q.E.F.

14. *A hollow steel shaft* 12 *ft. long is* 18 *in. in exterior diameter and* 8 *in. in interior diameter. Find the weight to the nearest pound, allowing* 0.29 *lb. per cubic inch.*

Solution. The volume of any right circular cylinder is expressed by the formula $v = \pi r^2 a$. § 590

Therefore the volume of a right circular cylinder 12 ft. long and 18 in. in diameter $= (\frac{18}{2})^2 \times 12 \times 12\,\pi$ cu. in. Ax. 9

$= 11{,}664\,\pi$ cu. in.

And the volume of a right circular cylinder 12 ft. long and 8 in. in diameter $= (\frac{8}{2})^2 \times 12 \times 12\,\pi$ cu. in. Ax. 9

$= 2304\,\pi$ cu. in.

Since the volume of the given hollow steel shaft is the difference of the volumes of two right circular cylinders each 12 ft. long, the one being 18 in. in diameter, the other 8 in. in diameter,

∴ the volume of the given steel shaft

$$= 11{,}664\,\pi \text{ cu. in.} - 2304\,\pi \text{ cu. in.}$$
$$= 9360\,\pi \text{ cu. in., or } 29{,}405.376 \text{ cu. in.}$$

But the weight of the steel is 0.29 lb. per cubic inch.

Therefore the weight of the given steel shaft

$$= 29{,}405.376 \times 0.29 \text{ lb., or } 8528 \text{ lb.}$$ Q.E.F.

15. *Find the expense, at 70 cents a square foot, of polishing the curved surface of a marble column in the shape of the frustum of a right circular cone whose slant height is 12 ft. and the radii of whose bases are 3 ft. 6 in. and 2 ft. 4 in. respectively.*

Solution. 3 ft. 6 in. $= 3\frac{1}{2}$ ft.,

and 2 ft. 4 in. $= 2\frac{1}{3}$ ft.

Since the lateral surface of any frustum of a right circular cone is expressed by the formula $l = \frac{1}{2}(c + c')s$, § 615

∴ the lateral surface of the given marble column in the shape of a frustum of a right circular cone

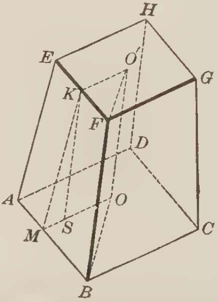
$$= \tfrac{1}{2}(2 \times 3\tfrac{1}{2}\,\pi + 2 \times 2\tfrac{1}{3}\,\pi) \times 12 \text{ sq. ft.} \quad \text{Ax. 9}$$
$$= 12\,(3\tfrac{1}{2}\,\pi + 2\tfrac{1}{3}\,\pi) \text{ sq. ft.}$$
$$= 70\,\pi \text{ sq. ft., or } 219.912 \text{ sq. ft.}$$

Therefore the expense of polishing the given marble column at 70 cents a square foot $= 219.912 \times \$0.70$

$$= \$153.94.$$ Q.E.F.

Exercise 98. Page 378

1. *The slant height of the frustum of a regular pyramid is 25 ft., and the sides of its square bases are 54 ft. and 24 ft. respectively. Find the volume.*



Given, in the frustum *ABCD-EFGH* of a regular pyramid, the sides of the square bases *ABCD* and *EFGH* equal to 54 ft. and 24 ft. respectively, and the slant height equal to 25 ft.

Required to find the volume of the frustum *ABCD-EFGH*.

Solution. Let OO' be a segment of the axis of the regular pyramid of which *ABCD-EFGH* is a frustum.

Draw $O'K \perp$ to EF.

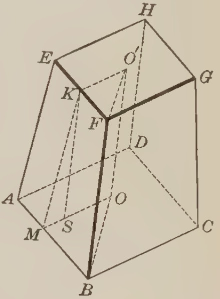
Through OO' and $O'K$ pass a plane (§ 425) cutting the base *ABCD* in OM, and the lateral face BE in KM. § 429

Then draw $KS \perp$ to OM.

The points O and O' are the centers of the bases $ABCD$ and $EFGH$ respectively. § 546

Then OO' is $\perp$ to each of the bases $ABCD$ and $EFGH$.

$\therefore$ the plane MO' is $\perp$ to the base $EFGH$. § 477



Now $O'K$ is $\perp$ to EF. Const.

$\therefore$ EF is $\perp$ to the plane MO'. § 432

$\therefore$ EF is $\perp$ to KM. § 430

Then KM is the slant height of the given frustum $ABCD\text{-}EFGH$. § 552

Since $O'K$ is $\parallel$ to OS, § 453

and OO' is $\parallel$ to KS, § 95

$\therefore$ $OSKO'$ is a ▱. § 118

$\therefore$ $OS = O'K$. § 125

But $SM = OM - OS$.

$\therefore$ $SM = OM - O'K$. Ax. 9

Now AB is $\parallel$ to EF. § 551

Since EF is $\perp$ to the plane MO', Proved

$\therefore$ AB is $\perp$ to the plane MO'. § 445

$\therefore$ AB is $\perp$ to OM. § 430

Draw OB and $O'F$.

Then OB and $O'F$ bisect the $\angle$s ABC and EFG respectively. § 363

$\therefore$ $\angle$s ABO and EFO' each equal 45°.

Then $\angle$s MOB and $KO'F$ each equal 45°. § 107

$\therefore$ $OM = MB$, and $O'K = KF$. § 76

But since OM is $\perp$ to AB, Proved

and $O'K$ is $\perp$ to EF, Const.

$\therefore$ $MB = \frac{1}{2}AB$, and $KF = \frac{1}{2}EF$. § 85

$\therefore$ $OM = \frac{1}{2}AB$, and $O'K = \frac{1}{2}EF$. Ax. 8

$\therefore$ $OM = \frac{1}{2} \times 54$ ft., or 27 ft.; and $O'K = \frac{1}{2} \times 24$ ft., or 12 ft. Ax. 9

Since $SM = OM - O'K$,

$\therefore$ $SM = 27$ ft. $- 12$ ft., or 15 ft. Ax. 9

In the rt. $\triangle KSM$, $\overline{KS}^2 = \overline{KM}^2 - \overline{SM}^2$. § 338

$\therefore$ $\overline{KS}^2 = 25^2 - 15^2$ Ax. 9

$= 625 - 225$, or 400.

$\therefore$ $KS = 20$. Ax. 5

The volume of any frustum of a pyramid is expressed by the formula $v = \frac{1}{3}a\left(b + b' + \sqrt{bb'}\right)$. § 565

$\therefore$ $v = \frac{1}{3} \times 20\left(54^2 + 24^2 + \sqrt{54^2 \times 24^2}\right)$ Ax. 9

$= \frac{20}{3} \times 4788$

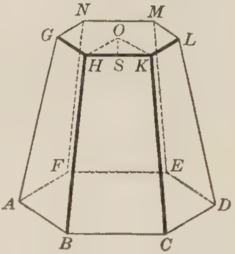
$= 31{,}920$.

Therefore the required volume is 31,920 cu. ft. Q. E. F.

2. *If the bases of the frustum of a pyramid are regular hexagons whose sides are* 1 *ft. and* 2 *ft. respectively, and the volume of the frustum is* 12 *cu. ft., find the altitude.*

Given, in the frustum *ABCDEF-GHKLMN*, the bases *AD* and *GL* regular hexagons whose sides are 2 ft. and 1 ft. respectively, and the volume equal to 12 cu. ft.

Required to find the altitude of the given frustum *ABCDEF-GHKLMN*.



Solution. Let O be the center of the base GL.

Draw OH and OK, and $OS \perp$ to HK.

$\measuredangle$s OHK and OKH each equal 60°. § 363

Then $\angle HOK = 60°$. § 107

Since OS is $\perp$ to HK, Const.

$\therefore HS = SK$. § 85

$\therefore HS = \frac{1}{2} HK$.

Then $HS = \frac{1}{2} \times 1$ ft., or $\frac{1}{2}$ ft. Ax. 9

$OH = HK = 1$ ft. § 77

Now $\overline{OS}^2 = \overline{OH}^2 - \overline{HS}^2$. § 338

$\therefore \overline{OS}^2 = 1^2 - (\frac{1}{2})^2$ Ax. 9

$= \frac{3}{4}$.

$\therefore OS = \frac{1}{2}\sqrt{3}$. Ax. 5

But the area of the base $GL = \frac{1}{2} OS \times 6\,HK$. § 386

$\therefore$ the area of $GL = \frac{1}{2} \times \frac{1}{2}\sqrt{3} \times 6$ Ax. 9

$= \frac{3}{2}\sqrt{3}$.

The regular polygons GL and AD are similar. § 373

$\therefore$ polygon GL : polygon $AD = \overline{HK}^2 : \overline{BC}^2$. § 334

$\therefore \frac{3}{2}\sqrt{3} : AD = 1^2 : 2^2$. Ax. 9

Whence the area of $AD = 6\sqrt{3}$. § 261

Since the volume of a frustum of any pyramid is expressed by the formula $v = \frac{1}{3} a\left(b + b' + \sqrt{bb'}\right)$, § 565

$\therefore 12 = \frac{1}{3} a\left(\frac{3}{2}\sqrt{3} + 6\sqrt{3} + \sqrt{\frac{3}{2} \times 6\left(\sqrt{3}\right)^2}\right)$ Ax. 9

$= \frac{1}{3} a\left(\frac{15}{2}\sqrt{3} + 3\sqrt{3}\right)$

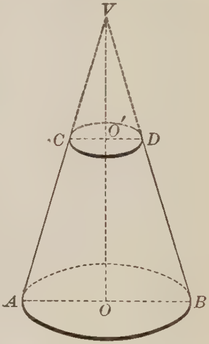
$= \frac{7}{2}\sqrt{3} \cdot a$.

$\therefore a = \dfrac{12}{\frac{7}{2}\sqrt{3}}$ Ax. 4

$= \frac{8}{7}\sqrt{3}$, or $1\frac{1}{7}\sqrt{3}$.

Therefore the altitude of the given frustum *ABCDEF-GHKLMN* is $1\frac{1}{7}\sqrt{3}$ ft. Q.E.F.

3. *From a right circular cone whose slant height is 30 ft., and the circumference of whose base is 10 ft., there is cut off by a plane parallel to the base a cone whose slant height is 6 ft. Find the lateral area and the volume of the frustum.*



Given the right circular cone $V\text{-}AB$, the circumference of whose base is 10 ft. and whose slant height is 30 ft., cut by the plane $CD \parallel$ to the base AB, so that VC is 6 ft.

Required to find the lateral area and the volume of the frustum $AB\text{-}CD$.

Solution. The section CD is a $\odot$. § 601

Let VO be the axis of the right circular cone $V\text{-}AB$. § 597

Then VO is $\perp$ to the plane of the $\odot\, CD$ at its center O'. § 456

$\therefore VO$ is $\perp$ to AO and CO'. § 430

Now the rt. $\triangle$ VOA and $VO'C$ are similar. § 287

$\therefore VA : VC = AO : CO'$ § 282

$= 2\pi AO : 2\pi CO'$.

$\therefore 30 : 6 = 10 : 2\pi CO'$. Ax. 9

$\therefore 30 \times 2\pi CO' = 60$. § 261

$\therefore 2\pi CO' = 2$. Ax. 4

Since the lateral area of a frustum of a circular cone is expressed by the formula $l = \frac{1}{2}(c + c')s$, § 615

$\therefore l = \frac{1}{2}(10 + 2)(30 - 6)$ Ax. 9

$= \frac{1}{2} \times 12 \times 24$

$= 144$.

Therefore the lateral area of the frustum $AB\text{-}CD$ is 144 sq. ft.

Furthermore, since the rt. $\triangle$ VOA and $VO'C$ are similar, Proved

$\therefore VO : VO' = AO : CO'$ § 282

$= 2\pi AO : 2\pi CO'$

$= 10 : 2$ Ax. 9

$= 5 : 1$.

$\therefore VO - VO' : VO' = 4 : 1$. § 268

That is, $OO' : VO' = 4 : 1$.

In the rt. $\triangle\, VO'C$, $\overline{VO'}^2 = \overline{VC}^2 - \overline{CO'}^2$. § 338

$\therefore \overline{VO'}^2 = 6^2 - \left(\frac{2}{2\pi}\right)^2$ Ax. 9

$= 36 - \frac{1}{\pi^2}$

$= \frac{36\pi^2 - 1}{\pi^2}$.

$$\therefore\ VO' = \frac{1}{\pi}\sqrt{36\pi^2 - 1}. \qquad \text{Ax. 5}$$

$$\therefore\ OO' : \frac{1}{\pi}\sqrt{36\pi^2 - 1} = 4 : 1. \qquad \text{Ax. 9}$$

$$\therefore\ OO' = \frac{4}{\pi}\sqrt{36\pi^2 - 1}. \qquad \S\ 261$$

Now the volume of a frustum of a cone is expressed by the formula

$$v = \frac{1}{3}\pi a(r^2 + r'^2 + rr'). \qquad \S\ 618$$

$$\therefore\ v = \frac{1}{3}\pi \times \frac{4}{\pi}\sqrt{36\pi^2 - 1} \times \left[\left(\frac{10}{2\pi}\right)^2 + \left(\frac{2}{2\pi}\right)^2 + \frac{5}{\pi^2}\right] \qquad \text{Ax. 9}$$

$$= \frac{4}{3}\sqrt{36\pi^2 - 1} \times \left(\frac{25}{\pi^2} + \frac{1}{\pi^2} + \frac{5}{\pi^2}\right)$$

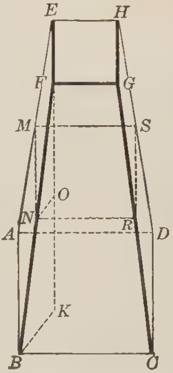
$$= \frac{4}{3}\sqrt{36\pi^2 - 1} \times \frac{31}{\pi^2}$$

$$= \frac{124}{3\pi^2}\sqrt{36\pi^2 - 1}.$$

Therefore the volume of the frustum $AB\text{-}CD$ is $\frac{124}{3\pi^2}\sqrt{36\pi^2 - 1}$ cu. ft.

Q. E. F.

4. *Find the difference between the volume of the frustum of a pyramid whose altitude is 9 ft. and whose bases are squares, 8 ft. and 6 ft. respectively on a side, and the volume of a prism of the same altitude whose base is a section of the frustum parallel to its bases and equidistant from them.*



Solution. The volume of a frustum of a pyramid is expressed by the formula

$$v = \tfrac{1}{3}a\left(b + b' + \sqrt{bb'}\right). \qquad \S\ 565$$

Therefore, for the given frustum $ABCD\text{-}EFGH$,

$$v = \tfrac{1}{3} \times 9\left(8^2 + 6^2 + \sqrt{8^2 \times 6^2}\right)$$

$$= 3(64 + 36 + 48)$$

$$= 3 \times 148, \text{ or } 444.$$

Let $MNRS$ be the section of the given frustum $ABCD\text{-}EFGH$ ∥ to the bases and midway between them.

From F let fall the ⊥ FK upon the base $ABCD$, cutting the mid-section $MNRS$ in O.

Draw BK and NO.

The lines BK and NO are ∥. § 453

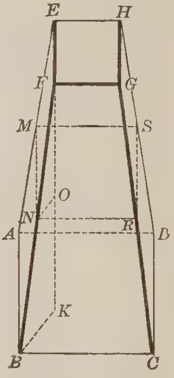
$\therefore\ FN : FB = FO : FK.$ § 273

But $FO : FK = 1 : 2.$ Given

$\therefore\ FN : FB = 1 : 2.$ Ax. 8

That is, N is the mid-point of BF.

Similarly, it may be shown that R, S, and M are the mid-points of CG, DH, and AE respectively.



Then each side of the mid-section $MNRS$ equals half the sum of a side of square $ABCD$ and a side of square $EFGH$. § 137

That is, a side of $MNRS = \frac{1}{2}(AB + EF)$

$= \frac{1}{2}(8 + 6)$ ft. Ax. 9

$= \frac{1}{2} \times 14$ ft., or 7 ft.

Moreover the $\angle$s of $MNRS$ are rt. $\angle$s. § 461

$\therefore$ $MNRS$ is a square 7 ft. on a side.

Now the volume of a prism is expressed by the formula $v = ba$. § 539

Therefore, for the given prism whose base is the mid-section $MNRS$, $v = 7^2 \times 9$ cu. ft. Ax. 9

$= 49 \times 9$ cu. ft.

$= 441$ cu. ft.

Therefore the difference between the volumes of the given frustum of a pyramid and the given prism

$= 444$ cu. ft. $-$ 441 cu. ft.

$= 3$ cu. ft. Q. E. F.

5. *A Dutch stone windmill in the shape of the frustum of a right cone is* 12 *meters high. The outer diameters at the bottom and the top are* 16 *meters and* 12 *meters, the inner diameters* 12 *meters and* 10 *meters. How many cubic meters of stone were required to build it?*

Solution. The volume of stone required to build the given Dutch windmill equals the difference in the volumes of two frustums of right cones each being 12 meters high, the one having for its diameters 16 meters and 12 meters, the other 12 meters and 10 meters.

But the volume of a frustum of a right circular cone is expressed by the formula $v = \frac{1}{3}\pi a(r^2 + r'^2 + rr')$. § 618

Therefore the volume of the greater given frustum

$= \frac{1}{3} \times 12\pi\left[\left(\frac{16}{2}\right)^2 + \left(\frac{12}{2}\right)^2 + \left(\frac{16}{2}\right)\cdot\left(\frac{12}{2}\right)\right]$ Ax. 9

$= 4 \times 148\pi$, or 592π;

and the volume of the smaller given frustum

$= \frac{1}{3} \times 12\pi\left[\left(\frac{12}{2}\right)^2 + \left(\frac{10}{2}\right)^2 + \left(\frac{12}{2}\right)\cdot\left(\frac{10}{2}\right)\right]$ Ax. 9

$= 4 \times 91\pi$, or 364π.

Therefore the volume of stone required to build the given Dutch windmill $= 592\pi$ cu. m. $-$ 364π cu. m.

$= 228\pi$ cu. m., or 716.2848 cu. m. Q. E. F.

6. *The chimney of a factory has the shape of a frustum of a regular pyramid. Its height is* 180 *ft., and its upper and lower bases are squares whose sides are* 10 *ft. and* 16 *ft. respectively. The flue throughout is a square whose side is* 7 *ft. How many cubic feet of material does the chimney contain?*

Solution. The volume of material in the given chimney equals the difference in volume of two geometric solids, the one a frustum of a regular pyramid with square bases 10 ft. and 16 ft. on a side and height 180 ft., the other a rectangular parallelepiped with a square base 7 ft. on a side and height 180 ft.

Now the volume of a frustum of a pyramid is expressed by the formula $v = \frac{1}{3}a\left(b + b' + \sqrt{bb'}\right)$. § 565

Therefore the volume of the given frustum of a regular pyramid

$$= \tfrac{1}{3} \times 180\left(10^2 + 16^2 + \sqrt{10^2 \times 16^2}\right) \text{ cu. ft.} \quad \text{Ax. 9}$$
$$= 60 \times 516 \text{ cu. ft., or } 30{,}960 \text{ cu. ft.}$$

Also the volume of a rectangular parallelepiped is equal to the product of its three dimensions. § 534

Therefore the volume of the given rectangular parallelepiped

$$= 180 \times 7^2 \text{ cu. ft., or } 8820 \text{ cu. ft.} \quad \text{Ax. 9}$$

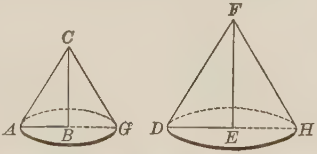
Therefore the volume of material in the given chimney

$$= 30{,}960 \text{ cu. ft.} - 8820 \text{ cu. ft.}$$
$$= 22{,}140 \text{ cu. ft.}$$ Q.E.F.

7. *Two right triangles with bases* 15 *in. and* 21 *in., and with hypotenuses* 25 *in. and* 35 *in. respectively, revolve about their third sides. Find the ratio of the total areas of the solids generated and find their volumes.*

Given the right circular cones C-AG and F-DH generated by the revolution of the rt. △ ABC and DEF about their sides BC and EF respectively; also $AB = 15$ in., $DE = 21$ in., $AC = 25$ in., and $DF = 35$ in.

Required to find (1) the ratio of the total areas of the right circular cones C-AG and F-DH; (2) the volumes of the right circular cones C-AG and F-DH.



Solution. (1) Since $t : t' = r^2 : r'^2$, § 614

∴ the ratio of the total areas of the right circular cones C-AG and F-DH

$$= 15^2 : 21^2 \quad \text{Ax. 9}$$
$$= 5^2 : 7^2$$
$$= 25 : 49.$$ Q.E.F.

(2) In the rt. △ ABC, $\overline{BC}^2 = \overline{AC}^2 - \overline{AB}^2$. § 338

$\therefore \overline{BC}^2 = 25^2 - 15^2$ Ax. 9

$= 400.$

$\therefore BC = 20$ in. Ax. 5

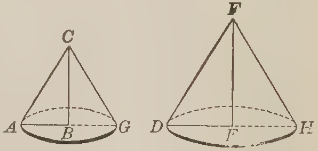
In the rt. △ DEF, $\overline{EF}^2 = \overline{DF}^2 - \overline{DE}^2$. § 338

$\therefore \overline{EF}^2 = 35^2 - 21^2$ Ax. 9

$= 784.$

$\therefore EF = 28$ in. Ax. 5

Now the volume of any right circular cone is expressed by the formula $v = \frac{1}{3}\pi r^2 a$. § 612



$\therefore$ the volume of $C\text{-}AG = \frac{1}{3} \times 15^2 \pi \times 20$ Ax. 9

$= 1500\,\pi$, or 4712.4;

and the volume of $F\text{-}DH = \frac{1}{3} \times 21^2 \pi \times 28$ Ax. 9

$= 4116\,\pi$, or 12,930.83.

Therefore the volumes of $C\text{-}AG$ and $F\text{-}DH$ required are 4712.4 cu. in. and 12,930.83 cu. in. respectively. Q.E.F.

Exercise 99. Page 379

1. *A cube each edge of which is* 12 *in. is transformed into a right prism whose base is a rectangle* 16 *in. long and* 12 *in. wide. Find the height of the prism and the difference between its total area and the total area of the cube.*

Solution. Let a = the number of inches in the height of the right prism equivalent to the given cube.

Then the volume of the right prism

$= 16 \times 12\,a$ cu. in., or $192\,a$ cu. in. § 539

But the volume of the given cube is 12^3 cu. in., or 1728 cu. in. § 534

Since the right prism and the cube are equivalent solids, Given

$\therefore 192\,a = 1728.$

$\therefore a = 9.$ Ax. 4

Therefore the height of the given right prism is 9 in.

Now the lateral area of a right prism is expressed by the formula

$l = ap.$ § 513

Therefore the lateral area of the given right prism

$= 9 \times 2\,(16 + 12)$ sq. in. Ax. 9

$= 504$ sq. in.

Since the area of one base of the right prism is 192 sq. in., § 320

$\therefore$ the entire area of the two bases

$= 2 \times 192$ sq. in., or 384 sq. in.

Therefore the total area of the right prism

$= 504$ sq. in. $+ 384$ sq. in., or 888 sq. in. Ax. 11

Furthermore, the total area of any cube is equal to six times the area of one of its faces. § 517

Therefore the total area of the given cube

$= 6 \times 12^2$ sq. in., or 864 sq. in. § 320

Hence the difference in the total areas of the right prism and the cube $= 888$ sq. in. $- 864$ sq. in.

$= 24$ sq. in. Q.E.F.

2. *The dimensions of a rectangular parallelepiped are* a, b, c. *Find the height of an equivalent right circular cylinder, having* a *for the radius of its base; the height of an equivalent right circular cone having* a *for the radius of its base.*

Solution. (1) Let h be the height of a right circular cylinder, with a base whose radius is a, equivalent to the given rectangular parallelepiped.

Then $\pi a^2 h =$ the volume of this right circular cylinder. § 590

But $a \times b \times c$, or abc, $=$ the volume of the given rectangular parallelepiped. § 534

$$\therefore \pi a^2 h = abc.$$

$$\therefore h = \frac{abc}{\pi a^2} \qquad \text{Ax. 4}$$

$$= \frac{bc}{\pi a}. \qquad \text{Q.E.F.}$$

(2) Let h_1 be the height of a right circular cone, with a base whose radius is a, equivalent to the given rectangular parallelepiped.

Then $\frac{1}{3}\pi a^2 h_1 =$ the volume of this right circular cone. § 612

Since the volume of the given rectangular parallelepiped $= abc$, § 534

$$\therefore \tfrac{1}{3}\pi a^2 h_1 = abc.$$

$$\therefore h_1 = \frac{abc}{\frac{1}{3}\pi a^2} \qquad \text{Ax. 4}$$

$$= \frac{3bc}{\pi a}. \qquad \text{Q.E.F.}$$

3. *A regular pyramid* 12 *ft. high is transformed into a regular prism with an equivalent base. Find the height of the prism.*

Solution. Let $x =$ the number of feet in the height of the regular prism equivalent to the given regular pyramid.

Denote the area of each of the bases of these geometric solids by b.

Then $\frac{1}{3} \times 12\,b$, or $4\,b =$ the volume of the given regular pyramid. § 561

Also $bx =$ the volume of the regular prism equivalent to the given regular pyramid. § 539

$$\therefore\ bx = 4\,b.$$

$$\therefore\ x = 4. \qquad \text{Ax. 4}$$

Therefore the height of the prism is 4 ft. Q.E.F.

4. *The diameter of a cylinder is* 14 *ft. and its height* 8 *ft. Find the height of an equivalent right prism the base of which is a square with a side* 4 *ft. long.*

Solution. Let $x =$ the number of feet in the height of a right prism with a square base 4 ft. on a side, equivalent to the given circular cylinder.

Then $4^2 \cdot x$, or $16\,x =$ the volume of this right prism. § 539

But $(\frac{14}{2})^2 \times 8\,\pi$, or $392\,\pi =$ the volume of the given circular cylinder. § 590

$$\therefore\ 16\,x = 392\,\pi.$$

$$\therefore\ x = 24.5\,\pi \qquad \text{Ax. 4}$$

$$= 76.97.$$

Therefore the height of the prism is 76.97 ft. Q.E.F.

5. *If one edge of a cube is* e, *what is the height* h *of an equivalent right circular cylinder whose radius is* r*?*

Solution. The volume of the right circular cylinder equivalent to the given cube $= \pi r^2 h$. § 590

Also the volume of the given cube $= e^3$. § 534

$$\therefore\ \pi r^2 h = e^3.$$

$$\therefore\ h = \frac{e^2}{\pi r^2}, \text{ by Ax. 4.}$$

Q.E.F.

6. *The heights of two equivalent right circular cylinders are in the ratio* 4 : 9. *If the diameter of the first is* 6 *ft., what is the diameter of the second?*

Solution. Let $x =$ the number of feet in the diameter of the second of the two given right circular cylinders.

Denote by $4\,h$ and $9\,h$ the heights of the two given cylinders respectively.

Then the volume of the first of these cylinders

$$= (\tfrac{6}{2})^2 \times 4\,\pi h \qquad \text{§ 590}$$

$$= 36\,\pi h,$$

and the volume of the second cylinder

$= \left(\frac{x}{2}\right)^2 \times 9\pi h$ § 590

$= \frac{9}{4}\pi h x^2.$

Since the two given cylinders are equivalent, Given

$\therefore \frac{9}{4}\pi h x^2 = 36\pi h.$

$\therefore x^2 = 16.$ Ax. 4

$\therefore x = 4.$ Ax. 5

Therefore the diameter of the second prism is 4 ft. Q.E.F.

7. *A right circular cylinder 6 ft. in diameter is equivalent to a right circular cone 7 ft. in diameter. If the height of the cone is 8 ft., what is the height of the cylinder?*

Solution. Let $x =$ the number of feet in the height of the given right circular cylinder.

Then the volume of this cylinder $= (\frac{6}{2})^2 \pi x$ § 590

$= 9\pi x.$

Furthermore the volume of the given right circular cone

$= \frac{1}{3}(\frac{7}{2})^2 \times 8\pi$ § 612

$= \frac{98}{3}\pi.$

Since the given cylinder and cone are equivalent, Given

$\therefore 9\pi x = \frac{98}{3}\pi.$

$\therefore x = \frac{98}{27}$, or $3\frac{17}{27}$. Ax. 4

Therefore the height of the given cylinder is $3\frac{17}{27}$ ft. Q.E.F.

8. *The frustum of a regular pyramid 6 ft. high has for bases squares 5 ft. and 8 ft. on a side. Find the height of an equivalent regular pyramid whose base is a square 12 ft. on a side.*

Solution. Let $x =$ the number of feet in the height of a regular pyramid with a square base 12 ft. on a side, equivalent to the given frustum of a regular pyramid.

Then the volume of this pyramid $= \frac{1}{3}(12)^2 \cdot x$ § 561

$= 48x.$

Since the volume of a frustum of a pyramid is expressed by the formula $v = \frac{1}{3}a\left(b + b' + \sqrt{bb'}\right)$, § 565

$\therefore$ the volume of the given frustum of a regular pyramid

$= \frac{1}{3} \times 6\left(5^2 + 8^2 + \sqrt{5^2 \times 8^2}\right)$ Ax. 9

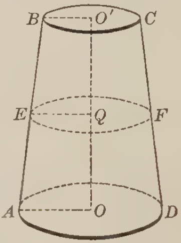
$= 2 \times 129$, or 258.

$\therefore 48x = 258.$

$\therefore x = \frac{258}{48}$, or $5\frac{3}{8}$. Ax. 4

Therefore the height of the regular pyramid is $5\frac{3}{8}$ ft. Q.E.F.

9. *The frustum of a cone of revolution is 5 ft. high and the diameters of its bases are 2 ft. and 3 ft. respectively. Find the height of an equivalent right circular cylinder whose base is equal in area to the section of the frustum made by a plane parallel to the bases and equidistant from them.*



Solution. Let AD-BC be the given frustum of a cone of revolution with bases BC and AD 2 ft. and 3 ft. in diameter respectively, and with height 5 ft. Let AB and OO' be segments of an element and of the axis respectively of the given cone of revolution of which AD-BC is a frustum. Let the plane AO' determined by the lines AB and OO' intersect the bases AD and BC in the lines AO and BO' respectively, and the section EF, ‖ to AD and BC and midway between them, in EQ.

Now OO' is $\perp$ to the base AD. § 597

Then OO' is $\perp$ to the base BC and to the section EF. § 456

Moreover the section EF is a ⊙. § 601

Then O, O', and Q are centers of the ⓢ AD, BC, and EF respectively. § 602

$\therefore$ AO, BO', and EQ are radii of the ⓢ AD, BC, and EF respectively.

The radii AO, BO', and EQ are ‖ lines. § 453

Then in the trapezoid $AOO'B$,

EQ bisects OO' at Q,

and is ‖ to its bases AO and BO'.

$\therefore$ EQ bisects AB. § 134

$\therefore EQ = \frac{1}{2}(AO + BO')$. § 137

$\therefore EQ = \frac{1}{2} \times \frac{1}{2}(3 + 2)$ Ax. 9

$= \frac{5}{4}$.

That is, EQ, or $\frac{5}{4}$ ft., is the radius of the base of a right circular cylinder equivalent to the given frustum of a cone of revolution, AD-BC.

Let x denote the height in feet of this right circular cylinder.

Then the volume of this cylinder $= (\frac{5}{4})^2 \cdot \pi x$, or $\frac{25}{16}\pi x$. § 590

But the volume of the given frustum of a cone of revolution

$= \frac{1}{3} \times 5\pi[(\frac{2}{2})^2 + (\frac{3}{2})^2 + \frac{2}{2} \times \frac{3}{2}]$ § 618

$= \frac{5}{3} \times \frac{19}{4}\pi$, or $\frac{95}{12}\pi$.

$\therefore \frac{25}{16}\pi x = \frac{95}{12}\pi$.

$\therefore 75x = 380$. Ax. 3

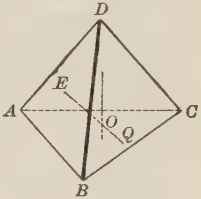
$\therefore x = 5\frac{1}{15}$. Ax. 4

Therefore the height of the right circular cylinder equivalent to the given frustum of a cone is $5\frac{1}{15}$ ft. Q.E.F.

BOOK VIII

Exercise 101. Page 390

1. *The four perpendiculars erected at the centers of the circles circumscribed about the faces of a tetrahedron meet in the same point.*



Given any tetrahedron D-ABC with ⊥s erected at the circumcenters of its four faces.

To prove that these ⊥s are concurrent lines.

Proof. Let O be the center of a sphere circumscribed about the tetrahedron D-ABC. § 650

Let EQ be the ⊥ erected at the circumcenter of the face ABD.

Then EQ is the locus of a point equidistant from A, B, and D. § 442

Since $OA = OB = OD$, § 621

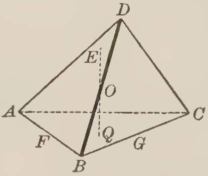
$\therefore$ O lies in EQ.

That is, EQ passes through O.

Similarly, it may be shown that each of the ⊥s erected at the circumcenters of the other three faces of the tetrahedron D-ABC passes through O.

Therefore the ⊥s erected at the circumcenters of the faces of a tetrahedron D-ABC are concurrent lines. Q. E. D.

2. *The six planes perpendicular to the edges of a tetrahedron at their mid-points intersect in the same point.*



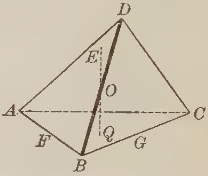
Given in any tetrahedron D-ABC the six planes ⊥ to its edges at their mid-points.

To prove that these planes intersect in the same point.

Proof. Let the two planes ⊥ to the edges AB and BC at their mid-points F and G respectively intersect in the line EQ. § 429

Then, since the plane ⊥ to AB at F is the locus of a point in space equidistant from A and B, and the plane ⊥ to BC at G is the locus of a point in space equidistant from B and C, § 443 therefore the intersection of these planes, line EQ, is the locus of a point in space equidistant from A, B, and C.

Then line EQ lies in the plane ⊥ to AC at its mid-point; that is, the plane ⊥ to AC at its mid-point passes through line EQ. § 443



Furthermore, since the planes ⊥ to AB and BC at F and G respectively are each ⊥ to the plane of the △ ABC, § 477

∴ the intersection of these planes, line EQ, is ⊥ to the plane of the △ ABC. § 478

Therefore line EQ is ⊥ to △ ABC at its circumcenter. § 243

In like manner it may be shown that the planes ⊥ to the sides of the face △s DAB, DBC, and DAC at their mid-points intersect in lines ⊥ to these △s respectively at their circumcenters.

Now let O denote the center of the sphere circumscribed about the tetrahedron $D\text{-}ABC$. § 650

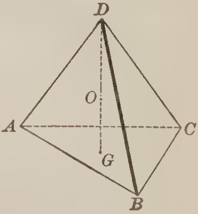
Then, since $OA = OB = OC$, § 621

∴ O lies in line EQ; that is, line EQ passes through O.

Similarly, it may be shown that each of the ⊥s to the face △s DAB, DBC, and DAC at their circumcenters passes through O.

Therefore the six planes ⊥ to the edges of a tetrahedron $D\text{-}ABC$ at their mid-points intersect in the same point. Q.E.D.

3. *The six planes which bisect the six dihedral angles of a tetrahedron intersect in the same point.*



Given in any tetrahedron $D\text{-}ABC$ the bisecting planes of the six dihedral ∠s.

To prove that these planes intersect in the same point.

Proof. Let the two planes bisecting the dihedral ∠s DA and DB intersect in the line DG. § 429

Since the bisecting plane of the dihedral ∠ DA is the locus of a point equidistant from the planes of the face △s DAB and DAC, and the bisecting plane of the dihedral ∠ DB is the locus of a point equidistant from the planes of the face △s DAB and DBC, § 479

∴ the intersection of these bisecting planes, line DG, is the locus of a point equidistant from the planes of the face △s DAB, DAC, and DBC.

Then line DG lies in the bisecting plane of the dihedral ∠ DC; that is, the bisecting plane of the dihedral ∠ DC passes through line DG. § 479

Similarly, it may be shown that the bisecting planes of the dihedral ∠s of the trihedral ∠s A, B, and C intersect in lines which are the loci of a point equidistant from the faces of these trihedral ∠s respectively.

Now let O denote the center of the inscribed sphere of the tetrahedron D-ABC. § 649

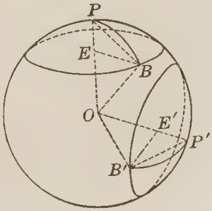
Since O is equally distant from the face $\triangle$s DAB, DAC, and DBC, § 647
$\therefore$ O lies in the line DG; that is, line DG passes through O.

For a like reason, the loci of points equidistant from the faces of the trihedral $\angle$s A, B, and C each pass through O.

Therefore the six bisecting planes of the six dihedral $\angle$s of the tetrahedron D-ABC pass through the same point. Q.E.D.

4. *Circles on the same sphere having equal polar distances are equal.*

Given, on the sphere whose center is O, the ⓢ whose centers are E and E', having the equal polar distances PB and $P'B'$ respectively.



To prove that the ⓢ whose centers are E and E' are equal.

Proof. Draw $PO, P'O, BO, B'O, BE$, and $B'E'$.

Then PO and $P'O$ are $\perp$ to the given ⓢ at E and E' respectively. § 441

$\therefore$ the $\angle$s OEB and $OE'B'$ are rt. $\angle$s. § 430

In the rt. $\triangle$s OEB and $OE'B'$,

$\angle POB = \angle P'OB'$, § 167

and $BO = B'O$. § 621

$\therefore$ the rt. $\triangle$s OEB and $OE'B'$ are congruent. § 91

$\therefore$ $BE = B'E'$. § 67

Therefore the ⓢ whose centers are E and E' and whose polar distances PB and $P'B'$ are equal, have equal radii and are consequently equal, by § 162. Q.E.D.

5. *Equal circles on the same sphere have equal polar distances.*

Given, on the sphere whose center is O, the equal ⓢ whose centers are E and E', and whose polar distances are PB and $P'B'$ respectively. (See Fig. in Ex. 4.)

To prove that the arcs PB and $P'B'$ are equal.

Proof. Draw PO, $P'O$, BO, $B'O$, BE, and $B'E'$.

Then PO and $P'O$ are $\perp$ to the given ⓢ at E and E' respectively. § 441

$\therefore$ the $\angle$s OEB and $OE'B'$ are rt. $\angle$s. § 430

In the rt. $\triangle$s OEB and $OE'B'$, $OB = OB'$, § 621

and $BE = B'E'$. § 162

$\therefore$ the rt. $\triangle$s OEB and $OE'B'$ are congruent. § 89

$\therefore$ the $\angle$s EOB and $E'OB'$ are equal. § 67

That is, the $\angle$s POB and $P'O'B'$ are equal.

$\therefore$ the arcs PB and $P'B'$ are equal, by § 166. Q.E.D.

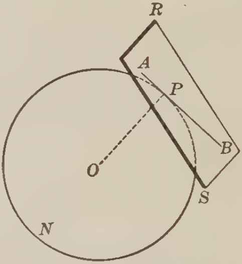
6. *Find the locus of a point in a plane at a given distance from a given point. Also of a point in a three-dimensional space.*

The locus of a point in a plane at a given distance from a given point is a ⊙ whose center is the given point and whose radius is the given distance. § 160

The locus of a point in space at a given distance from a given fixed point is a sphere whose center is the fixed point and whose radius is the given distance.

For every point on the surface of the sphere is at the given distance from the fixed point (§ 621), and every point not on the surface of the sphere is at a distance greater or less than the given distance from the fixed point. Ax. 11

7. *A line tangent to a great circle of a sphere lies in the plane tangent to the sphere at the point of contact.*



Given the line AB tangent at P to the great ⊙ PN of the sphere whose center is O; also the plane RS tangent to the sphere at P.

To prove that the line AB lies in the plane RS.

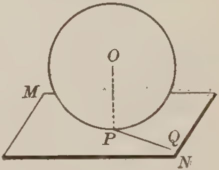
Proof. Draw OP.

AB is ⊥ to OP. § 185

Moreover the plane RS is ⊥ to OP. § 646

∴ AB lies in the plane RS, by § 432. Q. E. D.

8. *Any line in a tangent plane drawn through the point of contact is tangent to the sphere at that point.*



Given the plane MN tangent at P to the sphere whose center is O; and the line QP drawn in the plane MN through P.

To prove that QP is tangent to the given sphere.

Proof. Draw OP.

Since the plane MN is tangent to the sphere, Given

∴ the plane MN is ⊥ to OP. § 646

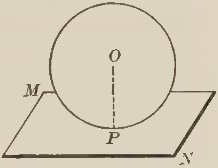
∴ OP is ⊥ to QP. § 430

Through QP and OP pass the plane QOP.

Then QP is tangent to the great ⊙ made by the intersection of the plane QOP with the sphere. § 184

Therefore QP is tangent to the given sphere, by § 643. Q. E. D.

9. *One plane and only one plane can be passed through a given point on a given sphere tangent to the sphere.*



Given the plane MN tangent at P to the sphere whose center is O.

To prove that MN is the only plane that can be passed through P tangent to the given sphere.

Proof. Draw OP.

The plane MN is ⊥ to OP. § 646

That is, OP is ⊥ to the plane MN. § 430

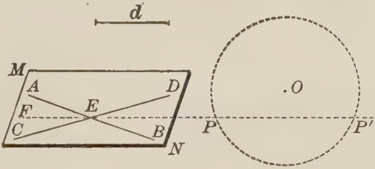
Now OP is the only line that can be drawn ⊥ to the plane MN at P. § 436

That is, the plane MN is the only plane that can be passed through P ⊥ to OP. § 433

Since every plane passed through P other than the plane MN is not ⊥ to OP, therefore every plane passed through P other than the plane MN is not tangent to the given sphere. § 646

Therefore the plane MN is the only plane that can be passed through P tangent to the given sphere. Q.E.D.

10. *Find a point in a plane equidistant from two intersecting lines in the plane, and at a given distance from a given point not in the plane. Discuss the solution.*



Given, in the plane MN, the lines AB and CD intersecting in E, O a fixed point without the plane MN, and d a line of fixed length.

Required to find a point in the plane MN equidistant from the intersecting lines AB and CD, and at the distance d from O.

Construction. With O as a center, and with a radius equal to d, describe a spherical surface.

Draw FE, the bisector of $\angle AEC$. § 231

Produce FE to intersect in P the spherical surface whose center is O.

Then P is the point required. Q.E.F.

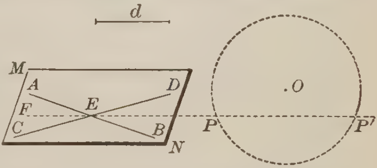
Proof. P is equidistant from AB and CD, since FE bisects $\angle AEC$. § 152

Furthermore, $PO = d$. § 621

Therefore P is a point in the plane MN equidistant from AB and CD, and at the given distance d from O. Q.E.D.

Discussion. If the plane MN intersects the sphere whose center is O so that point E lies within the sphere, it is clear that the bisectors of the two pairs of vertical $\angle$s included by AB and CD will intersect the surface of the sphere in *four* points, all of which fulfill the required conditions. Therefore the maximum number of points possible is four.

If the plane MN is tangent to the sphere so that one of the two bisectors of the two pairs of vertical $\angle$s included by AB and CD passes through the point of tangency, there is obviously but one point which will fulfill the required conditions.



In all cases where the plane MN is tangent to the sphere, or intersects the sphere, but where the bisectors of the two pairs of vertical $\angle$s included by AB and CD do not intersect or touch the sphere, there is no solution.

In the case where the plane MN does not intersect or touch the sphere there is no solution.

11. *How many points determine a straight line? a circle? a spherical surface? Prove that two spherical surfaces coincide if they have this number of points in common.*

Two points determine a straight line. § 54

Three points not in a straight line determine a ⊙. § 190

Four points not in the same plane determine a spherical surface. § 651

Given two spherical surfaces having four points in common.

To prove that the two given spherical surfaces coincide.

Proof. Join the four given points by straight lines.

These six lines determine planes which inclose a tetrahedron. § 566

Erect $\perp$s at the circumcenters of the face $\triangle$s of this tetrahedron.

Then these four $\perp$s are concurrent lines. Ex. 1, p. 390

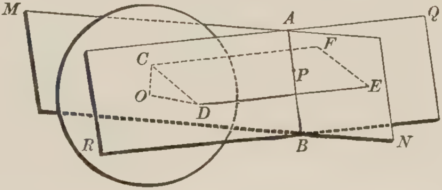
Therefore there is but one point in space equidistant from the four given points; namely, the intersection of the four $\perp$s. § 442

Hence every spherical surface in which the four given points lie must have its center at the intersection of the four $\perp$s. § 621

Therefore two spherical surfaces having four points in common must coincide. Q.E.D.

12. *If two planes which intersect in the line AB touch a sphere at the points C and D respectively, the line CD is perpendicular to AB in the sense mentioned in the discussion under* § 450, — *that a plane can be passed through CD perpendicular to AB.*

Given two planes MN and RQ, which intersect in the line AB, tangent at C and D respectively to the sphere whose center is O; also the line CD.



To prove that a plane can be passed through $CD \perp$ to AB.

Proof. Let the plane $CODEF$ determined by O, C, and D cut the line AB in P.

The radii OC and OD are $\perp$ to the planes MN and RQ respectively. § 646

Then the plane $CODEF$ is $\perp$ to the planes MN and RQ. § 477

$\therefore$ AB is $\perp$ to the plane $CODEF$. § 478

That is, the plane $CODEF$ is $\perp$ to AB.

But the line CD lies in the plane $CODEF$. § 422

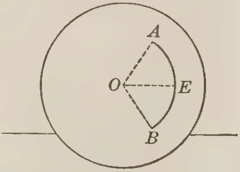
Therefore a plane can be passed through $CD \perp$ to AB. Q. E. D.

Exercise 102. Page 394

1. *To bisect a given great-circle arc.*

Given AB an arc of a great ⊙ on the sphere whose center is O.

Required to bisect the arc AB.



Construction. Pass a plane through the points O, A, and B. § 427

In the plane OAB draw OA and OB.

Then bisect the plane $\angle AOB$ by the line OE. § 231

Produce OE to intersect the arc AB in E.

Then point E bisects the arc AB. Q. E. F.

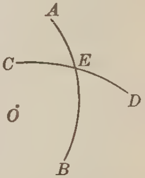
Proof. The arc AB lies in the plane OAB.

The $\angle$s AOE and BOE are equal. Const.

Therefore the arcs AE and BE are equal, by § 166. Q. E. D.

2. *If two great-circle arcs intersect, the vertical angles are equal.*

Given AB and CD arcs of great Ⓢ, intersecting at E, on the sphere whose center is O.



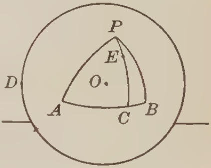
To prove that the spherical ∡ AEC and BED are equal; also that the spherical ∡ CEB and DEA are equal.

Proof. The planes of the given intersecting arcs intersect in the line OE. § 429

The dihedral ∡ $A\text{-}OE\text{-}C$ and $B\text{-}OE\text{-}D$ are equal; also the dihedral ∡ $C\text{-}OE\text{-}B$ and $D\text{-}OE\text{-}A$ are equal. § 470, 3

Therefore the spherical ∡ AEC and BED are equal; also the spherical ∡ CEB and DEA are equal, by § 655. Q. E. D.

3. *To describe an arc of a great circle through a given point and perpendicular to a given arc of a great circle.*



Given AB an arc of a great ⊙ and E a point on the sphere whose center is O.

Required to describe an arc of a great ⊙ through E ⊥ to the given arc AB.

Construction. From A and B as poles, with a quadrant radius arc, describe arcs of great Ⓢ intersecting at P.

From E and P as poles, with a quadrant radius arc, describe arcs of great Ⓢ intersecting at D.

From D as a pole, with a quadrant radius arc, describe PE, an arc of a great ⊙.

Produce the arc PE to intersect the arc AB in C.

Then PEC is the arc required. Q. E. F.

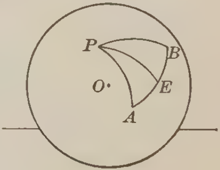
Proof. PEC is an arc of a great ⊙. Const.

Since PA and PB are quadrants, Const.

∴ P is a pole of the arc AB. § 641

Therefore the arc PEC is ⊥ to the given arc AB, by § 656. Q. E. D.

4. *Every point lying on a great circle which bisects a given arc of another great circle at right angles is equidistant* (§ 635) *from the extremities of the given arc.*



Given, on the sphere whose center is O, AB an arc of a great ⊙ bisected at rt. ∡ at E by PE an arc of a great ⊙.

To prove that every point on the arc PE is equidistant from A and B.

Proof. Take P any point on the arc PE.

Draw PA and PB arcs of great ⓢ.

In the spherical △ PEA and PEB,

side $PE =$ side PE, Iden.

side $AE =$ side BE, Given

and the spherical $\angle PEA =$ the spherical $\angle PEB$, being rt. spherical ∠s. Given

But the equal parts of these △ occur in the reverse order.

Therefore the spherical △ PEA and PEB are symmetric. § 672

Therefore the arcs PA and PB are equal.

Therefore P, any point on the arc PE, is equidistant from A and B. § 635

Therefore every point on the arc PE is equidistant from A and B.

Q. E. D.

5. *Two sides of a spherical triangle are respectively* 82° 47′ *and* 67° 39′. *What is known concerning the number of degrees in the third side?*

Solution. Let x denote the number of degrees in the third side of the given spherical △.

Then $x + 82^\circ\,47' + 67^\circ\,39' < 360^\circ$. § 664

$\therefore x < 360^\circ - 82^\circ\,47' - 67^\circ\,39'$ Ax. 6

$< 209^\circ\,34'$. Q. E. F.

6. *Three sides of a spherical quadrilateral are respectively* 86° 29′, 73° 47′, *and* 69° 54′. *What is known concerning the number of degrees in the fourth side?*

Solution. Let x denote the number of degrees in the fourth side of the given spherical quadrilateral.

Then $x + 86^\circ\,29' + 73^\circ\,47' + 69^\circ\,54' < 360^\circ$. § 664

That is, $x + 230^\circ\,10' < 360^\circ$.

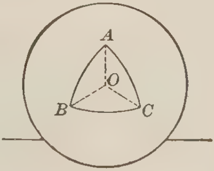
$\therefore x < 360^\circ - 230^\circ\,10'$ Ax. 6

$< 129^\circ\,50'$. Q. E. F.

7. *Draw a picture of a sphere, and of an equilateral spherical triangle on the sphere, each side being* 90°. *Then draw a picture of the polar triangle.*

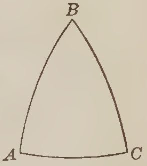
In the accompanying figure the spherical △ ABC is an equilateral spherical △, each side being 90°.

Therefore the polar △ of △ ABC coincides with △ ABC. § 665



Exercise 103. Page 398

1. *If two sides of a spherical triangle are quadrants, the third side measures the opposite angle.*



Given, in the spherical △ ABC, the arcs of great Ⓢ AB and CB each equal to a quadrant.

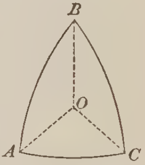
To prove that the arc of a great ⊙ AC is the measure of the spherical ∠ B.

Proof. Since the arcs AB and CB are quadrants, Given

∴ B is a pole of the arc AC. § 641

∴ the arc AC is the measure of the spherical ∠ B, by § 654. Q. E. D.

2. *In a birectangular spherical triangle the sides opposite the right angles are quadrants, and the side opposite the third angle measures that angle.*



Given the birectangular spherical △ ABC, the spherical ∡ A and C being rt. ∡.

To prove (1) that the arcs of great Ⓢ AB and CB are quadrants; (2) that the arc of a great ⊙ AC is the measure of the spherical ∠ B.

Proof. (1) Let O be the center of the given sphere.

Draw the radii OA, OB, and OC.

The planes of the arcs AB and CB are each ⊥ to the plane of the arc AC.

Then BO, the intersection of the planes BOA and BOC, is ⊥ to the plane AOC. § 478

∴ BO is ⊥ to OA and OC. § 430

∴ ∡ BOA and BOC are rt. ∡.

∴ the arcs AB and CB are quadrants, by § 213. Q. E. D.

(2) Since the arcs AB and CB are quadrants, Proved

∴ B is a pole of the arc AC. § 641

∴ the arc AC is the measure of the spherical ∠ B, by § 654. Q. E. D.

3. *Each side of a trirectangular spherical triangle is a quadrant.*

Given the trirectangular spherical △ ABC.

To prove that the arcs of great Ⓢ AB, AC, and BC are quadrants.

Proof. Let O be the center of the given sphere.

Draw the radii OA, OB, and OC.

The planes of the arcs AB, AC, and BC are each ⊥ to the other two.

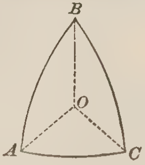
Then BO, the intersection of the planes BOA and BOC, is ⊥ to the plane AOC. § 478

Similarly, AO is ⊥ to the plane BOC.

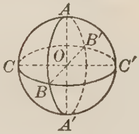
∴ BO is ⊥ to OA and OC, and AO is ⊥ to OC. § 430

∴ the ∠s BOA, BOC, and AOC are rt. ∠s.

∴ the arcs AB, AC, and BC are quadrants, by § 213. Q. E. D.



4. *Three planes passed through the center of a sphere, each perpendicular to the other two, divide the spherical surface into eight congruent trirectangular triangles.*



Given the three planes $ABA'B'$, $AC'A'C$, and $BC'B'C$ intersecting the sphere whose center is O, each ⊥ to the other two and passing through O.

To prove that the eight spherical △s into which the surface of the sphere is divided are congruent trirectangular △s.

Proof. The lines AA', BB', and CC' are each ⊥ to the other two at O. §§ 478, 430

∴ the ∠s AOB, AOC, and BOC are rt. ∠s.

∴ the arcs AB, AC, and BC are quadrants. § 213

Then the spherical △ ABC is a trirectangular △. § 654

For a like reason, each of the other seven spherical △s are trirectangular △s.

Therefore the eight trirectangular spherical △s into which the surface of the sphere is divided are congruent, by § 662. Q. E. D.

5. *Find the number of degrees in the sides of a spherical triangle if the angles of its polar triangle are* 82°, 77°, 69°.

Solution. The sides of the required spherical △ are

$180° - 82° = 98°$,

$180° - 77° = 103°$,

and $180° - 69° = 111°$, by § 667. Q. E. F.

6. *Find the number of degrees in the sides of a spherical triangle if the angles of its polar triangle are* $84\frac{1}{2}°$, $81\frac{3}{4}°$, $72\frac{1}{6}°$.

Solution. The sides of the required spherical △ are

$180° - 84\frac{1}{2}° = 95\frac{1}{2}°$,

$180° - 81\frac{3}{4}° = 98\frac{1}{4}°$,

and $180° - 72\frac{1}{6}° = 107\frac{5}{6}°$, by § 667. Q. E. F

7. *Find the number of degrees in the sides of a spherical triangle if the angles of its polar triangle are* 78° 30′, 89°, 102°.

Solution. The sides of the required spherical △ are

$$180^\circ - 78^\circ\,30' = 101^\circ\,30',$$

$$180^\circ - 89^\circ = 91^\circ,$$

and $180^\circ - 102^\circ = 78^\circ$, by § 667. Q.E.F.

8. *Find the number of degrees in the sides of a spherical triangle if the angles of its polar triangle are* 83° 40′, 48° 57′, 103° 43′.

Solution. The sides of the required spherical △ are

$$180^\circ - 83^\circ\,40' = 96^\circ\,20',$$

$$180^\circ - 48^\circ\,57' = 131^\circ\,3',$$

and $180^\circ - 103^\circ\,43' = 76^\circ\,17'$, by § 667. Q.E.F.

9. *Find the number of degrees in the sides of a spherical triangle if the angles of its polar triangle are* 96° 37′ 40″, 82° 29′ 30″, 68° 47′.

Solution. The sides of the required spherical △ are

$$180^\circ - 96^\circ\,37'\,40'' = 83^\circ\,22'\,20'',$$

$$180^\circ - 82^\circ\,29'\,30'' = 97^\circ\,30'\,30'',$$

and $180^\circ - 68^\circ\,47' = 111^\circ\,13'$, by § 667. Q.E.F.

10. *Find the number of degrees in the sides of a spherical triangle if the angles of its polar triangle are* 43° 29′ 37″, 98° 22′ 53″, 87° 36′ 39″.

Solution. The sides of the required spherical △ are

$$180^\circ - 43^\circ\,29'\,37'' = 136^\circ\,30'\,23'',$$

$$180^\circ - 98^\circ\,22'\,53'' = 81^\circ\,37'\,7'',$$

and $180^\circ - 87^\circ\,36'\,39'' = 92^\circ\,23'\,21''$, by § 667. Q.E.F.

11. *Find the number of degrees in the angles of a spherical triangle if the sides of its polar triangle are* 68° 42′ 39″, 93° 48′ 7″, 89° 38′ 14″.

Solution. The ∡s of the required spherical △ are

$$180^\circ - 68^\circ\,42'\,39'' = 111^\circ\,17'\,21'',$$

$$180^\circ - 93^\circ\,48'\,7'' = 86^\circ\,11'\,53'',$$

and $180^\circ - 89^\circ\,38'\,14'' = 90^\circ\,21'\,46''$, by § 667. Q.E.F.

12. *Find the number of degrees in the angles of a spherical triangle if the sides of its polar triangle are* 78° 47′ 29″, 106° 36′ 42″, *a quadrant.*

Solution. The ∡s of the required spherical △ are

$$180^\circ - 78^\circ\,47'\,29'' = 101^\circ\,12'\,31'',$$

$$180^\circ - 106^\circ\,36'\,42'' = 73^\circ\,23'\,18'',$$

and $180^\circ - \text{a quadrant} = 90^\circ$, by § 667. Q.E.F.

13. *Find the number of degrees in the angles of a spherical triangle if the sides of its polar triangle are a quadrant, half a quadrant, three fourths of a quadrant.*

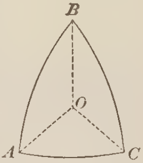
Solution. The ∠s of the required spherical △ are

$$180° - \text{a quadrant} = 90°,$$

$$180° - \text{half a quadrant} = 135°,$$

and $180° -$ three fourths of a quadrant $= 112° 30'$, by § 667. Q. E. F.

14. *From the center of a sphere are drawn three radii, each perpendicular to the other two. Find the number of degrees in the sides and angles of the spherical triangle determined by their extremities.*



Given, in the sphere whose center is O, the three radii OA, OB, and OC drawn ⊥ each to the other two; also the spherical △ ABC.

Required to find the number of degrees in the sides and ∠s of the spherical △ ABC.

Solution. Since AO is ⊥ to OB and OC, and BO is ⊥ to OC, Given

∴ the ∠s AOB, AOC, and BOC are rt. ∠s.

∴ the arcs of great ⊙s AB, AC, and BC are quadrants, or 90°. § 213

Again, since AO is ⊥ to OB and OC, AO is ⊥ to the plane BOC. § 431

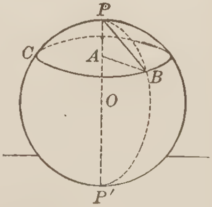
Also, since BO is ⊥ to OA and OC, BO is ⊥ to the plane AOC. § 431

∴ the planes AOB, AOC, and BOC are each ⊥ to the other two. § 477

∴ the spherical ∠s A, B, and C are rt. ∠s, or 90°, by § 655. Q. E. F.

Exercise 104. Page 405

1. *The radius of a sphere is 4 in. From any point on the surface as a pole a circle is described upon the sphere with an opening of the compasses equal to 3 in. Find the area of this circle.*



Given, on the surface of the sphere whose center is O, the ⊙ CB described from P as a pole, with an opening of the compasses equal to PB. $OP = 4$ in., and $PB = 3$ in.

Required to find the area of the ⊙ CB.

Solution. Draw the diameter PP'.

The diameter PP' is ⊥ to the plane of the ⊙ CB at its center A. § 627

Draw the line AB.

$\therefore PP'$ is $\perp$ to AB. § 430

The section PBP' is a semicircle.

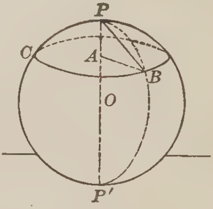
$\therefore PA : PB = PB : PP'$. § 298

Whence $\overline{PB}^2 = PA \times PP'$. § 261

$\therefore 3^2 = 2 \times 4 \times PA$. Ax. 9

$\therefore 9 = 8\,PA$.

$\therefore PA = \frac{9}{8}$, or $1\frac{1}{8}$. Ax. 4



In the rt. $\triangle PAB$,

$\overline{AB}^2 = \overline{PB}^2 - \overline{PA}^2$. § 338

$\therefore \overline{AB}^2 = 3^2 - (1\frac{1}{8})^2$ Ax. 9

$= 9 - \frac{81}{64}$

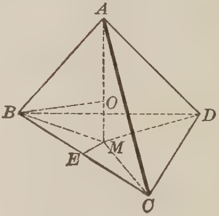
$= \frac{576}{64} - \frac{81}{64}$, or $\frac{495}{64}$.

But $\pi \cdot \overline{AB}^2 =$ the area of the $\odot\, CB$. § 389

Therefore the area of the $\odot\, CB = \frac{495}{64}\pi$ sq. in., or 24.2983 sq. in., by Ax. 9. Q. E. F.

2. *The edge of a regular tetrahedron is a. Find the radii r, r′ of the inscribed and circumscribed spheres.*

Given O, the common center of the inscribed and circumscribed spheres, having the radii r and r' respectively, of the regular tetrahedron A-BCD whose edge is a.



Required to find r and r' in terms of a.

Solution. Let fall $AM \perp$ to the face $\triangle BCD$ of the given tetrahedron at M.

Draw the lines MB, MC, and MD.

Draw $ME \perp$ to BC.

Finally, draw OB.

Since $AB = AC = AD$, § 568

$\therefore MB = MC = MD$. § 440

$\therefore M$ is the circumcenter of the $\triangle BCD$. § 243

$\therefore AM$ is the locus of a point equidistant from B, C, and D. § 442

Since O is equidistant from B, C, and D, § 621

$\therefore O$ lies in AM.

The line AM is $\perp$ to BM. § 430

$\therefore$ the $\triangle AMB$ is a rt. $\triangle$.

In the rt. $\triangle AMB$, $\overline{AM}^2 = \overline{AB}^2 - \overline{BM}^2$. (1) § 338

Now $\angle EBM = \frac{1}{2} \angle CBD$. § 363

But $\angle CBD = 60°$. § 145

$\therefore \angle EBM = \frac{1}{2}$ of $60°$, or $30°$. Ax. 9

$\therefore \angle EMB = 60°$. § 107

$\therefore EM = \frac{1}{2} BM$. Ex. 2, p. 80

Moreover since $MB = MC$, Proved

$\therefore BE = \frac{1}{2} BC$. § 85

In the rt. $\triangle MEB$, $\overline{BM}^2 - \overline{EM}^2 = \overline{BE}^2$. § 338

$\therefore \overline{BM}^2 - (\frac{1}{2} BM)^2 = (\frac{1}{2} BC)^2$. Ax. 9

$\therefore \overline{BM}^2 - \frac{1}{4} \overline{BM}^2 = \frac{1}{4} \overline{BC}^2$.

$\therefore \frac{3}{4} \overline{BM}^2 = \frac{1}{4} \overline{BC}^2$.

$\therefore 3 \overline{BM}^2 = \overline{BC}^2$ Ax. 3

$= \overline{AB}^2$. Ax. 9

$\therefore \overline{BM}^2 = \frac{1}{3} \overline{AB}^2$. Ax. 4

Substitute this value of $\overline{BM}^2$ in (1).

Then $\overline{AM}^2 = \overline{AB}^2 - \frac{1}{3} \overline{AB}^2$ Ax. 9

$= \frac{2}{3} \overline{AB}^2$.

$\therefore AM = \sqrt{\frac{2}{3} \overline{AB}^2}$ Ax. 5

$= \frac{1}{3} \sqrt{6}\, AB$.

But $AB = a$. Given

$\therefore AM = \frac{1}{3} \sqrt{6}\, a$. Ax. 9

Now $OA + OM = AM$. Ax. 11

$\therefore r' + r = \frac{1}{3} \sqrt{6}\, a$. Ax. 9

In the rt. $\triangle OMB$, $\overline{OB}^2 - \overline{OM}^2 = \overline{BM}^2$ § 338

$= \frac{1}{3} \overline{AB}^2$. Ax. 9

$\therefore r'^2 - r^2 = \frac{1}{3} a^2$. Ax. 9

Next solve the system of simultaneous equations for r and r':

$$\begin{cases} r' + r = \frac{1}{3} \sqrt{6}\, a, & (2) \\ r'^2 - r^2 = \frac{1}{3} a^2. & (3) \end{cases}$$

Divide (3) by (2).

Then $r' - r = \dfrac{a}{\sqrt{6}}$ Ax. 4

$= \frac{1}{6} \sqrt{6}\, a$. (4)

Subtract (4) from (2); also add (4) and (2).

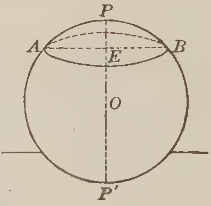
Then $2r = \frac{1}{6} \sqrt{6}\, a$, Ax. 2

and $2r' = \frac{1}{2} \sqrt{6}\, a$. Ax. 1

Whence $r = \frac{1}{12} \sqrt{6}\, a$,

and $r' = \frac{1}{4} \sqrt{6}\, a$, by Ax. 4. Q. E. F.

3. *Find the diameter of the section of a sphere of diameter* 10 *in. made by a plane* 3 *in. from the center.*



Given the sphere PBP', whose diameter PP' is 10 in., cut in the circular section AB by a plane 3 in. from the center O.

Required to find the diameter of the $\odot\ AB$.

Solution. Let PP' be $\perp$ to the plane of the $\odot\ AB$.

Then PP' intersects the $\odot\ AB$ in its center E. § 440

The section PBP' is a semicircle.

$\therefore\ PE : BE = BE : EP'$. § 297

Whence $\overline{BE}^2 = PE \times EP'$. § 261

Now $PE = OP - OE$

$= \frac{1}{2} \times 10 - 3$ Ax. 9

$= 5 - 3$, or 2;

and $EP' = PP' - PE$

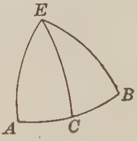
$= 10 - 2$, or 8. Ax. 9

$\therefore\ \overline{BE}^2 = 2 \times 8$, or 16. Ax. 9

$\therefore\ BE = \sqrt{16}$, or 4. Ax. 5

Therefore diameter of $\odot\ AB = 2 \times 4$ in., or 8 in., by § 163. Q. E. F.

4. *The arc of a great circle drawn from the vertex of an isosceles spherical triangle to the mid-point of the base bisects the vertical angle, is perpendicular to the base, and divides the triangle into two symmetric triangles.*



Given EC, an arc of a great $\odot$, joining the vertex E of the isosceles spherical $\triangle\ AEB$ to C, the mid-point of its base AB.

To prove that the spherical $\triangle$s AEC and BEC are symmetric, that $\angle AEC = \angle BEC$, and that CE is $\perp$ to AB.

Proof. In the spherical $\triangle$s AEC and BEC,

side AE = side BE, Given

side AC = side BC, Given

and side EC = side EC. Iden.

Moreover the equal parts are arranged in the reverse order.

Therefore the spherical $\triangle$s AEC and BEC are symmetric. § 677

Therefore $\angle AEC = \angle BEC$.

Also $\angle ECA = \angle ECB$.

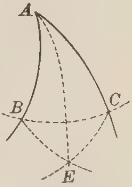
Therefore CE is $\perp$ to AB. Q. E. D.

Exercise 105. Page 406

1. *To bisect a given spherical angle.*

Given BAC any spherical ∠.

Required to bisect the spherical ∠ BAC.



Construction. From A as a pole, with a quadrant radius arc, describe arcs intersecting AB and AC at B and C respectively.

From B and C as poles, with equal radius arcs greater than one half the spherical distance BC, describe arcs intersecting at E.

Through A and E draw an arc of a great ⊙.

Then AE bisects the spherical ∠ BAC. Q.E.F.

Proof. Draw BE and CE, arcs of great Ⓢ.

In the spherical △ AEB and AEC,

side AE = side AE, Iden.

side AB = side AC, Const.

and side BE = side CE. Const.

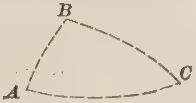
But the equal parts occur in the reverse order.

Therefore the spherical △ AEB and AEC are symmetric. § 677

Then the spherical ∠ BAE and CAE are equal.

Therefore AE bisects the spherical ∠ BAC. Q.E.D.

2. *To construct a spherical triangle, given two sides and the included angle.*



Construction. Draw AB, an arc of a great ⊙, equal to one of the given sides.

Draw BC, an arc of a great ⊙, so that the spherical ∠ ABC shall be equal to the given spherical ∠.

Then take BC equal to the second of the two given sides.

Finally, draw AC, an arc of a great ⊙.

Then obviously the △ ABC is the spherical △ required. Q.E.F.

3. *To construct a spherical triangle, given two angles and the included side.*

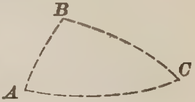
Construction. Draw an arc of a great ⊙ AB equal to the given side (see figure in Ex. 2).

From A and B draw AC and BC, arcs of great Ⓢ, so that the ∠ A and B shall be equal to the given ∠.

Produce the arcs AC and BC to intersect in C.

Then obviously the △ ABC is the spherical △ required. Q.E.F.

4. *To construct a spherical triangle, given the three sides.*



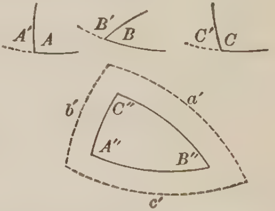
Construction. Draw AB, an arc of a great ⊙, equal to one of the given sides.

From A and B as poles, with radius arcs equal to the other two sides respectively, describe arcs intersecting at C.

Through A and C and through B and C pass arcs of great Ⓢ.

Then obviously the △ ABC is the spherical △ required. Q.E.F.

5. *To construct a spherical triangle, given the three angles.*



Given the three spherical $\angle$s A, B, and C.

Required to construct a spherical △ having $\angle$s equal to the $\angle$s A, B, and C respectively.

Construction. Construct the spherical $\angle$s A', B', and C' supplementary to the spherical $\angle$s A, B, and C respectively.

From the vertices of the spherical $\angle$s A', B', and C' as poles, with a quadrant radius arc, describe arcs of great Ⓢ.

Produce the sides of the spherical $\angle$s A', B', and C' to intercept arcs on these arcs of great Ⓢ.

Let a', b', and c' denote the arcs of great Ⓢ intercepted by the sides of the spherical $\angle$s A', B', and C' respectively.

Construct the spherical △ whose sides are the arcs a', b', and c'. Ex. 4, p. 406

Then construct $A''B''C''$, the polar △ of this △. § 665

Then the △ $A''B''C''$ is the △ required. Q.E.F.

Proof. $\angle A + \angle A' = 180°$. Const.

Since $\angle A'$ is measured by the arc a', § 654

$\therefore \angle A + \text{arc } a' = 180°$. Ax. 9

$\therefore \angle A = 180° - \text{arc } a'$. Ax. 2

Since △ $A''B''C''$ is the polar △ of the △ whose sides are the arcs a', b', and c' respectively, Const.

$\therefore \angle A'' + \text{arc } a' = 180°$. § 667

$\therefore \angle A'' = 180° - \text{arc } a'$. Ax. 2

$\therefore \angle A'' = \angle A$. Ax. 8

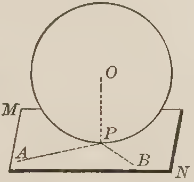
Similarly, it can be shown that $\angle B'' = \angle B$, and $\angle C'' = \angle C$.

Therefore the spherical △ $A''B''C''$ has its $\angle$s equal to the given $\angle$s respectively. Q.E.D.

6. *To pass a plane tangent to a given sphere at a given point on the surface of the sphere.*

Given the sphere whose center is O, and P a point on the surface of the sphere.

Required to pass a plane through P, tangent to the given sphere.



Construction. Draw the radius OP.

Erect the $\perp$s AP and BP to OP at P. § 228

Through AP and BP pass the plane MN. § 425

Then the plane MN is tangent to the given sphere. Q. E. F.

Proof. The lines AP and BP are each $\perp$ to OP. Const.

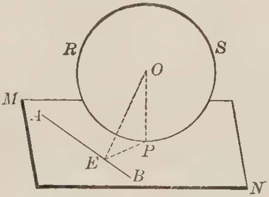
Therefore the plane MN, determined by the lines AP and BP, is $\perp$ to OP. § 431

Therefore the plane MN is tangent to the given sphere, by § 645. Q. E. D.

7. *To pass a plane tangent to a given sphere through a given straight line without the sphere.*

Given the straight line AB without the sphere whose center is O.

Required to pass through the line AB a plane tangent to the given sphere.



Construction. Through the point O pass a plane $\perp$ to AB. § 434

Let this plane intersect the line AB in the point E and let it cut the surface of the sphere whose center is O in the great circle RPS. § 625

Through E draw a line tangent to the $\odot$ RPS at P. § 246

Through P and the line AB pass the plane MN. § 426

Then the plane MN is tangent to the given sphere. Q. E. F.

Proof. Draw the radius OP.

The plane OEP is $\perp$ to AB. Const.

Therefore the plane MN is $\perp$ to the plane OEP. § 477

Now OP is $\perp$ to EP. § 185

$\therefore$ OP is $\perp$ to the plane MN. § 474

Therefore the plane MN is $\perp$ to OP. § 474

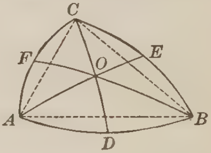
Therefore the plane MN is tangent to the given sphere, by § 645.

Q. E. D.

Exercise 106. Page 409

1. *The three medians of a spherical triangle are concurrent.*

Given AE, BF, and CD, the three medians of the spherical $\triangle ABC$ on the sphere whose center is O''.



To prove that the medians AE, BF, and CD are concurrent.

Proof. Draw the chords AB, BC, and AC, thereby forming the plane $\triangle ABC$.

Let D', E', and F' be the mid-points of the sides AB, BC, and AC respectively of the plane $\triangle ABC$.

Draw the medians AE', BF', and CD' of the plane $\triangle ABC$.

Let O' be the point of intersection of the medians AE', BF', and CD'. Ex. 5, p. 78

Through the edge $O''A$ of the trihedral $\angle O''$, at the center of the given sphere, and the straight line AE' pass the plane $AO''E'$ (§ 425), cutting the face $\angle CO''B$ of the trihedral $\angle O''$ in the line $O''E'$. § 429

In the plane ⊿s $BO''E'$ and $CO''E'$,

$$BE' = CE',$$ Const.

$$O''B = O''C,$$ § 621

and $$O''E' = O''E'.$$ Iden.

Therefore the plane ⊿s $BO''E'$ and $CO''E'$ are congruent. § 80

$$\therefore \angle BO''E' = \angle CO''E'.$$ § 67

Produce the straight line $O''E'$ to meet the arc BC of a great ⊙.

The straight line $O''E'$ will bisect the arc BC; that is, $O''E'$ will pass through E. § 166

Therefore the plane $AO''E'$ cuts the given spherical surface in the arc AE of a great ⊙.

Similarly, it can be shown that the planes $BO''F'$ and $CO''D'$ cut the given spherical surface in the arcs BF and CD, respectively, of great Ⓢ.

Draw $O''O'$.

Then $O''O'$ lies in the plane $AO''E'$. § 422

Produce $O''O'$ to meet the arc AE in O.

Since the point O' lies also in each of the planes $BO''F'$ and $CO''D'$, therefore the line $O''O'$ lies in each of the planes $BO''F'$ and $CO''D'$. § 422

Then the line $O''O'$ produced will meet the arc BF in a certain point, and the arc CD in a certain point.

But the line $O''O'$ can meet the given spherical surface in but one point within the spherical $\triangle ABC$.

Therefore O is the point of intersection of $O''O'$ with the given spherical surface.

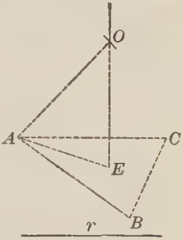
Hence $O''O'$ must meet the arcs BF and CD each in O.

That is, the arcs AE, BF, and CD pass through O.

Therefore the medians AE, BF, and CD are concurrent, by § 154.

Q. E. D.

2. *To construct with a given radius a spherical surface that passes through three given points.*



Given the points A, B, and C; also the fixed distance r.

Required to construct with the radius r a spherical surface passing through A, B, and C.

Construction. Construct the $\triangle ABC$.

Find E, the center of the circumscribed ⊙ of the $\triangle ABC$. § 240

Erect $EO \perp$ to the plane of the $\triangle ABC$ at E.

From A as a center, with r as a radius, describe an arc cutting EO in O.

About O as a center, with r as a radius, describe a spherical surface.

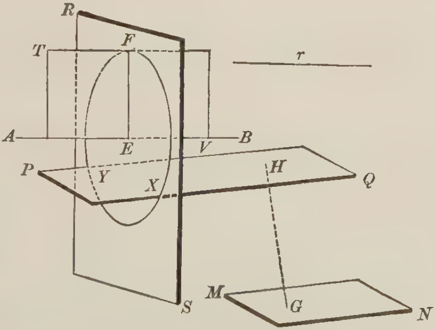
Then this spherical surface is the spherical surface required. Q. E. F.

Proof. The line EO is the locus of a point equidistant from A, B, and C. § 442

Therefore O is equidistant from A, B, and C.

Now $OA = r$. Const.

Therefore the spherical surface described about O as a center, with r as a radius, will pass through A, B, and C, by § 621. Q. E. D.



3. *To construct with a given radius a spherical surface that passes through two given points and is tangent to a given plane.*

Given the points A and B, the plane MN, and the fixed distance r.

Required to construct with the radius r a spherical surface, passing through A and B, tangent to the plane MN.

Construction. Draw AB.

Bisect AB in E. § 229

Through E pass the plane $RS \perp$ to AB. § 433

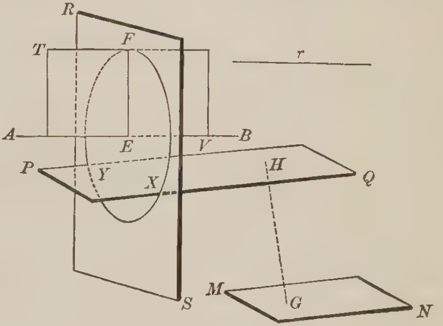
Through AB pass a plane TV intersecting the plane RS in EF. § 429

From A as a center, with r as a radius, describe an arc in the plane TV intersecting the line EF in F.

From E as a center, with EF as a radius, describe the circle FXY in the plane RS.

At any point G in the plane MN erect GH $\perp$ to the plane MN.

Take $GH = r$.



Through H pass the plane $PQ \parallel$ to the plane MN (§ 457), cutting the $\odot$ FXY in the points X and Y.

Then the spherical surface described about X or Y as a center, with r as a radius, is the spherical surface required. Q.E.F.

Proof. The points X and Y lie in the $\odot$ FXY in the plane RS, and also in the plane PQ. Const.

Since $AX = AY = r$, § 441

the points X and Y are each at the distance r from A and B. § 443

Moreover X and Y are each at the distance r from the plane MN. § 455

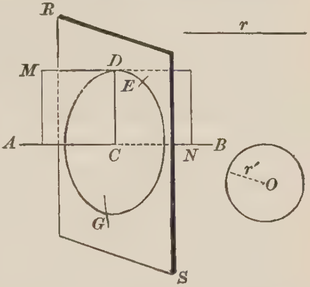
Therefore the spherical surface described about X or Y as a center, with r as a radius, passes through the points A and B and is tangent to the plane MN. Q.E.D.

Discussion. If $r < AE$, or if the plane PQ does not intersect or touch the $\odot$ FXY, there is no solution.

If the plane PQ is tangent to the $\odot$ FXY, only one spherical surface can be described which will satisfy the given conditions.

If two planes $\parallel$ to MN, one on each side of MN and at the distance r from it, are tangent to the $\odot$ FXY, there will be two spherical surfaces satisfying the conditions; and if the two planes $\parallel$ to MN intersect the $\odot$ FXY, four spherical surfaces can be described which will satisfy the conditions.

4. *To construct with a given radius a spherical surface that passes through two given points and is tangent to a given sphere.*



Given the two points A and B, the sphere whose center is O and radius r', and the fixed distance r.

Required to construct with the radius r a spherical surface passing through A and B, tangent to the given sphere.

Construction. Draw AB.

Bisect AB in C. § 229

Through C pass the plane $RS \perp$ to AB. § 433

Through AB pass a plane MN, cutting the plane RS in CD. § 429

From A as a center, with r as a radius, describe an arc in the plane MN, cutting CD in D.

From C as a center, with the radius CD, describe the $\odot\, DEG$ in the plane RS.

About O as a center, with a radius equal to $r + r'$, describe a spherical surface, cutting the $\odot\, DEG$ in the points E and G.

Then the spherical surface described about E or G as a center, with r as a radius, is the spherical surface required. Q.E.F.

Proof. The points E and G lie in the $\odot\, DEG$ in the plane RS, and also in the spherical surface described about O as a center with $r + r'$ as a radius. Const.

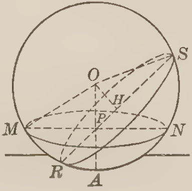
Since $AE = AG = r$, § 441

the points E and G are each at the distance r from A and B. § 443

Moreover E and G are each at the distance r from the spherical surface described about O as a center with r' as a radius. Const.

Hence a spherical surface described about E or G as center, with r as radius, passes through A and B and is tangent to the sphere. Q.E.D.

5. *The smallest circle on a given sphere whose plane passes through a given point within the sphere is the circle whose plane is perpendicular to the radius through the given point.*



Given the two circular sections MN and RS of the sphere, whose center is O, passing through a point P within the given sphere, the plane of the $\odot\, MN$ being $\perp$ to the radius OPA at P.

To prove that the $\odot\, MN <$ the $\odot\, RS$.

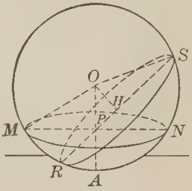
Proof. Draw $OH \perp$ to the plane of the ⊙ RS at H.

The points P and H are the centers of the ⓢ MN and RS respectively.

Draw the radii PM and HS of the ⓢ MN and RS respectively.

Also draw the radii OM and OS of the given sphere.

Now OP is $\perp$ to PM, and OH is $\perp$ to HS. § 430



In the rt. △ OPM,

$$\overline{OP}^2 + \overline{PM}^2 = \overline{OM}^2.$$ § 337

In the rt. △ OHS,

$$\overline{OH}^2 + \overline{HS}^2 = \overline{OS}^2.$$ § 337

But $OM = OS$. § 621

Whence $\overline{OM}^2 = \overline{OS}^2$. Ax. 5

$\therefore \overline{OP}^2 + \overline{PM}^2 = \overline{OH}^2 + \overline{HS}^2$. Ax. 8

Since $OP > OH$, § 438

$\therefore \overline{OP}^2 > \overline{OH}^2$. Ax. 5

$\therefore \overline{PM}^2 < \overline{HS}^2$. Ax. 7

$\therefore PM < HS$.

Therefore the ⊙ $MN <$ the ⊙ RS, by § 390. Q.E.D.

Exercise 107. Page 415

1. *Using* $\pi = 3.1416$, *find the area of a sphere whose radius is* 2 *in.*

Solution. $a = 4\pi r^2$. § 689

$\therefore a = 4 \times 2^2\pi$ Ax. 9

$= 16\pi$, or 50.2656.

$\therefore$ the area of the given sphere is 50.2656 sq. in. Q.E.F.

2. *Using* $\pi = 3.1416$, *find the area of a sphere whose radius is* 7 *in.*

Solution. $a = 4\pi r^2$. § 689

$\therefore a = 4 \times 7^2\pi$ Ax. 9

$= 196\pi$, or 615.7536.

$\therefore$ the area of the given sphere is 615.7536 sq. in. Q.E.F.

3. *Using* $\pi = 3.1416$, *find the area of a sphere whose radius is* $3\frac{1}{2}$ *in.*

Solution. $a = 4\pi r^2$. § 689

$\therefore a = 4 \times (3\frac{1}{2})^2\pi$ Ax. 9

$= 49\pi$, or 153.9384.

$\therefore$ the area of the given sphere is 153.9384 sq. in. Q.E.F.

4. *Using* $\pi = 3.1416$, *find the area of a sphere whose radius is* $5\frac{7}{8}$ *in.*

Solution. $a = 4\pi r^2$. § 689

$$\therefore\ a = 4 \times (5\tfrac{7}{8})^2\pi \qquad \text{Ax. 9}$$

$$= \frac{2209\,\pi}{16},\ \text{or}\ 433.73715.$$

Therefore the area of the given sphere is 433.73715 sq. in. Q.E.F.

5. *Using* $\pi = 3.1416$, *find the area of a sphere whose radius is* 2 *ft.* 1 *in.*

Solution. 2 ft. 1 in. = $2\tfrac{1}{12}$ ft.

Now

$$a = 4\,\pi r^2. \qquad \text{§ 689}$$

$$\therefore\ a = 4 \times (2\tfrac{1}{12})^2\pi \qquad \text{Ax. 9}$$

$$= \frac{625\,\pi}{36},\ \text{or}\ 54.542.$$

Therefore the area of the given sphere is 54.542 sq. ft. Q.E.F.

6. *Using* $\pi = 3.1416$, *find the area of a sphere whose radius is* 3 *ft.* 6 *in.*

Solution. 3 ft. 6 in. = $3\tfrac{1}{2}$ ft.

Now

$$a = 4\,\pi r^2. \qquad \text{§ 689}$$

$$\therefore\ a = 4 \times (3\tfrac{1}{2})^2\pi \qquad \text{Ax. 9}$$

$$= 49\,\pi,\ \text{or}\ 153.9384.$$

Therefore the area of the given sphere is 153.9384 sq. ft. Q.E.F.

7. *Using* $\pi = 3.1416$, *find the area of a sphere whose radius is* 48.8 *in.*

Solution.

$$a = 4\,\pi r^2. \qquad \text{§ 689}$$

$$\therefore\ a = 4 \times 48.8^2\,\pi \qquad \text{Ax. 9}$$

$$= 9525.76\,\pi,\ \text{or}\ 29{,}926.127596.$$

Therefore the area of the given sphere is 29,926.128 sq. in. Q.E.F.

8. *Using* $\pi = 3.1416$, *find the area of a sphere whose radius is* 4000 *mi.*

Solution.

$$a = 4\,\pi r^2. \qquad \text{§ 689}$$

$$\therefore\ a = 4 \times 4000^2\,\pi \qquad \text{Ax. 9}$$

$$= 64{,}000{,}000\,\pi,\ \text{or}\ 201{,}062{,}400.$$

Therefore the area of the given sphere is 201,062,400 sq. mi. Q.E.F.

9. *Using* $\pi = 3.1416$, *find the radius of a sphere whose area is* 12.5664 *sq. in.*

Solution.

$$a = 4\,\pi r^2. \qquad \text{§ 689}$$

$$\therefore\ r^2 = \frac{a}{4\,\pi}. \qquad \text{Ax. 4}$$

$$\therefore\ r = \frac{1}{2\,\pi}\sqrt{\pi a}. \qquad \text{Ax. 5}$$

$$\therefore\ r = \frac{1}{2\,\pi}\sqrt{12.5664\,\pi} \qquad \text{Ax. 9}$$

$$= \frac{1}{2\,\pi}\sqrt{4\,\pi^2},\ \text{or}\ 1.$$

Therefore the radius of the given sphere is 1 in. Q.E.F

10. *Using* $\pi = 3.1416$, *find the radius of a sphere whose area is* 50.2656 *sq. in.*

Solution. $a = 4\pi r^2$. § 689

$\therefore r^2 = \frac{a}{4\pi}$. Ax. 4

$\therefore r = \frac{1}{2\pi}\sqrt{\pi a}$. Ax. 5

$\therefore r = \frac{1}{2\pi}\sqrt{50.2656\,\pi}$ Ax. 9

$= \frac{1}{2\pi}\sqrt{16\pi^2}$, or 2.

Therefore the radius of the given sphere is 2 in. Q.E.F.

11. *Using* $\pi = 3.1416$, *find the radius of a sphere whose area is* 1 *sq. ft.*

Solution. $a = 4\pi r^2$. § 689

$\therefore r^2 = \frac{a}{4\pi}$. Ax. 4

$\therefore r = \frac{1}{2\pi}\sqrt{\pi a}$. Ax. 5

$\therefore r = \frac{1}{2\pi}\sqrt{\pi}$, or 0.282. Ax. 9

Therefore the radius of the given sphere is 0.282 ft. Q.E.F.

12. *Using* $\pi = 3.1416$, *find the radius of a sphere whose area is* 100π *sq. in.*

Solution. $a = 4\pi r^2$. § 689

$\therefore r^2 = \frac{a}{4\pi}$. Ax. 4

$\therefore r = \frac{1}{2\pi}\sqrt{\pi a}$. Ax. 5

$\therefore r = \frac{1}{2\pi}\sqrt{100\pi^2}$, or 5. Ax. 9

Therefore the radius of the given sphere is 5 in. Q.E.F.

13. *Using* $\pi = 3.1416$, *find the radius of a sphere whose area is s.*

Solution. $a = 4\pi r^2$. § 689

$\therefore r^2 = \frac{a}{4\pi}$. Ax. 4

$\therefore r = \frac{1}{2\pi}\sqrt{\pi a}$. Ax. 5

$\therefore r = \frac{1}{2\pi}\sqrt{\pi s}$, by Ax. 9. Q.E.F

14. *Using* $\pi = 3.1416$, *find the radius of a sphere whose area is* $4\,\pi^3$.

Solution. $a = 4\,\pi r^2$. § 689

$\therefore r^2 = \frac{a}{4\,\pi}$. Ax. 4

$\therefore r = \frac{1}{2\,\pi}\sqrt{\pi a}$. Ax. 5

$\therefore r = \frac{1}{2\,\pi}\sqrt{4\,\pi^4}$. Ax. 9

$\therefore r = \pi$, or 3.1416. Q.E.F.

15. *On a sphere whose radius is* 20 *in. find the area of a zone whose altitude is* 2 *in., using* $\pi = 3.1416$.

Solution. Since the area of a zone is equal to the product of its altitude by the circumference of a great ⊙, § 691

$\therefore$ the area of the zone $= 2\,(2 \times 20\,\pi)$ sq. in. Ax. 9

$= 80\,\pi$ sq. in., or 251.328 sq. in. Q.E.F.

16. *On a sphere whose radius is* 20 *in. find the area of a zone whose altitude is* 3 *in., using* $\pi = 3.1416$.

Solution. Since the area of a zone is equal to the product of its altitude by the circumference of a great ⊙, § 691

$\therefore$ the area of the zone $= 3\,(2 \times 20\,\pi)$ sq. in. Ax. 9

$= 120\,\pi$ sq. in., or 376.992 sq. in. Q.E.F.

17. *On a sphere whose radius is* 20 *in. find the area of a zone whose altitude is* 7 *in., using* $\pi = 3.1416$.

Solution. Since the area of a zone is equal to the product of its altitude by the circumference of a great ⊙, § 691

$\therefore$ the area of the zone $= 7\,(2 \times 20\,\pi)$ sq. in. Ax. 9

$= 280\,\pi$ sq. in., or 879.648 sq. in. Q.E.F.

18. *On a sphere whose radius is* 20 *in. find the area of a zone whose altitude is* 10 *in., using* $\pi = 3.1416$.

Solution. Since the area of a zone is equal to the product of its altitude by the circumference of a great ⊙, § 691

$\therefore$ the area of the zone $= 10\,(2 \times 20\,\pi)$ sq. in. Ax. 9

$= 400\,\pi$ sq. in., or 1256.64 sq. in. Q.E.F.

19. *On a sphere whose radius is* 20 *in. find the area of a zone whose altitude is* 1 *ft., using* $\pi = 3.1416$.

Solution. Since the area of a zone is equal to the product of its altitude by the circumference of a great ⊙, § 691

$\therefore$ the area of the zone $= 12\,(2 \times 20\,\pi)$ sq. in. Ax. 9

$= 480\,\pi$ sq. in., or 1507.968 sq. in. Q.E.F.

20. *On a sphere whose radius is 20 in. find the area of a zone whose altitude is $2\frac{1}{2}$ in., using $\pi = 3.1416$.*

Solution. Since the area of a zone is equal to the product of its altitude by the circumference of a great ⊙, § 691

$\therefore$ the area of the zone $= 2\frac{1}{2}(2 \times 20\pi)$ sq. in. Ax. 9

$= 100\pi$ sq. in., or 314.16 sq. in. Q.E.F.

21. *On a sphere whose radius is 20 in. find the area of a zone whose altitude is 3.45 in., using $\pi = 3.1416$.*

Solution. Since the area of a zone is equal to the product of its altitude by the circumference of a great ⊙, § 691

$\therefore$ the area of the zone $= 3.45(2 \times 20\pi)$ sq. in. Ax. 9

$= 138\pi$ sq. in., or 433.5408 sq. in. Q.E.F.

22. *On a sphere whose radius is 20 in. find the area of a zone whose altitude is 6.83 in., using $\pi = 3.1416$.*

Solution. Since the area of a zone is equal to the product of its altitude by the circumference of a great ⊙, § 691

$\therefore$ the area of the zone $= 6.83(2 \times 20\pi)$ sq. in. Ax. 9

$= 273.2\pi$ sq. in., or 858.285 sq. in. Q.E.F.

23. *On a sphere whose radius is 10 in. find the area of a lune whose angle is 30°, using $\pi = 3.1416$.*

Solution. $l : s = a : 4$ rt. $\measuredangle$s. § 694

Here $s = 4\pi r^2$ § 689

$= 400\pi.$ Ax. 9

$\therefore l : 400\pi = 30 : 360$ Ax. 9

$= 1 : 12.$

$\therefore 12\,l = 400\pi.$ § 261

$\therefore l = 33\frac{1}{3}\pi$, or 104.72. Ax. 4

Therefore the area of the given lune is 104.72 sq. in. Q.E.F.

24. *On a sphere whose radius is 10 in. find the area of a lune whose angle is 45°, using $\pi = 3.1416$.*

Solution. $l : s = a : 4$ rt. $\measuredangle$s. § 694

Here $s = 4\pi r^2$ § 689

$= 400\pi.$ Ax. 9

$\therefore l : 400\pi = 45 : 360$ Ax. 9

$= 1 : 8.$

$\therefore 8\,l = 400\pi.$ § 261

$\therefore l = 50\pi$, or 157.08. Ax. 4

Therefore the area of the given lune is 157.08 sq. in. Q.E.F.

25. *On a sphere whose radius is* 10 *in. find the area of a lune whose angle is* 90°, *using* $\pi = 3.1416$.

Solution. $l : s = a : 4$ rt. ∠s. § 694
Here $s = 4\pi r^2$ § 689
$= 400\pi$. Ax. 9
$\therefore l : 400\pi = 90 : 360$ Ax. 9
$= 1 : 4$.
$\therefore 4l = 400\pi$. § 261
$\therefore l = 100\pi$, or 314.16. Ax. 4

Therefore the area of the given lune is 314.16 sq. in. Q.E.F.

26. *On a sphere whose radius is* 10 *in. find the area of a lune whose angle is* 180°, *using* $\pi = 3.1416$.

Solution. $l : s = a : 4$ rt. ∠s. § 694
Here $s = 4\pi r^2$ § 689
$= 400\pi$. Ax. 9
$\therefore l : 400\pi = 180 : 360$ Ax. 9
$= 1 : 2$.
$\therefore 2l = 400\pi$. § 261
$\therefore l = 200\pi$, or 628.32. Ax. 4

Therefore the area of the given lune is 628.32 sq. in. Q.E.F.

27. *On a sphere whose radius is* 10 *in. find the area of a lune whose angle is* 22° 30′, *using* $\pi = 3.1416$.

Solution. $l : s = a : 4$ rt. ∠s. § 694
Here $s = 4\pi r^2$ § 689
$= 400\pi$. Ax. 9
$\therefore l : 400\pi = 22\frac{1}{2} : 360$ Ax. 9
$= 1 : 16$.
$\therefore 16l = 400\pi$. § 261
$\therefore l = 25\pi$, or 78.54. Ax. 4

Therefore the area of the given lune is 78.54 sq. in. Q.E.F.

28. *On a sphere whose radius is* 10 *in. find the area of a lune whose angle is* 7° 30′, *using* $\pi = 3.1416$.

Solution. $l : s = a : 4$ rt. ∠s. § 694
Here $s = 4\pi r^2$ § 689
$= 400\pi$. Ax. 9
$\therefore l : 400\pi = 7\frac{1}{2} : 360$ Ax. 9
$= 1 : 48$.
$\therefore 48l = 400\pi$. § 261
$\therefore l = \frac{400\pi}{48}$, or 26.18. Ax. 4

Therefore the area of the given lune is 26.18 sq. in Q.E.F.

29. *On a sphere whose radius is* 10 *in. find the area of a lune whose angle is* 52° 20′ 20″, *using* $\pi = 3.1416$.

Solution. $l : s = a : 4$ rt. ∡s. § 694

Here $s = 4\pi r^2$ § 689

$= 400\pi$. Ax. 9

Moreover $a = 52° 20' 20'' = 188{,}420''$,

and 4 rt. ∡s $= 360° = 1{,}296{,}000''$.

$\therefore l : 400\pi = 188{,}420 : 1{,}296{,}000$ Ax. 9

$= 9421 : 64{,}800$.

$\therefore 64{,}800\, l = 3{,}768{,}400\,\pi$. § 261

$$\therefore l = \frac{37684\,\pi}{648}$$ Ax. 4

$$= \frac{9421\,\pi}{162}, \text{ or } 182.6976.$$

Therefore the area of the given lune is 182.6976 sq. in. Q.E.F.

30. *On a sphere whose radius is* 10 *in. find the area of a lune whose angle is* 48° 35′ 10″, *using* $\pi = 3.1416$.

Solution $l : s = a : 4$ rt. ∡s. § 694

Here $s = 4\pi r^2$ § 689

$= 400\pi$. Ax. 9

Moreover $a = 48° 35' 10'' = 174{,}910''$,

and 4 rt. ∡s $= 360° = 1{,}296{,}000''$.

$\therefore l : 400\pi = 174{,}910 : 1{,}296{,}000$ Ax. 9

$= 17{,}491 : 129{,}600$.

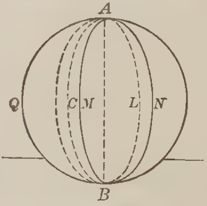
$\therefore 129{,}600\, l = 6{,}996{,}400\,\pi$. § 261

$$\therefore l = \frac{69964\,\pi}{1296}$$ Ax. 4

$$= \frac{17491\,\pi}{324}, \text{ or } 169.5979.$$

Therefore the area of the given lune is 169.5979 sq. in. Q.E.F.

31. *Two lunes on the same sphere or on equal spheres have the same ratio as their angles.*



Given the lunes $AQBM$ and $AMBN$ on the sphere AB, having the ∡s QAM and MAN.

To prove that $\dfrac{\text{lune } AQBM}{\text{lune } AMBN} = \dfrac{\angle QAM}{\angle MAN}$.

Proof. Case 1. When the ∡s QAM and MAN are commensurable.

Let the $\angle CAM$ be a common measure of the ∡s QAM and MAN.

Suppose the $\angle CAM$ is contained m times in the $\angle QAM$, and n times in the $\angle MAN$.

Then $$\frac{\angle QAM}{\angle MAN} = \frac{m}{n}.$$

Apply the $\angle CAM$ as a unit of measure to the $\angle$s QAM and MAN, and at the points of division draw arcs of great Ⓢ to B.

Then the lunes $AQBM$ and $AMBN$ are divided into m and n equal lunes respectively.

$$\therefore \frac{\text{lune } AQBM}{\text{lune } AMBN} = \frac{m}{n}.$$

$$\therefore \frac{\text{lune } AQBM}{\text{lune } AMBN} = \frac{\angle QAM}{\angle MAN}. \qquad \text{Ax. 8}$$

The above proof holds true when the given lunes are on equal spheres. Q. E. D.

Case 2. When the $\angle$s QAM and MAN are incommensurable.

Let the $\angle QAM$ be divided into any number of equal $\angle$s.

Apply one of these equal $\angle$s as a unit of measure to the $\angle MAN$.

Since the $\angle$s QAM and MAN are incommensurable, the $\angle MAN$ will contain the unit of measure a certain number of times, leaving a remainder, the $\angle LAN$, smaller than the unit of measure.

Draw the arc of a great ⊙ ALB.

Then, since the $\angle$s QAM and MAL are commensurable,

$$\therefore \frac{\text{lune } AQBM}{\text{lune } AMBL} = \frac{\angle QAM}{\angle MAL}. \qquad \text{Case 1}$$

Now suppose the number of $\angle$s into which the $\angle QAM$ is divided to be indefinitely increased.

Then the unit of measure will be indefinitely diminished, and the remainder, the $\angle LAN$, will approach zero as a limit.

$\therefore$ the lune $AMBL$ will approach the lune $AMBN$ as a limit, and the $\angle MAL$ will approach the $\angle MAN$ as a limit.

$$\therefore \frac{\text{lune } AQBM}{\text{lune } AMBL} \text{ will approach } \frac{\text{lune } AQBM}{\text{lune } AMBN} \text{ as a limit,}$$

and $$\frac{\angle QAM}{\angle MAL} \text{ will approach } \frac{\angle QAM}{\angle MAN} \text{ as a limit.}$$

But $\dfrac{\text{lune } AQBM}{\text{lune } AMBL}$ is always equal to $\dfrac{\angle QAM}{\angle MAL}$, as the $\angle MAL$ varies in value and approaches the $\angle MAN$ as a limit. Case 1

$$\therefore \frac{\text{lune } AQBM}{\text{lune } AMBN} = \frac{\angle QAM}{\angle MAN}. \qquad \S\ 207$$

The above proof holds true when the given lunes are on equal spheres. Q. E. D.

32. *The area of a lune is equal to one ninetieth of the area of a great circle multiplied by the number of degrees in the angle of the lune.*

Given l the area, and a the number of degrees in the angle, of a lune on a sphere whose radius is r.

To prove that $l = \frac{1}{90}\pi r^2 \cdot a$.

Proof. $l : s = a : 4 \text{ rt. } \angle\text{s}$. § 694

$\therefore\ l : 4\pi r^2 = a : 360$. Ax. 9

$\therefore\ 360\, l = 4\pi r^2 \cdot a$. § 261

$\therefore\ 90\, l = \pi r^2 \cdot a$. Ax. 4

$\therefore\ l = \frac{1}{90}\pi r^2 \cdot a$, by Ax. 4. Q. E. D.

33. *Zones on the same sphere or on equal spheres are to each other as their altitudes.*

Given z and z' the areas of two zones, with altitudes a and a' respectively, on the same sphere of radius r.

To prove that $\frac{z}{z'} = \frac{a}{a'}$.

Proof. $z = 2\pi ra$,

and $z' = 2\pi ra'$. § 691

$$\therefore\ \frac{z}{z'} = \frac{2\pi ra}{2\pi ra'} = \frac{a}{a'}.$$

The above proof holds true also for zones on equal spheres. Q. E. D.

34. *Given the radius of a sphere* 15 *in., find the area of a lune whose angle is* 30°.

Solution. $l : s = a : 4 \text{ rt. } \angle\text{s}$. § 694

Here $s = 4\pi r^2$ § 689

$= 900\pi$. Ax. 9

$\therefore\ l : 900\pi = 30 : 360$ Ax. 9

$= 1 : 12$.

$\therefore\ 12\, l = 900\pi$. § 261

$\therefore\ l = 75\pi$, or 235.62. Ax. 4

Therefore the area of the given lune is 235.62 sq. in. Q. E. F.

35. *Given the diameter of a sphere* 16 *in., find the area of a lune whose angle is* 75°.

Solution. $l : s = a : 4 \text{ rt. } \angle\text{s}$. § 694

Here $s = 4\pi r^2$ § 689

$= 4 \times (\frac{16}{2})^2 \pi$ Ax. 9

$= 256\pi$.

$\therefore\ l : 256\pi = 75 : 360$ Ax. 9

$= 5 : 24$.

$\therefore 24\,l = 1280\,\pi.$ § 261

$\therefore l = \frac{1280\,\pi}{24}$, or 167.552. Ax. 4

Therefore the area of the given lune is 167.552 sq. in. Q.E.F.

36. *What is the spherical excess of a trirectangular triangle?*

Solution. The required spherical excess $= 3 \times 90° - 180°$, or 90°, by § 693. Q.E.F.

Exercise 108. Page 417

1. *Compute the area of a triangle on a sphere of diameter* 16 *in., the angles being* 100°, 120°, 140°.

Solution. The spherical excess of the given △

$= 100° + 120° + 140° - 180° = 180°.$ § 693

Therefore the given spherical △ is equivalent to a lune whose angle is half 180°, or 90°. § 695

But the area of such a lune $= \frac{90}{360}$ of the spherical surface § 694

$= \frac{1}{4} \times 4\,\pi r^2$ Ax. 9

$= (\frac{16}{2})^2\,\pi$ Ax. 9

$= 64\,\pi$, or 201.0624.

Therefore the area of the given spherical △ is 201.0624 sq. in. Q.E.F.

2. *Compute the area of a triangle on a sphere of diameter* 10 *in., the angles being* 105°, 130°, 125°.

Solution. The spherical excess of the given △

$= 105° + 130° + 125° - 180° = 180°.$ § 693

Therefore the given spherical △ is equivalent to a lune whose angle is half 180°, or 90°. § 695

But the area of such a lune $= \frac{90}{360}$ of the spherical surface § 694

$= \frac{1}{4} \times 4\,\pi r^2$ Ax. 9

$= (\frac{10}{2})^2\,\pi$ Ax. 9

$= 25\,\pi$, or 78.54.

Therefore the area of the given spherical △ is 78.54 sq. in. Q.E.F.

3. *Compute the area of a triangle on a sphere of diameter* 20 *in., the angles being* 127°, 132°, 90°.

Solution. The spherical excess of the given △

$= 127° + 132° + 90° - 180° = 169°.$ § 693

Therefore the given spherical △ is equivalent to a lune whose angle is half 169°, or $84\frac{1}{2}°$. § 695

But the area of such a lune $= \frac{84\frac{1}{2}}{360}$ of the spherical surface § 694

$= \frac{84\frac{1}{2}}{360} \times 4\pi r^2$ Ax. 9

$= \frac{169}{720} \times 4 \times (\frac{20}{2})^2 \pi$ Ax. 9

$= \frac{1690\,\pi}{18}$, or 294.961.

Therefore the area of the given spherical △ is 294.961 sq. in. Q.E.F.

4. *Compute the area of a triangle on a sphere of diameter* 30 *in., the angles being* 115°, 124°, 85°.

Solution. The spherical excess of the given △

$= 115° + 124° + 85° - 180° = 144°$. § 693

Therefore the given spherical △ is equivalent to a lune whose angle is half 144°, or 72°. § 695

But the area of such a lune $= \frac{72}{360}$ of the spherical surface § 694

$= \frac{1}{5} \times 4\pi r^2$ Ax. 9

$= \frac{4}{5} \times (\frac{30}{2})^2 \pi$ Ax. 9

$= 180\,\pi$, or 565.488.

Therefore the area of the given spherical △ is 565.488 sq. in. Q.E.F.

5. *Compute the area of a triangle on a sphere of diameter* 40 *in., the angles being* 135°, 110°, 92°.

Solution. The spherical excess of the given △

$= 135° + 110° + 92° - 180° = 157°$. § 693

Therefore the given spherical △ is equivalent to a lune whose angle is half 157°, or $78\frac{1}{2}$°. § 695

But the area of such a lune $= \frac{78\frac{1}{2}}{360}$ of the spherical surface § 694

$= \frac{157}{720} \times 4\pi r^2$ Ax. 9

$= \frac{157}{180} \times (\frac{40}{2})^2 \pi$ Ax. 9

$= \frac{3140\,\pi}{9}$, or 1096.069.

Therefore the area of the given spherical △ is 1096.069 sq. in. Q.E.F.

6. *Compute the area of a triangle on a sphere of diameter* 25.8 *in., the angles being* 148°, 93°, 68°.

Solution. The spherical excess of the given △

$= 148° + 93° + 68° - 180° = 129°$. § 693

Therefore the given spherical △ is equivalent to a lune whose angle is half 129°, or $64\frac{1}{2}°$. § 695

But the area of such a lune $= \dfrac{64\frac{1}{2}}{360}$ of the spherical surface § 694

$$= \tfrac{129}{720} \times 4\pi r^2 \qquad \text{Ax. 9}$$

$$= \frac{129}{180} \times \left(\frac{25.8}{2}\right)^2 \pi \qquad \text{Ax. 9}$$

$$= \tfrac{43}{60} \times 166.41\,\pi$$

$$= 374.6688.$$

Therefore the area of the given spherical △ is 374.6688 sq. in. Q.E.F.

7. *Compute the area of a triangle on a sphere of diameter* 8000 *mi., the angles being* 115° 27′ 30″, 102° 32′ 48″, 68° 27′ 39″.

Solution. The spherical excess of the given △

$$= 115°\,27'\,30'' + 102°\,32'\,48'' + 68°\,27'\,39'' - 180°$$

$$= 106°\,27'\,57''. \qquad \S\,693$$

Therefore the given spherical △ is equivalent to a lune whose angle is half 106° 27′ 57″, or 53° 13′ 58.5″. § 695

But the area of such a lune $= \dfrac{53°\,13'\,58.5''}{360°}$

$$= \tfrac{1916385}{12960000} \text{ of the spherical surface} \qquad \S\,694$$

$$= \tfrac{1916385}{12960000} \times 4\pi r^2 \qquad \text{Ax. 9}$$

$$= \tfrac{1916385}{3240000} \times \left(\tfrac{8000}{2}\right)^2 \pi \qquad \text{Ax. 9}$$

$$= \tfrac{1916385}{3240000} \times 16{,}000{,}000\,\pi$$

$$= \tfrac{1916385}{81} \times 400\,\pi$$

$$= 29{,}730{,}938.844.$$

Therefore the area of the given spherical △ is 29,730,938.844 sq. mi.
Q.E.F.

8. *Compute the area of a triangle on a sphere of radius* 9 *in., the angles being* 120°, 100°, 90°.

Solution. The spherical excess of the given △

$$= 120° + 100° + 90° - 180° = 130°. \qquad \S\,693$$

Therefore the given spherical △ is equivalent to a lune whose angle is half 130°, or 65°. § 695

But the area of such a lune $= \tfrac{65}{360}$ of the spherical surface § 694

$$= \tfrac{13}{72} \times 4\pi r^2 \qquad \text{Ax. 9}$$

$$= \tfrac{13}{18} \times 9^2 \pi \qquad \text{Ax. 9}$$

$$= \frac{117\,\pi}{2}, \text{ or } 183.7836.$$

Therefore the area of the given spherical △ is 183.7836 sq. in. Q.E.F.

9. *Compute the area of a triangle on a sphere of radius* 10 *in., the angles being* 130°, 90°, 80°.

Solution. The spherical excess of the given △

$= 130° + 90° + 80° - 180° = 120°$. § 693

Therefore the given spherical △ is equivalent to a lune whose angle is half 120°, or 60°. § 695

But the area of such a lune $= \frac{60}{360}$ of the spherical surface § 694

$= \frac{1}{6} \times 4\pi r^2$ Ax. 9

$= \frac{2}{3} \times 10^2 \pi$ Ax. 9

$= \frac{200\pi}{3}$, or 209.44.

Therefore the area of the given spherical △ is 209.44 sq. in. Q. E. F.

10. *Compute the area of a triangle on a sphere of radius* 18 *in., the angles being* 105°, 75°, 65°.

Solution. The spherical excess of the given △

$= 105° + 75° + 65° - 180° = 65°$. § 693

Therefore the given spherical △ is equivalent to a lune whose angle is half 65°, or $32\frac{1}{2}°$. § 695

But the area of such a lune $= \frac{32\frac{1}{2}}{360}$ of the spherical surface § 694

$= \frac{13}{144} \times 4\pi r^2$ Ax. 9

$= \frac{13}{36} \times 18^2 \pi$ Ax. 9

$= 117\pi$, or 367.5672.

Therefore the area of the given spherical △ is 367.5672 sq. in. Q. E. F.

11. *Compute the area of a triangle on a sphere of radius* 36 *in., the angles being* 115°, 102°, 30°.

Solution. The spherical excess of the given △

$= 115° + 102° + 30° - 180° = 67°$. § 693

Therefore the given spherical △ is equivalent to a lune whose angle is half 67°, or $33\frac{1}{2}°$. § 695

But the area of such a lune $= \frac{33\frac{1}{2}}{360}$ of the spherical surface § 694

$= \frac{67}{720} \times 4\pi r^2$ Ax. 9

$= \frac{67}{180} \times 36^2 \pi$ Ax. 9

$= \frac{67}{5} \times 36\pi$

$= 1515.50784$.

Therefore the area of the given spherical △ is 1515.50784 sq. in. Q. E. F.

12. *Compute the area of a triangle on a sphere of radius* 90 *in., the angles being* 140°, 120°, 85°.

Solution. The spherical excess of the given △

$= 140° + 120° + 85° - 180° = 165°$. § 693

Therefore the given spherical △ is equivalent to a lune whose angle is half 165°, or $82\frac{1}{2}$°. § 695

But the area of such a lune $= \frac{82\frac{1}{2}}{360}$ of the spherical surface § 694

$= \frac{165}{720} \times 4\pi r^2$ Ax. 9

$= \frac{11}{12} \times 90^2\pi$ Ax. 9

$= 7425\pi$, or 23,326.38.

Therefore the area of the given spherical △ is 23,326.38 sq. in. Q.E.F.

13. *Compute the area of a triangle on a sphere of radius* 1.8 *in., the angles being* 136°, 117°, 93°.

Solution. The spherical excess of the given △

$= 136° + 117° + 93° - 180° = 166°$. § 693

Therefore the given spherical △ is equivalent to a lune whose angle is half 166°, or 83°. § 695

But the area of such a lune $= \frac{83}{360}$ of the spherical surface § 694

$= \frac{83}{360} \times 4\pi r^2$ Ax. 9

$= \frac{83}{90} \times 1.8^2\pi$ Ax. 9

$= \frac{83}{50} \times 1.8\pi$

$= .83 \times 3.6\pi$

$= 9.387.$

Therefore the area of the given spherical △ is 9.387 sq. in. Q.E.F.

14. *Compute the area of a triangle on a sphere of circumference* 31.416 *in., the angles being* 93°, 94°, 120°.

Solution. The spherical excess of the given △

$= 93° + 94° + 120° - 180° = 127°$. § 693

Therefore the given spherical △ is equivalent to a lune whose angle is half 127°, or $63\frac{1}{2}$°. § 695

But the area of such a lune $= \frac{63\frac{1}{2}}{360}$ of the spherical surface § 694

$= \frac{127}{720} \times 4\pi r^2$ Ax. 9

$= \frac{127}{720} \times 4\left(\frac{c}{2\pi}\right)^2\pi$ Ax. 9

$= \frac{127}{720} \times \frac{c^2}{\pi}$

$= \frac{127}{720} \times \frac{31.416^2}{\pi}$

$= \frac{127}{720} \times 314.16$, or 55.414.

Therefore the area of the given spherical △ is 55.414 sq. in. Q.E.F.

15. *Compute the area of a triangle on a sphere of circumference* 62.832 *in., the angles being* 82°, 105°, 98°.

Solution. The spherical excess of the given △

$$= 82^\circ + 105^\circ + 98^\circ - 180^\circ = 105^\circ. \qquad \S\ 693$$

Therefore the given spherical △ is equivalent to a lune whose angle is half 105°, or $52\frac{1}{2}^\circ$. § 695

But the area of such a lune $= \dfrac{52\frac{1}{2}}{360}$ of the spherical surface § 694

$$= \tfrac{7}{48} \times 4\pi r^2 \qquad \text{Ax. 9}$$

$$= \frac{7}{48} \times 4\left(\frac{c}{2\pi}\right)^2 \pi \qquad \text{Ax. 9}$$

$$= \frac{7}{48} \times \frac{c^2}{\pi}$$

$$= \frac{7}{48} \times \frac{62.832^2}{\pi}$$

$$= \tfrac{7}{48} \times 1256.64, \text{ or } 183.26.$$

Therefore the area of the given spherical △ is 183.26 sq. in. Q.E.F.

16. *Compute the area of a triangle on a sphere of circumference* 15.708 *in., the angles being* 148°, 27°, 125°.

Solution. The spherical excess of the given △

$$= 148^\circ + 27^\circ + 125^\circ - 180^\circ = 120^\circ. \qquad \S\ 693$$

Therefore the given spherical △ is equivalent to a lune whose angle is half 120°, or 60°. § 695

But the area of such a lune $= \frac{60}{360}$ of the spherical surface § 694

$$= \tfrac{1}{6} \times 4\pi r^2 \qquad \text{Ax. 9}$$

$$= \frac{1}{6} \times 4\left(\frac{c}{2\pi}\right)^2 \pi \qquad \text{Ax. 9}$$

$$= \frac{1}{6} \times \frac{c^2}{\pi}$$

$$= \frac{1}{6} \times \frac{15.708^2}{\pi}$$

$$= \tfrac{1}{6} \times 78.54, \text{ or } 13.09.$$

Therefore the area of the given spherical △ is 13.09 sq. in. Q.E.F.

17. *Compute the area of a triangle on a sphere of circumference* 78.54 *in., the angles being* 162°, 39°, 120°.

Solution. The spherical excess of the given △

$$= 162^\circ + 39^\circ + 120^\circ - 180^\circ = 141^\circ. \qquad \S\ 693$$

Therefore the given spherical △ is equivalent to a lune whose angle is half 141°, or $70\frac{1}{2}$°. § 695

But the area of such a lune $= \frac{70\frac{1}{2}}{360}$ of the spherical surface § 694

$= \frac{47}{240} \times 4\pi r^2$ Ax. 9

$= \frac{47}{240} \times 4\left(\frac{c}{2\pi}\right)^2 \pi$ Ax. 9

$= \frac{47}{240} \times \frac{c^2}{\pi}$

$= \frac{47}{240} \times \frac{78.54^2}{\pi}$

$= \frac{47}{240} \times 1963.5$, or 384.52.

Therefore the area of the given spherical △ is 384.52 sq. in. Q.E.F.

18. *Compute the area of a triangle on a sphere of circumference* 39.27 *in., the angles being* 149°, 41°, 116°.

Solution. The spherical excess of the given △

$= 149° + 41° + 116° - 180° = 126°$. § 693

Therefore the given spherical △ is equivalent to a lune whose angle is half 126°, or 63°. § 695

But the area of such a lune $= \frac{63}{360}$ of the spherical surface § 694

$= \frac{7}{40} \times 4\pi r^2$ Ax. 9

$= \frac{7}{40} \times 4\left(\frac{c}{2\pi}\right)^2 \pi$ Ax. 9

$= \frac{7}{40} \times \frac{c^2}{\pi}$

$= \frac{7}{40} \times \frac{39.27^2}{\pi}$

$= \frac{7}{40} \times 490.875$, or 85.903.

Therefore the area of the given spherical △ is 85.903 sq. in. Q.E.F.

19. *Compute the area of a triangle on a sphere of circumference* 314.16 *in., the angles being* 126° 30′ 42″, 105° 26′ 15″, 63° 15′ 3″.

Solution. The spherical excess of the given △

$= 126° 30' 42'' + 105° 26' 15'' + 63° 15' 3'' - 180°$

$= 115° 12'$. § 693

Therefore the given spherical △ is equivalent to a lune whose angle is half 115° 12′, or 57° 36′. § 695

But the area of such a lune $= \dfrac{57^\circ\,36'}{360^\circ}$

$= \frac{3456}{21600}$ of the spherical surface § 694

$= 0.16 \times 4\,\pi r^2$ Ax. 9

$= 0.16 \times 4\left(\dfrac{c}{2\,\pi}\right)^2 \pi$ Ax. 9

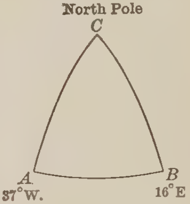
$= 0.16 \times \dfrac{c^2}{\pi}$

$= 0.16 \times \dfrac{314.16^2}{\pi}$

$= 0.16 \times 31{,}416$, or 5026.56.

Therefore the area of the given spherical △ is 5026.56 sq. in. Q.E.F.

20. *What is the area of a triangle on the earth's surface the vertices of which are the north pole and two points on the equator, one at* 37° *W. and the other at* 16° *E., the earth being considered a sphere with a radius of* 4000 *mi.?*



Given the spherical △ *ABC* on the earth's surface, the vertex *C* being at the north pole and the vertices *A* and *B* being on the equator at 37° W. and 16° E. longitude respectively; also the radius of the earth = 4000 mi.

Required to find the area of the spherical △ *ABC*.

Solution. The arcs *AC* and *BC* are quadrants. § 639

∴ *C* is a pole of the arc of a great ⊙ *AB*. § 641

∴ ∠*C* is measured by the arc *AB*. § 654

Since the arc $AB = 37^\circ + 16^\circ$, or 53°, Ax. 11

∴ $\angle C = 53^\circ$. Ax. 8

The ∠s *A* and *B* each equal 90°. § 656

Therefore the spherical excess of the △ *ABC*

$= 53^\circ + 90^\circ + 90^\circ - 180^\circ$, or 53°. § 693

Therefore the given spherical △ is equivalent to a lune whose angle is half 53°, or $26\frac{1}{2}^\circ$. § 695

But the area of such a lune $= \dfrac{26\frac{1}{2}}{360}$ of the earth's surface § 694

$= \frac{53}{720} \times 4\,\pi r^2$ Ax. 9

$= \frac{53}{180} \times 4000^2\,\pi$ Ax. 9

$= \frac{53}{180} \times 16{,}000{,}000 \times 3.1416$

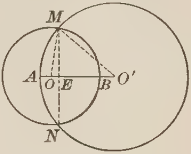
$= \frac{53}{3} \times 800{,}000 \times 1.0472$

$= 14{,}800{,}426.67$.

Therefore the area of the spherical △ *ABC* is 14,800,426.67 sq. mi.

Q.E.F.

21. *If the radii of two spheres are 6 in. and 4 in. respectively, and the distance between the centers is 5 in., what is the area of the circle of intersection of the spheres?*



Given the two spheres whose centers are O and O', intersecting in the $\odot\, MN$, their radii being $OB = 4$ in. and $O'A = 6$ in.; also $OO' = 5$ in.

Required to find the area of the $\odot\, MN$.

Solution. Draw the radii OM and $O'M$.

Also draw the diameter MN of the $\odot\, MN$.

The line OO' is $\perp$ to the $\odot\, MN$ at its center E. § 652

$\therefore OO'$ is $\perp$ to MN. § 430

That is, ME is $\perp$ to OO'.

In the rt. $\triangle\, OEM$, $\overline{OM}^2 - \overline{OE}^2 = \overline{ME}^2$. § 338

In the rt. $\triangle\, O'EM$, $\overline{O'M}^2 - \overline{O'E}^2 = \overline{ME}^2$. § 338

$\therefore \overline{OM}^2 - \overline{OE}^2 = \overline{O'M}^2 - \overline{O'E}^2$. Ax. 8

$\therefore \overline{O'E}^2 - \overline{OE}^2 = \overline{O'M}^2 - \overline{OM}^2$. Axs. 1, 2

$\therefore \overline{O'E}^2 - (OO' - O'E)^2 = \overline{O'M}^2 - \overline{OM}^2$. Ax. 9

$\therefore \overline{O'E}^2 - (5 - O'E)^2 = 6^2 - 4^2$. Ax. 9

$\therefore \overline{O'E}^2 - (25 - 10 \times O'E + \overline{O'E}^2) = 36 - 16$.

$\therefore \overline{O'E}^2 - 25 + 10 \times O'E - \overline{O'E}^2 = 20$.

$\therefore 10 \times O'E = 20 + 25$ Ax. 1

$= 45$.

$\therefore O'E = 4.5$. Ax. 4

Since $\overline{ME}^2 = \overline{O'M}^2 - \overline{O'E}^2$, Proved

$\therefore \overline{ME}^2 = 6^2 - 4.5^2$ Ax. 9

$= 36 - 20.25$

$= 15.75$.

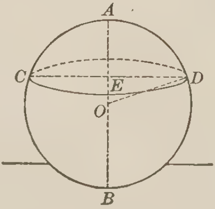
$\therefore$ the area of the $\odot\, MN$, which is equal to $\pi \cdot \overline{ME}^2$, § 389

$= 15.75 \times 3.1416$

$= 49.4802$.

Therefore the area of the $\odot\, MN$ is 49.4802 sq. in. Q. E. F.

22. *Find the radius of the circle determined on a sphere of 5 in. diameter by a plane 1 in. from the center.*



Given the sphere whose center is O and diameter 5 in., cut by a plane in the circular section CD, 1 in. from the center O.

Required to find the radius of the $\odot\, CD$.

Solution. Draw the diameter AB ⊥ to the plane of the $\odot\, CD$, cutting the $\odot\, CD$ in its center E.

Draw the radius ED of the ⊙ CD.

Also draw the radius OD of the sphere.

Now OE is ⊥ to the plane of the ⊙ CD. § 623

∴ OE is ⊥ to ED. § 430

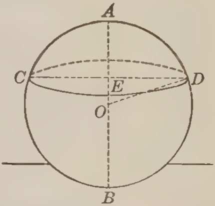
In the rt. △ OED, $\overline{ED}^2 = \overline{OD}^2 - \overline{OE}^2$ § 338

$= (\frac{5}{2})^2 - 1$ Ax. 9

$= \frac{25}{4} - 1$, or $\frac{21}{4}$.

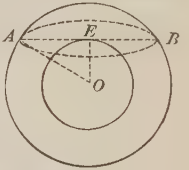
∴ $ED = \sqrt{\frac{21}{4}}$ Ax. 5

$= \frac{1}{2}\sqrt{21}$.



Therefore the radius of the ⊙ CD is $\frac{1}{2}\sqrt{21}$ in. Q.E.F.

23. *If the radii of two concentric spheres are r and r', and if a plane is passed tangent to the interior sphere, what is the area of the section made in the other sphere?*



Given O, the common center of the two concentric spheres, r and r' being the radii of the exterior and interior spheres respectively; also a plane cutting the exterior sphere in the circular section AB, which is tangent to the interior sphere at E.

Required to find the area of the ⊙ AB in terms of r and r'.

Solution. Draw the radius OE of the interior sphere.

Also draw EA and OA.

Now OE is ⊥ to the plane of the ⊙ AB. § 646

∴ OE is ⊥ to AB. § 430

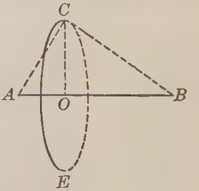
In the rt. △ OEA, $\overline{EA}^2 = \overline{OA}^2 - \overline{OE}^2$ § 338

$= r^2 - r'^2$. Ax. 9

∴ the area of the ⊙ AB, which is equal to $\pi \cdot \overline{EA}^2$, § 389

$= \pi(r^2 - r'^2)$, by Ax. 9. Q.E.F.

24. *Two points A and B are 8 in. apart. Find the locus in space of a point 5 in. from A and 7 in. from B.*



Given the points A and B 8 in. apart.

Required to find the locus in space of a point 5 in. from A and 7 in. from B.

Construction. Draw AB.

Construct the △ ACB, so that $AC = 5$ in., and $BC = 7$ in.

Let fall CO ⊥ to AB at O. § 227

Revolve the △ ACB about AB as an axis.

Then point C will describe the ⊙ CE, whose center is O.

Therefore the ⊙ CE is the locus required. Q.E.F.

Proof. Every point in the ⊙ CE is 5 in. from A and 7 in. from B. Const.

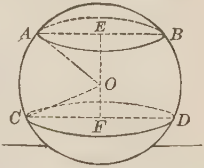
Every point without the circumference of the ⊙ CE is not 5 in. from A and 7 in. from B. Ax. 11

Therefore the circumference of the ⊙ CE is the locus of a point 5 in. from A and 7 in. from B. Q.E.D.

25. *Two points A and B are* 10 *in. apart. Find the locus in space of a point* 7 *in. from A and* 3 *in. from B.*

Solution. Obviously there is but one point fulfilling the required condition. That point lies in the line AB 3 in. from B. Q.E.F.

26. *The radii of two parallel sections of the same sphere are a and b respectively, and the distance between the sections is d. Find the radius of the sphere.*



Given the sphere whose center is O cut by parallel planes in the sections AB and CD, the radii of the Ⓢ AB and CD being a and b respectively, and the distance between the Ⓢ AB and CD being d.

Required to find r, the radius of the given sphere, in terms of a, b, and d.

Solution. Draw OE to the plane of the ⊙ AB at its center E.

Produce EO to meet the plane of the ⊙ CD in F.

The line OE is ⊥ to the plane of the ⊙ AB. § 623

∴ EF is ⊥ to the plane of the ⊙ CD. § 456

∴ F is the center of the ⊙ CD.

Draw the radii OA and OC of the sphere.

Also draw AE and CF.

The line EF is ⊥ to AE and CF. § 430

In the rt. △ OEA, $\overline{OE}^2 = \overline{OA}^2 - \overline{AE}^2$ § 338

$= r^2 - a^2.$ Ax. 9

$\therefore\ OE = \sqrt{r^2 - a^2}.$ Ax. 5

In the rt. △ OFC, $\overline{OF}^2 = \overline{OC}^2 - \overline{CF}^2$ § 338

$= r^2 - b^2.$ Ax. 9

$\therefore\ OF = \sqrt{r^2 - b^2}.$ Ax. 5

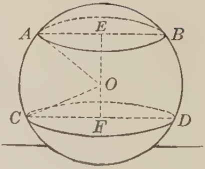
But $OE + OF = EF = d.$ Ax. 11

$\therefore\ \sqrt{r^2 - a^2} + \sqrt{r^2 - b^2} = d.$ Ax. 9

$\therefore\ r^2 - a^2 + r^2 - b^2 + 2\sqrt{(r^2 - a^2)(r^2 - b^2)} = d^2.$ Ax. 5

$\therefore\ 2\sqrt{(r^2 - a^2)(r^2 - b^2)} = a^2 + b^2 + d^2 - 2r^2.$ Axs. 1, 2

$\therefore\ \sqrt{(r^2 - a^2)(r^2 - b^2)} = \dfrac{a^2 + b^2 + d^2}{2} - r^2.$ Ax. 4



$$\therefore (r^2 - a^2)(r^2 - b^2)$$

$$= \left(\frac{a^2 + b^2 + d^2}{2}\right)^2 - 2r^2\left(\frac{a^2 + b^2 + d^2}{2}\right) + r^4. \quad \text{Ax. 5}$$

$$\therefore r^4 - r^2(a^2 + b^2) + a^2b^2$$

$$= \left(\frac{a^2 + b^2 + d^2}{2}\right)^2 - r^2(a^2 + b^2 + d^2) + r^4.$$

$$\therefore r^2d^2 = \left(\frac{a^2 + b^2 + d^2}{2}\right)^2 - a^2b^2 \quad \text{Axs. 1, 2}$$

$$= \frac{a^4 + b^4 + d^4 + 2a^2b^2 + 2a^2d^2 + 2b^2d^2}{4} - \frac{4a^2b^2}{4}$$

$$= \frac{a^4 + b^4 + d^4 - 2a^2b^2 + 2a^2d^2 + 2b^2d^2}{4}.$$

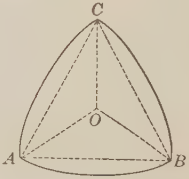
$$\therefore r^2 = \frac{a^4 + b^4 + d^4 - 2a^2b^2 + 2a^2d^2 + 2b^2d^2}{4d^2}. \quad \text{Ax. 4}$$

$$\therefore r = \sqrt{\frac{a^4 + b^4 + d^4 - 2a^2b^2 + 2a^2d^2 + 2b^2d^2}{4d^2}} \quad \text{Ax. 5}$$

$$= \frac{1}{2d}\sqrt{a^4 + b^4 + d^4 - 2a^2b^2 + 2a^2d^2 + 2b^2d^2}.$$

By analogous reasoning the same result may be obtained for the case in which the parallel planes lie on the same side of the center of the sphere. Q. E. F.

27. *The diameter of a certain sphere is* $\sqrt{2}$. *The chords of the arcs that form the sides of a triangle on the surface of the sphere are respectively* 1, 1, *and* $\frac{1}{2}\sqrt{2}$. *Find the area of the spherical triangle.*



Given the spherical $\triangle ABC$ on the surface of the sphere whose center is O and diameter $\sqrt{2}$, the chords of the arcs of great ⓢ AC, BC, and AB being 1, 1, and $\frac{1}{2}\sqrt{2}$ respectively.

Required to find the area of the spherical $\triangle ABC$.

Solution. Draw the radii OA, OB, and OC.

Then $OA = OB = OC = \frac{1}{2}\sqrt{2}$. § 621

Since the chord $AB = \frac{1}{2}\sqrt{2}$, Given

$\therefore$ the $\triangle AOB$ is equilateral.

$\therefore$ the $\triangle AOB$ is equiangular. § 75

$\therefore \angle AOB = 60°$. § 107

$\therefore$ the arc $AB = 60°$. § 213

Since $\left(\frac{1}{2}\sqrt{2}\right)^2 + \left(\frac{1}{2}\sqrt{2}\right)^2 = 1$,

$\therefore \overline{OA}^2 + \overline{OC}^2 = \overline{AC}^2$, Ax. 9

and $\overline{OB}^2 + \overline{OC}^2 = \overline{BC}^2$. Ax. 9

$\therefore$ $\triangle$s AOC and BOC are right-angled at O. § 337, Converse
$\therefore$ the arcs AC and BC are quadrants. § 213
$\therefore$ C is a pole of the arc AB. § 641
$\therefore$ $\angle C$ is measured by the arc AB § 654
$\therefore$ $\angle C = 60°$. Ax. 8
Moreover $\angle$s A and B each $= 90°$. § 656
$\therefore$ the spherical excess of $\triangle ABC = 60° + 90° + 90° - 180°$, or $60°$. § 693

Therefore the spherical $\triangle ABC$ is equivalent to a lune whose angle is half 60°, or 30°. § 695

But the area of such a lune $= \frac{30}{360}$ of the spherical surface § 694
$= \frac{1}{12} \times 4\pi r^2$ Ax. 9
$= \frac{1}{3} \times \left(\frac{1}{2}\sqrt{2}\right)^2 \pi$ Ax. 9
$= \frac{1}{6}\pi$.

Therefore the area of the given spherical $\triangle ABC$ is $\frac{1}{6}\pi$. Q.E.F.

Exercise 109. Page 420

1. *Find the area of a spherical polygon on a sphere whose area is* 2 *sq. ft., the angles being* 30°, 90°, 120°, 130°.

Solution. The spherical excess of the polygon
$= 30° + 90° + 120° + 130° - 2 \times 180°$, or $10°$. § 698

Therefore the spherical polygon is equivalent to a lune whose angle is half 10°, or 5°. § 699

But the area of such a lune $= \frac{5}{360}$ of the spherical surface § 694
$= \frac{1}{72}a$ Ax. 9
$= \frac{1}{72} \times 2$ sq. ft. Ax. 9
$= \frac{1}{36}$ sq. ft., or 4 sq. in.

Therefore the area of the spherical polygon is 4 sq. in. Q.E.F.

2. *Find the area of a spherical polygon on a sphere whose area is* 288 *sq. in., the angles being* 45°, 60°, 100°, 165°.

Solution. The spherical excess of the polygon
$= 45° + 60° + 100° + 165° - 2 \times 180°$, or $10°$. § 698

Therefore the spherical polygon is equivalent to a lune whose angle is half 10°, or 5°. § 699

But the area of such a lune $= \frac{5}{360}$ of the spherical surface § 694
$= \frac{1}{72}a$ Ax. 9
$= \frac{1}{72} \times 288$ sq. in. Ax. 9
$= 4$ sq. in.

Therefore the area of the spherical polygon is 4 sq. in. Q.E.F

3. *Find the area of a spherical polygon on a sphere whose area is* 500 *sq. in., the angles being* 70°, 168°, 92°, 120°.

Solution. The spherical excess of the polygon

$= 70° + 168° + 92° + 120° - 2 \times 180°$, or $90°$. § 698

Therefore the spherical polygon is equivalent to a lune whose angle is half 90°, or 45°. § 699

But the area of such a lune $= \frac{45}{360}$ of the spherical surface § 694

$= \frac{1}{8}\,a$ Ax. 9

$= \frac{1}{8} \times 500$ sq. in. Ax. 9

$= 62\frac{1}{2}$ sq. in.

Therefore the area of the spherical polygon is $62\frac{1}{2}$ sq. in. Q.E.F.

4. *Find the area of a spherical polygon on a sphere whose area is* 750 *sq. in., the angles being* 68° 30′, 149° 50′, 96° 54′, 136° 52′.

Solution. The spherical excess of the polygon

$= 68°\,30' + 149°\,50' + 96°\,54' + 136°\,52' - 2 \times 180°$, or $92°\,6'$. § 698

Therefore the spherical polygon is equivalent to a lune whose angle is half 92° 6′, or 46° 3′. § 699

But the area of such a lune $= \dfrac{46°\,3'}{360°}$ of the spherical surface § 694

$= \frac{2763}{21600}$ of the spherical surface

$= \frac{2763}{21600}\,a$ Ax. 9

$= \frac{2763}{21600} \times 750$ sq. in. Ax. 9

$= \frac{1535}{16}$ sq. in., or $95\frac{15}{16}$ sq. in.

Therefore the area of the spherical polygon is $95\frac{15}{16}$ sq. in. Q.E.F.

5. *Find the area of a spherical polygon on a sphere whose area is* 600 *sq. in., the angles being* 122° 27′ 40″, 130° 32′ 50″, 98° 31′ 30″, 96° 48′.

Solution. The spherical excess of the polygon $= 122°\,27'\,40'' + 130°\,32'\,50'' + 98°\,31'\,30'' + 96°\,48' - 2 \times 180°$, or $88°\,20'$. § 698

Therefore the spherical polygon is equivalent to a lune whose angle is half 88° 20′, or 44° 10′. § 699

But the area of such a lune $= \dfrac{44°\,10'}{360°}$ of the spherical surface § 694

$= \frac{2650}{21600}$ of the spherical surface

$= \frac{265}{2160}\,a$ Ax. 9

$= \frac{265}{2160} \times 600$ sq. in. Ax. 9

$= \frac{1325}{18}$ sq. in., or $73\frac{11}{18}$ sq. in.

Therefore the area of the spherical polygon is $73\frac{11}{18}$ sq. in. Q.E.F.

6. *Find the area of a spherical polygon on a sphere whose area is* 3 *sq. ft.* 120 *sq. in., the angles being* 132°, 96°, 154°, 120°, 150°.

Solution. The spherical excess of the polygon

$= 132° + 96° + 154° + 120° + 150° - 3 \times 180°$, or $112°$. § 698

Therefore the spherical polygon is equivalent to a lune whose angle is half 112°, or 56°. § 699

But the area of such a lune $= \frac{56}{360}$ of the spherical surface § 694

$= \frac{7}{45}\,a$ Ax. 9

$= \frac{7}{45} \times 3$ sq. ft. 120 sq. in. Ax. 9

$= \frac{7}{45} \times 552$ sq. in., or $85\frac{13}{15}$ sq. in.

Therefore the area of the spherical polygon is $85\frac{13}{15}$ sq. in. Q.E.F.

7. *Find the area of a spherical polygon on a sphere whose area is* 157.2 *sq. in., the angles being* 130°, 156°, 172°, 95°, 120°, 100°.

Solution. The spherical excess of the polygon

$= 130° + 156° + 172° + 95° + 120° + 100° - 4 \times 180°$, or 53°. § 698

Therefore the spherical polygon is equivalent to a lune whose angle is half 53°, or $26\frac{1}{2}°$. § 699

But the area of such a lune $= \dfrac{26\frac{1}{2}}{360}$ of the spherical surface § 694

$= \frac{53}{720}\,a$ Ax. 9

$= \frac{53}{720} \times 157.2$ sq. in. Ax. 9

$= 11.5717$ sq. in.

Therefore the area of the spherical polygon is 11.5717 sq. in. Q.E.F.

8. *Find the area of a spherical polygon on a sphere of radius* 10 *in., the angles being* 130°, 150°, 80°, 90°.

Solution. The spherical excess of the polygon

$= 130° + 150° + 80° + 90° - 2 \times 180°$, or 90°. § 698

Therefore the spherical polygon is equivalent to a lune whose angle is half 90°, or 45°. § 699

But the area of such a lune $= \dfrac{45}{360} = \dfrac{1}{8}$ of the spherical surface. § 694

Now the area of the sphere $= 4\,\pi r^2$ § 689

$= 4 \times 10^2\pi$ sq. in. Ax. 9

$= 400\,\pi$ sq. in.

Then the area of a lune whose angle is 45°

$= \frac{1}{8} \times 400\,\pi$ sq. in. Ax. 9

$= 50\,\pi$ sq. in., or 157.08 sq. in.

Therefore the area of the spherical polygon is 157.08 sq. in. Q.E.F.

9. *Find the area of a spherical polygon on a sphere of radius* 20 *in., the angles being* 148°, 157°, 90°, 100°, 120°.

Solution. The spherical excess of the polygon

$= 148° + 157° + 90° + 100° + 120° - 3 \times 180°$, or 75°. § 698

Therefore the spherical polygon is equivalent to a lune whose angle is half 75°, or $37\frac{1}{2}°$. § 699

But the area of such a lune $= \frac{37\frac{1}{2}}{360} = \frac{5}{48}$ of the spherical surface. § 649

Now the area of the sphere $= 4\pi r^2$ § 689

$= 4 \times 20^2 \pi$ sq. in. Ax. 9

$= 1600\pi$ sq. in.

Then the area of a lune whose angle is $37\frac{1}{2}°$

$= \frac{5}{48} \times 1600\pi$ sq. in. Ax. 9

$= \frac{500\pi}{3}$ sq. in., or 523.6 sq. in.

Therefore the area of the spherical polygon is 523.6 sq. in. Q.E.F.

10. *Find the area of a spherical polygon on a sphere of radius* 24 *in., the angles being* 172°, 169°, 86°, 141°, 100°, 90°.

Solution. The spherical excess of the polygon

$= 172° + 169° + 86° + 141° + 100° + 90° - 4 \times 180°$, or 38°. § 698

Therefore the spherical polygon is equivalent to a lune whose angle is half 38°, or 19°. § 699

But the area of such a lune $= \frac{19}{360}$ of the spherical surface. § 694

Now the area of the sphere $= 4\pi r^2$ § 689

$= 4 \times 24^2 \pi$ sq. in. Ax. 9

$= 2304\pi$ sq. in.

Then the area of a lune whose angle is 19°

$= \frac{19}{360} \times 2304\pi$ sq. in. Ax. 9

$= 19 \times 6.4\pi$ sq. in., or 382.01856 sq. in.

Therefore the area of the spherical polygon is 382.01856 sq. in. Q.E.F.

11. *Find the area of a spherical polygon on a sphere of radius* 10 *in., the angles being* 135° 30′, 148° 42′, 96° 37′, 102° 11′.

Solution. The spherical excess of the polygon

$= 135° 30' + 148° 42' + 96° 37' + 102° 11' - 2 \times 180°$, or 123°. § 698

Therefore the spherical polygon is equivalent to a lune whose angle is half 123°, or $61\frac{1}{2}°$. § 699

But the area of such a lune $= \frac{61\frac{1}{2}}{360} = \frac{41}{240}$ of the spherical surface. § 694

Now the area of the sphere $= 4\pi r^2$ § 689

$= 4 \times 10^2 \pi$ sq. in. Ax. 9

$= 400\pi$ sq. in.

Then the area of a lune whose angle is $61\frac{1}{2}°$

$= \frac{41}{240} \times 400\pi$ sq. in. Ax. 9

$= \frac{205\pi}{3}$ sq. in., or 214.676 sq. in.

Therefore the area of the spherical polygon is 214.676 sq. in. Q.E.F.

12. *Find the area of a spherical polygon on a sphere of diameter* 10 *in., the angles being* 148°, 92°, 60°, 120°.

Solution. The spherical excess of the polygon

$= 148° + 92° + 60° + 120° - 2 \times 180°$, or 60°. § 698

Therefore the spherical polygon is equivalent to a lune whose angle is half 60°, or 30°. § 699

But the area of such a lune $= \frac{30}{360} = \frac{1}{12}$ of the spherical surface. § 694

Now the area of the sphere $= 2\pi rd$ § 688

$= 2\pi\left(\frac{d}{2}\right)d$ Ax. 9

$= \pi d^2$

$= 10^2\pi$ sq. in., or 100π sq. in. Ax. 9

Then the area of a lune whose angle is 30°

$= \frac{1}{12} \times 100\pi$ sq. in. Ax. 9

$= \frac{1}{12} \times 314.16$ sq. in., or 26.18 sq. in.

Therefore the area of the spherical polygon is 26.18 sq. in. Q. E. F.

13. *Find the area of a spherical polygon on a sphere of diameter* 22 *in., the angles being* 172°, 168°, 93°, 37°, 100°.

Solution. The spherical excess of the polygon

$= 172° + 168° + 93° + 37° + 100° - 3 \times 180°$, or 30°. § 698

Therefore the spherical polygon is equivalent to a lune whose angle is half 30°, or 15°. § 699

But the area of such a lune $= \frac{15}{360} = \frac{1}{24}$ of the spherical surface. § 694

Now the area of the sphere $= 2\pi rd$ § 688

$= 2\pi\left(\frac{d}{2}\right)d$ Ax. 9

$= \pi d^2$

$= 22^2\pi$ sq. in., or 484π sq. in. Ax. 9

Then the area of a lune whose angle is 15°

$= \frac{1}{24} \times 484\pi$ sq. in. Ax. 9

$= 20\frac{1}{6}\pi$ sq. in., or 63.3556 sq. in.

Therefore the area of the spherical polygon is 63.3556 sq. in. Q. E. F.

14. *Find the area of a spherical polygon on a sphere of diameter* 20 *in., the angles being* 102°, 162°, 139°, 141°, 138°, 126°.

Solution. The spherical excess of the polygon

$= 102° + 162° + 139° + 141° + 138° + 126° - 4 \times 180°$, or 88°. § 698

Therefore the spherical polygon is equivalent to a lune whose angle is half 88°, or 44°. § 699

But the area of such a lune $= \frac{44}{360} = \frac{11}{90}$ of the spherical surface. § 694

Now the area of the sphere $= 2\pi rd$ § 688

$= 2\pi\left(\frac{d}{2}\right)d$ Ax. 9

$= \pi d^2$

$= 20^2\pi$ sq. in., or 400π sq. in. Ax. 9

Then the area of a lune whose angle is 44°

$= \frac{11}{90} \times 400\pi$ sq. in. Ax. 9

$= \frac{440}{9}\pi$ sq. in., or 153.589 sq. in.

Therefore the area of the spherical polygon is 153.589 sq. in. Q.E.F.

15. *Find the area of a spherical polygon on a sphere of diameter* 20 *in., the angles being* $82°\,50'\,42''$, $120°\,29'\,18''$, $98°\,37'\,15''$, $141°\,22'\,45''$.

Solution. The spherical excess of the polygon $= 82°\,50'\,42'' + 120°\,29'\,18'' + 98°\,37'\,15'' + 141°\,22'\,45'' - 2 \times 180°$, or $83°\,20'$. § 698

Therefore the spherical polygon is equivalent to a lune whose angle is half $83°\,20'$, or $41°\,40'$. § 699

But the area of such a lune $= \frac{41°\,40'}{360°} = \frac{2500}{21600}$

$= \frac{25}{216}$ of the spherical surface. § 649

Now the area of the sphere $= 2\pi rd$ § 688

$= 2\pi\left(\frac{d}{2}\right)d$ Ax. 9

$= \pi d^2$

$= 20^2\pi$ sq. in., or 400π sq. in. Ax. 9

Then the area of a lune whose angle is $41°\,40'$

$= \frac{25}{216} \times 400\pi$ sq. in. Ax. 9

$= \frac{2500}{54}\pi$ sq. in., or 145.44 sq. in.

Therefore the area of the spherical polygon is 145.44 sq. in. Q.E.F.

16. *Find the area of a spherical polygon on a sphere of circumference* 3.1416 *in., the angles being* 39°, 148°, 172°, 168°.

Solution. The spherical excess of the polygon

$= 39° + 148° + 172° + 168° - 2 \times 180°$, or 167°. § 698

Therefore the spherical polygon is equivalent to a lune whose angle is half 167°, or $83\frac{1}{2}°$. § 699

But the area of such a lune $= \frac{83\frac{1}{2}}{360} = \frac{167}{720}$ of the spherical surface. § 694

Now the area of the sphere $= 4\pi r^2$ § 689

$= 4\pi\left(\frac{c}{2\pi}\right)^2$ Ax. 9

$= \frac{c^2}{\pi}$

$= \frac{3.1416^2}{\pi}$ sq. in. Ax. 9

$= 3.1416$ sq. in.

Then the area of a lune whose angle is $83\frac{1}{2}°$

$= \frac{167}{720} \times 3.1416$ sq. in., or 0.7287 sq. in. Ax. 9

Therefore the area of the spherical polygon is 0.7287 sq. in. Q.E.F.

17. *Find the area of a spherical polygon on a sphere of circumference* 31.416 *in., the angles being* 128°, 92°, 168°, 109°.

Solution. The spherical excess of the polygon

$= 128° + 92° + 168° + 109° - 2 \times 180°$, or 137°. § 698

Therefore the spherical polygon is equivalent to a lune whose angle is half 137°, or $68\frac{1}{2}°$. § 699

But the area of such a lune $= \dfrac{68\frac{1}{2}}{360} = \dfrac{137}{720}$ of the spherical surface. § 694

Now the area of the sphere $= 4\,\pi r^2$ § 689

$$= 4\,\pi\left(\frac{c}{2\,\pi}\right)^2 \quad \text{Ax. 9}$$

$$= \frac{c^2}{\pi}$$

$$= \frac{31.416^2}{\pi} \text{ sq. in., or } 314.16 \text{ sq. in.} \quad \text{Ax. 9}$$

Then the area of a lune whose angle is $68\frac{1}{2}°$

$= \frac{137}{720} \times 314.16$ sq. in., or 59.778 sq. in. Ax. 9

Therefore the area of the spherical polygon is 59.778 sq. in. Q.E.F.

18. *Find the area of a spherical polygon on a sphere of circumference* 6.2832 *in., the angles being* 146°, 129°, 102°, 137°, 100°.

Solution. The spherical excess of the polygon

$= 146° + 129° + 102° + 137° + 100° - 3 \times 180°$, or 74°. § 698

Therefore the spherical polygon is equivalent to a lune whose angle is half 74°, or 37°. § 699

But the area of such a lune $= \frac{37}{360}$ of the spherical surface. § 694

Now the area of the sphere $= 4\,\pi r^2$ § 689

$$= 4\,\pi\left(\frac{c}{2\,\pi}\right)^2 \quad \text{Ax. 9}$$

$$= \frac{c^2}{\pi}$$

$$= \frac{6.2832^2}{\pi} \text{ sq. in., or } 12.5664 \text{ sq. in.} \quad \text{Ax. 9}$$

Then the area of a lune whose angle is 37°

$= \frac{37}{360} \times 12.5664$ sq. in., or 1.2915 sq. in. Ax. 9

Therefore the area of the spherical polygon is 1.2915 sq. in. Q.E.F.

19. *Find the area of a spherical polygon on a sphere of circumference* 18.8496 *in., the angles being* 128°, 145°, 139°, 82°, 161°, 137°.

Solution. The spherical excess of the polygon

$= 128° + 145° + 139° + 82° + 161° + 137° - 4 \times 180°$, or 72°. § 698

Therefore the spherical polygon is equivalent to a lune whose angle is half 72°, or 36°. § 699

But the area of such a lune $= \frac{36}{360} = \frac{1}{10}$ of the spherical surface. § 694

Now the area of the sphere $= 4\pi r^2$ § 689

$$= 4\pi\left(\frac{c}{2\pi}\right)^2 \qquad \text{Ax. 9}$$

$$= \frac{c^2}{\pi}$$

$$= \frac{18.8496^2}{\pi}\text{ sq. in., or } 113.0976 \text{ sq. in.} \qquad \text{Ax. 9}$$

Then the area of a lune whose angle is 36°

$$= \tfrac{1}{10} \times 113.0976 \text{ sq. in., or } 11.30976 \text{ sq. in.} \qquad \text{Ax. 9}$$

Therefore the area of the spherical polygon is 11.30976 sq. in. Q. E. F.

Exercise 110. Page 424

1. *Find the volume of a sphere whose radius is 3 in.*

Solution. $v = \frac{4}{3}\pi r^3.$ § 706

$\therefore v = \frac{4}{3} \times 3^3\pi$ Ax. 9

$= 36\pi$, or 113.0976.

$\therefore$ the volume of the given sphere is 113.0976 cu. in. Q. E. F.

2. *Find the volume of a sphere whose radius is 5 in.*

Solution. $v = \frac{4}{3}\pi r^3.$ § 706

$\therefore v = \frac{4}{3} \times 5^3\pi$ Ax. 9

$= \frac{500}{3}\pi$, or 523.6.

$\therefore$ the volume of the given sphere is 523.6 cu. in. Q. E. F.

3. *Find the volume of a sphere whose radius is 7 in.*

Solution. $v = \frac{4}{3}\pi r^3.$ § 706

$\therefore v = \frac{4}{3} \times 7^3\pi$ Ax. 9

$= \frac{1372}{3}\pi$, or 1436.7584.

$\therefore$ the volume of the given sphere is 1436.7584 cu. in. Q. E. F.

4. *Find the volume of a sphere whose radius is $2\frac{1}{2}$ in.*

Solution. $v = \frac{4}{3}\pi r^3.$ § 706

$\therefore v = \frac{4}{3}(2\frac{1}{2})^3\pi$ Ax. 9

$= \frac{4}{3} \times \frac{125}{8}\pi$

$= \frac{125}{6}\pi$, or 65.45.

$\therefore$ the volume of the given sphere is 65.45 cu. in. Q. E. F.

5. *Find the volume of a sphere whose radius is* $4\frac{3}{8}$ *in.*

Solution.

$$\begin{aligned} v &= \tfrac{4}{3}\pi r^3. &&\text{§ 706}\\ \therefore\ v &= \tfrac{4}{3}(4\tfrac{3}{8})^3\pi &&\text{Ax. 9}\\ &= \tfrac{4}{3}\times\tfrac{42875}{512}\pi\\ &= \tfrac{42875}{384}\pi,\text{ or } 350.77. \end{aligned}$$

∴ the volume of the given sphere is 350.77 cu. in. Q.E.F.

6. *Find the volume of a sphere whose radius is* $9\frac{7}{8}$ *in.*

Solution.

$$\begin{aligned} v &= \tfrac{4}{3}\pi r^3. &&\text{§ 706}\\ \therefore\ v &= \tfrac{4}{3}(9\tfrac{7}{8})^3\pi &&\text{Ax. 9}\\ &= \tfrac{4}{3}\times\tfrac{493039}{512}\pi\\ &= \tfrac{493039}{384}\pi,\text{ or } 4033.675. \end{aligned}$$

∴ the volume of the given sphere is 4033.675 cu. in. Q.E.F.

7. *Find the volume of a sphere whose radius is* 20.7 *ft.*

Solution.

$$\begin{aligned} v &= \tfrac{4}{3}\pi r^3. &&\text{§ 706}\\ \therefore\ v &= \tfrac{4}{3}\times 20.7^3\pi &&\text{Ax. 9}\\ &= \tfrac{4}{3}\times 8869.743\,\pi\\ &= 11{,}826.324\,\pi,\text{ or } 37{,}153.579. \end{aligned}$$

∴ the volume of the given sphere is 37,153.579 cu. ft. Q.E.F.

8. *Find the volume of a sphere whose radius is* 2 *ft.* 3 *in.*

Solution. 2 ft. 3 in. = $2\frac{1}{4}$ ft.

Now

$$\begin{aligned} v &= \tfrac{4}{3}\pi r^3. &&\text{§ 706}\\ \therefore\ v &= \tfrac{4}{3}(2\tfrac{1}{4})^3\pi &&\text{Ax. 9}\\ &= \tfrac{4}{3}\times\tfrac{729}{64}\pi\\ &= \tfrac{243}{16}\pi,\text{ or } 47.713. \end{aligned}$$

∴ the volume of the given sphere is 47.713 cu. ft. Q.E.F.

9. *Find the volume of a sphere whose radius is* 4000 *mi.*

Solution.

$$\begin{aligned} v &= \tfrac{4}{3}\pi r^3. &&\text{§ 706}\\ \therefore\ v &= \tfrac{4}{3}\times 4000^3\pi &&\text{Ax. 9}\\ &= 25{,}600{,}000\times 10{,}472\\ &= 268{,}083{,}200{,}000. \end{aligned}$$

∴ the volume of the given sphere is 268,083,200,000 cu. mi. Q.E.F.

10. *Find the volume of a sphere whose diameter is* 24 *in.*

Solution.

$$\begin{aligned} v &= \tfrac{1}{6}\pi d^3. &&\text{§ 706}\\ \therefore\ v &= \tfrac{1}{6}\times 24^3\pi &&\text{Ax. 9}\\ &= 2304\,\pi,\text{ or } 7238.2464. \end{aligned}$$

∴ the volume of the given sphere is 7238.2464 cu. in. Q.E.F.

11. *Find the volume of a sphere whose diameter is 36 in.*

Solution. $v = \frac{1}{6}\pi d^3.$ § 706

$\therefore v = \frac{1}{6} \times 36^3 \pi$ Ax. 9

$= 7776\pi$, or $24,429.0816$.

$\therefore$ the volume of the given sphere is 24,429.0816 cu. in. Q.E.F.

12. *Find the volume of a sphere whose diameter is 48 in.*

Solution. $v = \frac{1}{6}\pi d^3.$ § 706

$\therefore v = \frac{1}{6} \times 48^3 \pi$ Ax. 9

$= 18,432\pi$, or $57,905.9712$.

$\therefore$ the volume of the given sphere is 57,905.9712 cu. in. Q.E.F.

13. *Find the volume of a sphere whose diameter is 2.8 in.*

Solution. $v = \frac{1}{6}\pi d^3.$ § 706

$\therefore v = \frac{1}{6} \times 2.8^3 \pi$ Ax. 9

$= \frac{1}{6} \times 21.952\pi$

$= 21.952 \times 0.5236$, or 11.494.

$\therefore$ the volume of the given sphere is 11.494 cu. in. Q.E.F.

14. *Find the volume of a sphere whose diameter is 3.4 in.*

Solution. $v = \frac{1}{6}\pi d^3.$ § 706

$\therefore v = \frac{1}{6} \times 3.4^3 \pi$ Ax. 9

$= \frac{1}{6} \times 39.304\pi$

$= 39.304 \times 0.5236$, or 20.5796.

$\therefore$ the volume of the given sphere is 20.5796 cu. in. Q.E.F.

15. *Find the volume of a sphere whose diameter is 4.5 in.*

Solution. $v = \frac{1}{6}\pi d^3.$ § 706

$\therefore v = \frac{1}{6} \times 4.5^3 \pi$ Ax. 9

$= \frac{1}{6} \times 91.125\pi$

$= 91.125 \times 0.5236$, or 47.713.

$\therefore$ the volume of the given sphere is 47.713 cu. in. Q.E.F.

16. *Find the volume of a sphere whose diameter is 2 ft. 1 in.*

Solution. 2 ft. 1 in. $= 2\frac{1}{12}$ ft.

Now $v = \frac{1}{6}\pi d^3.$ § 706

$\therefore v = \frac{1}{6}\left(2\frac{1}{12}\right)^3 \pi$ Ax. 9

$= \frac{1}{6} \times \frac{15625}{1728}\pi$

$= \frac{15625}{1728} \times 0.5236$, or 4.7345.

$\therefore$ the volume of the given sphere is 4.7345 cu. ft. Q.E.F.

17. *Find the volume of a sphere whose diameter is 3 ft. 4 in.*

Solution. 3 ft. 4 in. $= 3\frac{1}{3}$ ft.

Now $v = \frac{1}{6}\pi d^3.$ § 706

$\therefore v = \frac{1}{6}(3\frac{1}{3})^3\pi$ Ax. 9

$= \frac{1}{6} \times \frac{1000}{27}\pi$

$= \frac{1000}{27} \times 0.5236$, or 19.3926.

$\therefore$ the volume of the given sphere is 19.3926 cu. ft. Q.E.F.

18. *Find the volume of a sphere whose diameter is 8 ft. 6 in.*

Solution. 8 ft. 6 in. $= 8\frac{1}{2}$ ft.

Now $v = \frac{1}{6}\pi d^3.$ § 706

$\therefore v = \frac{1}{6}(8\frac{1}{2})^3\pi$ Ax. 9

$= \frac{1}{6} \times \frac{4913}{8}\pi$

$= \frac{4913}{8} \times 0.5236$, or 321.5559.

$\therefore$ the volume of the given sphere is 321.5559 cu. ft. Q.E.F.

19. *Find the volume of a sphere whose circumference is 6.2832 in.*

Solution. $v = \frac{1}{6}\pi d^3.$ § 706

But $d = \frac{c}{\pi}.$ § 384

$\therefore v = \frac{1}{6}\pi\left(\frac{c}{\pi}\right)^3$ Ax. 9

$= \frac{c^3}{6\pi^2}.$

$\therefore v = \frac{6.2832^3}{6\pi^2}$, or 4.1888. Ax. 9

$\therefore$ the volume of the given sphere is 4.1888 cu. in. Q.E.F.

20. *Find the volume of a sphere whose circumference is 12.5664 in.*

Solution. $v = \frac{1}{6}\pi d^3.$ § 706

But $d = \frac{c}{\pi}.$ § 384

$\therefore v = \frac{1}{6}\pi\left(\frac{c}{\pi}\right)^3$ Ax. 9

$= \frac{c^3}{6\pi^2}.$

$\therefore v = \frac{12.5664^3}{6\pi^2}$, or 33.5104. Ax. 9

$\therefore$ the volume of the given sphere is 33.5104 cu. in. Q.E.F.

21. *Find the volume of a sphere whose circumference is 18.8496 in.*

Solution. $v = \frac{1}{6}\pi d^3.$ § 706

But $d = \frac{c}{\pi}.$ § 384

$\therefore v = \frac{1}{6}\pi\left(\frac{c}{\pi}\right)^3$ Ax. 9

$= \frac{c^3}{6\pi^2}.$

$\therefore v = \frac{18.8496^3}{6\pi^2}$, or 113.0976. Ax. 9

$\therefore$ the volume of the given sphere is 113.0976 cu. in. Q. E. F.

22. *Find the volume of a sphere whose surface area is 12.5664 sq. in.*

Solution. $v = \frac{4}{3}\pi r^3.$ § 706

But $4\pi r^2 = a.$ § 689

$\therefore r^2 = \frac{a}{4\pi}.$ Ax. 4

Whence $r = \sqrt{\frac{a}{4\pi}}$ Ax. 5

$= \frac{1}{2\pi}\sqrt{\pi a}.$

$\therefore r^3 = \frac{1}{8\pi^3}(\sqrt{\pi a})^3$ Ax. 5

$= \frac{\pi a}{8\pi^3}\sqrt{\pi a}$

$= \frac{a}{8\pi^2}\sqrt{\pi a}.$

$\therefore v = \frac{4\pi}{3} \times \frac{a}{8\pi^2}\sqrt{\pi a}$ Ax. 9

$= \frac{a}{6\pi}\sqrt{\pi a}.$

$\therefore v = \frac{12.5664}{6\pi}\sqrt{12.5664\,\pi}$ Ax. 9

$= \frac{2}{3} \times 2\pi = \frac{4}{3}\pi$, or 4.1888.

$\therefore$ the volume of the given sphere is 4.1888 cu. in. Q. E. F.

23. *Find the volume of a sphere whose surface area is 50.2656 sq. in.*

Solution. $v = \frac{4}{3}\pi r^3.$ § 706

But $4\pi r^2 = a.$ § 689

$\therefore r^2 = \frac{a}{4\pi}.$ Ax. 4

Whence $$r = \sqrt{\frac{a}{4\pi}} \qquad \text{Ax. 5}$$

$$= \frac{1}{2\pi}\sqrt{\pi a}.$$

$$\therefore r^3 = \frac{1}{8\pi^3}(\sqrt{\pi a})^3 \qquad \text{Ax. 5}$$

$$= \frac{\pi a}{8\pi^3}\sqrt{\pi a}$$

$$= \frac{a}{8\pi^2}\sqrt{\pi a}.$$

$$\therefore v = \frac{4\pi}{3} \times \frac{a}{8\pi^2}\sqrt{\pi a} \qquad \text{Ax. 9}$$

$$= \frac{a}{6\pi}\sqrt{\pi a}.$$

$$\therefore v = \frac{50.2656}{6\pi}\sqrt{50.2656\pi} \qquad \text{Ax. 9}$$

$$= \tfrac{8}{3} \times 4\pi = \tfrac{32}{3}\pi, \text{ or } 33.5104.$$

∴ the volume of the given sphere is 33.5104 cu. in. Q. E. F.

24. *Find the volume of a sphere whose surface area is* 113.0976 *sq. in.*

Solution. $$v = \tfrac{4}{3}\pi r^3. \qquad \S\ 706$$

But $$4\pi r^2 = a. \qquad \S\ 689$$

$$\therefore r^2 = \frac{a}{4\pi}. \qquad \text{Ax. 4}$$

Whence $$r = \sqrt{\frac{a}{4\pi}} \qquad \text{Ax. 5}$$

$$= \frac{1}{2\pi}\sqrt{\pi a}.$$

$$\therefore r^3 = \frac{1}{8\pi^3}(\sqrt{\pi a})^3 \qquad \text{Ax. 5}$$

$$= \frac{\pi a}{8\pi^3}\sqrt{\pi a}$$

$$= \frac{a}{8\pi^2}\sqrt{\pi a}.$$

$$\therefore v = \frac{4\pi}{3} \times \frac{a}{8\pi^2}\sqrt{\pi a} \qquad \text{Ax. 9}$$

$$= \frac{a}{6\pi}\sqrt{\pi a}.$$

$$\therefore v = \frac{113.0976}{6\pi}\sqrt{113.0976\pi} \qquad \text{Ax. 9}$$

$$= 6 \times 6\pi$$

$$= 36\pi, \text{ or } 113.0976.$$

∴ the volume of the given sphere is 113.0976 cu. in. Q. E. F.

25. *Find the radius of a sphere whose volume is* 4.1888 *cu. in.*

Solution. $v = \frac{4}{3}\pi r^3.$ § 706

$\therefore r^3 = \frac{v}{\frac{4}{3}\pi}$ Ax. 4

$= \frac{3v}{4\pi}.$

$\therefore r = \sqrt[3]{\frac{3v}{4\pi}}$ Ax. 5

$= \frac{1}{2\pi}\sqrt[3]{6\pi^2 v}.$

$\therefore r = \frac{1}{2\pi}\sqrt[3]{6\pi^2 \cdot 4.1888}$ Ax. 9

$= \frac{1}{2\pi}\sqrt[3]{6\pi^2 \cdot 1\frac{1}{3}\pi}$

$= \frac{1}{2\pi}\sqrt[3]{8\pi^3}$, or 1.

Therefore the radius of the given sphere is 1 in. Q.E.F.

26. *Find the radius of a sphere whose volume is* 33.5104 *cu. in.*

Solution. $v = \frac{4}{3}\pi r^3.$ § 706

$\therefore r^3 = \frac{v}{\frac{4}{3}\pi}$ Ax. 4

$= \frac{3v}{4\pi}.$

$\therefore r = \sqrt[3]{\frac{3v}{4\pi}}$ Ax. 5

$= \frac{1}{2\pi}\sqrt[3]{6\pi^2 v}.$

$\therefore r = \frac{1}{2\pi}\sqrt[3]{6\pi^2 \cdot 33.5104}$ Ax. 9

$= \frac{1}{2\pi}\sqrt[3]{6\pi^2 \cdot 10\frac{2}{3}\pi}$

$= \frac{1}{2\pi}\sqrt[3]{64\pi^3}$, or 2.

Therefore the radius of the given sphere is 2 in. Q.E.F.

27. *Find the radius of a sphere whose volume is* 113.0976 *cu. in.*

Solution. $v = \frac{4}{3}\pi r^3.$ § 706

$\therefore r^3 = \frac{v}{\frac{4}{3}\pi}$ Ax. 4

$= \frac{3v}{4\pi}.$

$$\therefore r = \sqrt[3]{\frac{3v}{4\pi}} \qquad \text{Ax. 5}$$

$$= \frac{1}{2\pi}\sqrt[3]{6\pi^2 v}.$$

$$\therefore r = \frac{1}{2\pi}\sqrt[3]{6\pi^2 \cdot 113.0976} \qquad \text{Ax. 9}$$

$$= \frac{1}{2\pi}\sqrt[3]{6\pi^2 \cdot 36\pi}$$

$$= \frac{1}{2\pi}\sqrt[3]{216\pi^3}, \text{ or } 3.$$

Therefore the radius of the given sphere is 3 in. Q.E.F.

28. *The circumference of a hemispherical dome is 66 ft. How many square feet of lead are required to cover it?*

Solution. $a = 4\pi r^2.$ § 689

But $r = \frac{c}{2\pi}.$ § 385

$$\therefore a = 4\pi\left(\frac{c}{2\pi}\right)^2 \qquad \text{Ax. 9}$$

$$= \frac{c^2}{\pi}.$$

Since $c = 66$ ft., Given

$$\therefore a = \frac{66^2}{\pi} \text{ sq. ft.} \qquad \text{Ax. 9}$$

$$\therefore \tfrac{1}{2}a = \frac{66^2}{2\pi} \text{ sq. ft., or } 693.277 \text{ sq. ft.} \qquad \text{Ax. 4}$$

Therefore the number of square feet of lead required is 693.277. Q.E.F.

29. *If the ball on the top of St. Paul's Cathedral in London is 6 ft. in diameter, how much would it cost to gild it at 9 cents per square inch?*

Solution. $a = 4\pi r^2.$ § 689

$$\therefore a = 4\pi\left(\frac{d}{2}\right)^2. \qquad \text{Ax. 9}$$

$$\therefore a = 4\pi\left(\frac{6}{2}\right)^2 \text{ sq. ft., or } 36\pi \text{ sq. ft.} \qquad \text{Ax. 9}$$

Therefore the cost of gilding the ball would be $36\pi \times 144 \times \$0.09$, or \$1465.74. Q.E.F.

30. *The dihedral angles made by the faces of a spherical pyramid are* 80°, 100°, 120°, *and* 150°, *and the length of a lateral edge is* 42 *ft. Find the area of the base.*

Solution. The base of the given spherical pyramid is a spherical quadrilateral whose $\angle$s are 80°, 100°, 120°, and 150°. § 655

The spherical excess of this quadrilateral

$= 80° + 100° + 120° + 150° - 2 \times 180°$, or 90°. § 698

Therefore the base of the given spherical pyramid is equivalent to a lune whose angle is half 90°, or 45°. § 699

But the area of such a lune $= \frac{45}{360} = \frac{1}{8}$ of the spherical surface § 694

$= \frac{1}{8} \times 4\pi r^2$ Ax. 9

$= \frac{1}{2} \times 42^2 \pi$ Ax. 9

$= 882\pi$, or 2770.8912.

Therefore the area of the base of the spherical pyramid is 2770.8912 sq. ft. Q.E.F.

31. *The dihedral angles made by the faces of a spherical pyramid are* 60°, 80°, *and* 100°, *and the area of the base is* 4π *sq. ft. Find the radius.*

Solution. The base of the given spherical pyramid is a spherical $\triangle$ whose $\angle$s are 60°, 80°, and 100°. § 655

The spherical excess of this $\triangle$

$= 60° + 80° + 100° - 180°$, or 60°. § 693

Therefore the base of the given spherical pyramid is equivalent to a lune whose angle is half 60°, or 30°. § 695

But the area of such a lune $= \frac{30}{360} = \frac{1}{12}$ of the spherical surface. § 694

Therefore, since the area of the base of the given spherical pyramid is 4π sq. ft., Given

$\frac{1}{12}$ of the spherical surface $= 4\pi$ sq. ft. Ax. 8

Then the area of the sphere $= 12 \times 4\pi$ sq. ft., or 48π sq. ft. Ax. 3

That is, $4\pi r^2 = 48\pi$. Ax. 9

$\therefore r^2 = 12$. Ax. 4

$\therefore r = \sqrt{12}$, or $2\sqrt{3}$. Ax. 5

Therefore the radius of the sphere is $2\sqrt{3}$ ft. Q.E.F.

32. *What is the area of the surface of the earth?*

Solution. $a = 4\pi r^2$. § 689

Here $r = 4000$ mi. Given

Then $a = 4\pi \times 4000^2$ Ax. 9

$= 64{,}000{,}000\pi$

$= 201{,}062{,}400$.

Therefore the area of the earth's surface is 201,062,400 sq. mi. Q.E.F.

33. *The altitude of the torrid zone is* 3200 *mi. Find its area.*

Solution. Let z denote the area of the torrid zone in square miles.

Then $z = 3200 \times 2 \times 4000\,\pi$ § 691

$= 25{,}600{,}000\,\pi$

$= 80{,}424{,}960.$

Therefore the area of the torrid zone is 80,424,960 sq. mi. Q.E.F.

34. *What is the area of the north temperate zone if its altitude is* 1800 *mi.?*

Solution. Let z denote the area of the north temperate zone in square miles.

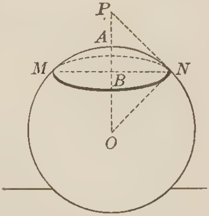
Then $z = 1800 \times 2 \times 4000\,\pi$ § 691

$= 14{,}400{,}000\,\pi$

$= 45{,}239{,}040.$

Therefore the area of the north temperate zone is 45,239,040 sq. mi. Q.E.F.

35. *Find the number of square miles of the earth's surface that can be seen from an aëroplane* 1500 *ft. above the surface.*



Given P the position of the aëroplane 1500 ft. above the surface of the earth, represented by the sphere whose center is O; also AMN the zone whose surface can be seen from P.

Required to find the number of square miles in the area of the zone AMN.

Solution. Let the $\odot\, MN$ be the base of the zone AMN.

Draw the line PO, intersecting the surface of the given sphere in A, and the $\odot\, MN$ in B.

Now P and O are each equidistant from all points in the $\odot\, MN$.

Therefore the line PO, determined by the points P and O, bisects perpendicularly every line drawn through its foot in the plane of the $\odot\, MN$ and terminated by the circumference. § 151

$\therefore\ PO$ is $\perp$ to the $\odot\, MN$ at its center B. § 431

Draw PN tangent to the sphere at N, a point in the $\odot\, MN$.

Also draw ON and BN.

Now ON is $\perp$ to PN. § 185

Moreover, PB is $\perp$ to BN. § 430

Then in the rt. $\triangle\, PNO$,

$OB : ON = ON : PO.$ § 294, 3

$\therefore\ OB \times PO = \overline{ON}^2.$ § 261

$\therefore\ OB \times (PA + OA) = \overline{ON}^2.$ Ax. 11

Since $PA = 1500 \text{ ft.} = \frac{1500}{5280} \text{ mi.} = \frac{25}{88} \text{ mi.},$

and $OA = ON = 4000 \text{ mi.},$

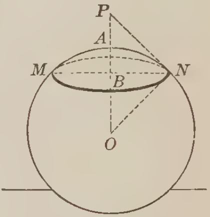
$$\therefore\ 4000\tfrac{25}{88} \times OB = 4000^2. \qquad \text{Ax. 9}$$

$$\therefore\ \tfrac{352025}{88} \times OB = 16{,}000{,}000.$$

$$\therefore\ OB = \frac{16000000}{\frac{352025}{88}} \qquad \text{Ax. 4}$$

$$= \tfrac{56320000}{14081}$$

$$= 3999\tfrac{10081}{14081}.$$



Now $AB = OA - OB$

$$= 4000 - 3999\tfrac{10081}{14081} \qquad \text{Ax. 9}$$

$$= \tfrac{4000}{14081}.$$

Since the area of any zone

$$= 2\pi ra, \qquad \text{§ 691}$$

∴ the area of the zone AMN

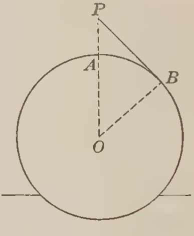
$$= 2 \times 4000\pi \times \tfrac{4000}{14081} \qquad \text{Ax. 9}$$

$$= \frac{32000000\pi}{14081}$$

$$= 7139.493.$$

Therefore the area of the zone AMN is 7139.493 sq. mi. Q.E.F.

36. *How far in one direction can a man see from the deck of an ocean steamer if his eye is 40 ft. above the water?*



Given P the position of the observer's eye 40 ft. above the surface of the earth, represented by the sphere whose center is O, and PB a tangent to the given sphere at B.

Required to find the length of PB.

Solution. Draw PO and OB.

Now OB is ⊥ to PB. § 185

In the rt. △ PBO,

$$\overline{PB}^2 = \overline{PO}^2 - \overline{OB}^2 \qquad \text{§ 338}$$

$$= (PA + OA)^2 - \overline{OB}^2 \qquad \text{Ax. 11}$$

$$= (\tfrac{40}{5280} + 4000)^2 - 4000^2 \qquad \text{Ax. 9}$$

$$= (\tfrac{1}{132} + 4000)^2 - 4000^2$$

$$= (4000\tfrac{1}{132})^2 - 4000^2$$

$$= (4000\tfrac{1}{132} + 4000)(4000\tfrac{1}{132} - 4000)$$

$$= 8000\tfrac{1}{132} \times \tfrac{1}{132}$$

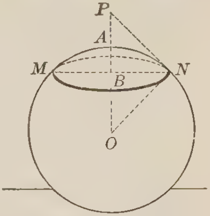
$$= 1{,}056{,}001 \times (\tfrac{1}{132})^2.$$

$$\therefore\ PB = \tfrac{1}{132}\sqrt{1{,}056{,}001} \qquad \text{Ax. 5}$$

$$= \frac{1027.62}{132}, \text{ or } 7.78.$$

Therefore the distance the observer can see in one direction is 7.78 mi. Q.E.F.

37. *To what height must a man be raised above the earth in order to see one sixth of its surface?*



Given P the position of the observer from which the area of the zone AMN, which is one sixth of the surface of the earth, represented by the sphere whose center is O, can be seen.

Required to find the height of the observer above the earth's surface.

Solution. Draw the line PO, intersecting the surface of the given sphere in A, and the circular base MN of the zone AMN in B.

Draw a tangent PN to the given sphere at any point N in the $\odot MN$.

Draw BN and ON.

Now P and O are each equidistant from all points in the $\odot MN$.

Therefore the line PO, determined by the points P and O, bisects perpendicularly every line drawn through its foot in the plane of the $\odot MN$ and terminated by the circumference. § 151

$\therefore$ PO is $\perp$ to the $\odot MN$ at its center B. § 431

The area of the earth's surface $= 4\pi r^2$ § 689

$= 4 \times 4000^2\pi$ Ax. 9

$= 64{,}000{,}000\pi$.

Now the area of the zone $AMN = \frac{1}{6} \times 64{,}000{,}000\pi$ Given

$= \dfrac{32000000\pi}{3}$.

But the area of the zone $AMN = 2\pi r \times AB$. § 691

$\therefore 2\pi r \times AB = \dfrac{32000000\pi}{3}$. Ax. 8

$\therefore r \times AB = \frac{16000000}{3}$. Ax. 4

$\therefore 4000 \times AB = \frac{16000000}{3}$. Ax. 9

$\therefore AB = \frac{4000}{3}$. Ax. 4

Since $OB = OA - AB$,

$\therefore OB = 4000 - \frac{4000}{3}$ Ax. 9

$= \frac{8000}{3}$.

Now PO is $\perp$ to BN. § 430

Also ON is $\perp$ to PN. § 185

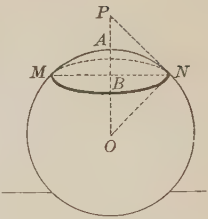
In the rt. $\triangle OBN$, $\overline{BN}^2 = \overline{ON}^2 - \overline{OB}^2$ § 338

$= 4000^2 - (\frac{8000}{3})^2$ Ax. 9

$= 4000^2 \times [1 - (\frac{2}{3})^2]$

$= 16{,}000{,}000 \times \frac{5}{9}$

$= \frac{80000000}{9}$.



Since $OB : BN = BN : PB$, § 294, 2

$\therefore OB \times PB = \overline{BN}^2$. § 261

$\therefore \frac{8000}{3} \times PB = \frac{80000000}{9}$. Ax. 9

$\therefore PB = \frac{10000}{3}$. Ax. 4

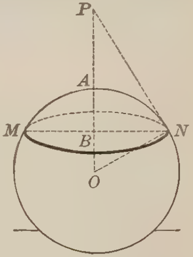
Since $PA = PB - AB$,

$\therefore PA = \frac{10000}{3} - \frac{4000}{3}$ Ax. 9

$= \frac{6000}{3}$, or 2000.

Therefore the height of the observer above the earth's surface is 2000 mi. Q. E. F.

38. *How much of the earth's surface would a man see if he were raised to the height of the radius above it?*



Given P the position of the observer 4000 mi. above the surface of the earth, represented by the sphere whose center is O; also AMN the zone whose surface can be seen from P.

Required to find what part of the earth's surface the area of the given zone is.

Solution. Let the ⊙ MN be the base of the zone AMN.

Draw the line PO, intersecting the surface of the given sphere in A, and the ⊙ MN in B.

Now P and O are each equidistant from all points in the ⊙ MN.

Therefore the line PO, determined by the points P and O, bisects perpendicularly every line drawn through its foot in the plane of the ⊙ MN and terminated by the circumference. § 151

$\therefore PO$ is ⊥ to the ⊙ MN at its center B. § 431

Draw PN tangent to the sphere at N, a point in the ⊙ MN.

Also draw ON and BN.

Now ON is ⊥ to PN. § 185

Moreover PB is ⊥ to BN. § 430

Then in the rt. △ PNO,

$\overline{PN}^2 = \overline{PO}^2 - \overline{ON}^2$. § 338

Since $ON = 4000$ mi., Given

and $PO = PA + OA$ Ax. 11

$= 4000$ mi. $+ 4000$ mi., or 8000 mi., Ax. 9

$\therefore \overline{PN}^2 = 8000^2 - 4000^2$ Ax. 9

$= 64{,}000{,}000 - 16{,}000{,}000$, or 48,000,000.

Since $PO = 2 \times ON$, Given

$\therefore \angle OPN = 30°$. Ex. 2, p. 80, Converse

$\therefore \angle PNB = 60°$. § 107

Then in the rt. △ PBN,

$PN = 2\,BN.$ Ex. 2, p. 80

Whence $BN = \frac{1}{2}\,PN.$ Ax. 4

Also $\overline{PB}^2 = \overline{PN}^2 - \overline{BN}^2.$ § 338

$\therefore \overline{PB}^2 = \overline{PN}^2 - (\frac{1}{2}\,PN)^2$ Ax. 9

$= \overline{PN}^2 - \frac{1}{4}\,\overline{PN}^2$

$= \frac{3}{4}\,\overline{PN}^2$

$= \frac{3}{4} \times 48{,}000{,}000$ Ax. 9

$= 36{,}000{,}000.$

$\therefore PB = \sqrt{36000000}$ Ax. 5

$= 6000.$

Since $AB = PB - PA,$

$\therefore AB = 6000 - 4000$ Ax. 9

$= 2000.$

Now the area of the zone AMN

$= 2\,\pi r \times AB$ § 691

$= 2 \times 4000\,\pi \times 2000$ Ax. 9

$= 16{,}000{,}000\,\pi.$

But the area of the earth's surface

$= 4\,\pi r^2$ § 689

$= 4 \times 4000^2\,\pi$ Ax. 9

$= 64{,}000{,}000\,\pi.$

Since $16{,}000{,}000\,\pi = \frac{1}{4}$ of $64{,}000{,}000\,\pi,$

therefore the area of the zone AMN is $\frac{1}{4}$ of the area of the earth's surface. Q.E.F.

39. *If the atmosphere extends* 50 *mi. above the surface of the earth, find the volume of the atmosphere.*

Solution. The volume of the atmosphere is the difference between the volumes of two spheres, one with the radius 4050 mi., the other with the radius 4000 mi.

Therefore the volume of the atmosphere

$= \frac{4}{3} \times 4050^3\,\pi - \frac{4}{3} \times 4000^3\,\pi$ § 706

$= \frac{4}{3}\,\pi \times (4050^3 - 4000^3)$

$= \frac{4}{3}\,\pi \times (66430125000 - 64000000000)$

$= \frac{4}{3}\,\pi \times 2{,}430{,}125{,}000$

$= \frac{4}{3} \times 3.1416 \times 2{,}430{,}125{,}000$

$= 4188.8 \times 2{,}430{,}125$

$= 10{,}179{,}307{,}600.$

Therefore the atmosphere contains 10,179,307,600 cu. mi. Q.E.F.

40. *If an iron ball 4 in. in diameter weighs 9 lb., find the weight of a spherical iron shell 2 in. thick, the external diameter being 20 in.*

Solution. The volume of the given spherical iron shell is the difference between the volumes of two spheres, one with the diameter 20 in., the other with the diameter 20 in. – 4 in., or 16 in.

Therefore the volume of the given spherical iron shell

$$\begin{aligned} &= \tfrac{1}{6} \times 20^3 \pi - \tfrac{1}{6} \times 16^3 \pi && \S\ 706 \\ &= \tfrac{1}{6} \pi \times (20^3 - 16^3) \\ &= \tfrac{1}{6} \pi \times (8000 - 4096) \\ &= \tfrac{1}{6} \pi \times 3904. \end{aligned}$$

The volume of the given iron ball

$$\begin{aligned} &= \tfrac{1}{6} \times 4^3 \pi && \S\ 706 \\ &= \tfrac{1}{6} \pi \times 64. \end{aligned}$$

Denoting by x the number of pounds in the weight of the given spherical iron shell, we have

$$\frac{\frac{1}{6} \pi \times 64}{\frac{1}{6} \pi \times 3904} = \frac{9}{x}.$$

$$\therefore \frac{1}{61} = \frac{9}{x}.$$

Whence $x = 549.$ § 261

Therefore the weight of the given iron shell is 549 lb. Q. E. F.

41. *What is the angle of a spherical wedge if its volume is $1\frac{1}{4}$ cu. ft. and the volume of the entire sphere is $8\frac{3}{4}$ cu. ft.?*

Solution. Let x denote the number of degrees in the angle of the spherical wedge.

Then $$\frac{\text{vol. of wedge}}{\text{vol. of sphere}} = \frac{x}{360}.$$

$$\therefore \frac{1\frac{1}{4}}{8\frac{3}{4}} = \frac{x}{360}. \qquad \text{Ax. 9}$$

$$\therefore \frac{1}{7} = \frac{x}{360}.$$

Whence $7x = 360.$ § 261

$\therefore x = 51\frac{3}{7}.$ Ax. 4

Therefore the angle of the given spherical wedge is $51\frac{3}{7}°$. Q. E. F.

42. *The inside of a washbasin is in the shape of the segment of a sphere. The distance across the top is 16 in. and its greatest depth is 8 in. How many pints of water will it hold, allowing 7 gal. to the cubic foot?*

Solution. Since the greatest depth is 8 in. and the diameter is 16 in., the figure is a hemisphere.

Now $2r = 16$ in.,
and $r = 8$ in.

Therefore the volume of the hemisphere whose dimensions are those of the given washbasin

$= \frac{1}{2}$ of $\frac{4}{3}\pi r^3$
$= \frac{2}{3}\pi r^3$
$= \frac{2}{3}\pi\, 8^3$
$= \frac{2}{3}\pi \cdot 512$
$= 341\frac{1}{3}\pi$, or 1072.3328.

Therefore the number of pints the washbasin will contain

$= 7 \times 8 \times \frac{1}{1728} \times 1072.3328$
$= 34.7515.$ Q. E. F.

43. *Prove that the volume of a spherical pyramid is equal to the product of the base by one third of the radius, and find the volume if the base is one eighth of the surface of a sphere of radius* 10 *in.*

(1) Given a spherical pyramid of radius r with volume v' and base area a', the volume and area of the given sphere of radius r being v and a respectively.

To prove that $v' = a' \times \frac{1}{3}r$.

Proof. $v : v' = a : a'$.
Now $v = a \times \frac{1}{3}r$. § 705
$\therefore a \times \frac{1}{3}r : v' = a : a'$. Ax. 9
$\therefore a \times \frac{1}{3}r : a = v' : a'$. § 265
$\therefore \frac{1}{3}r : 1 = v' : a'$.
$\therefore v' = a' \times \frac{1}{3}r$, by § 261. Q. E. D.

(2) Given a spherical pyramid of radius 10 in. with base area equal to one eighth of the area of a sphere of radius 10 in.

Required to find the volume of this spherical pyramid.

Solution. The area of the given sphere

$= 4\pi r^2$ § 689
$= 4\pi \times 10^2$ Ax. 9
$= 400\pi$.

Therefore the area of the base of the given spherical pyramid

$= \frac{1}{8} \times 400\pi$ sq. in.
$= 50\pi$ sq. in., or 157.08 sq. in.

Therefore the volume of the given spherical pyramid

$= 157.08 \times \frac{1}{3} \times 10$ cu. in. (1)
$= 523.6$ cu. in. Q. E. F.

44. *Find the volume of a spherical sector whose base is a zone of area z, the radius of the sphere being r, following a process of reasoning similar to that in* § 705.

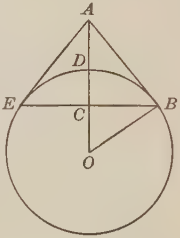
Solution. Since the volume of a sphere is equal to the product of the area of its surface by one third of its radius (§ 705), by analogous reasoning we have the volume of the given spherical sector

$$= z \times \tfrac{1}{3} r$$

$$= \tfrac{1}{3} zr, \text{ by § 708.} \qquad \text{Q. E. F.}$$

Exercise 111. Page 426

1. *Find the area z of the zone of a sphere of radius r, illuminated by a lamp placed at the height h above the surface.*



Given EDB the zone of the sphere whose center is O, illuminated by a lamp placed at A. $AD = h$, $OB = OD = r$, and the area of the zone $EDB = z$.

Required to find z in terms of r and h.

Solution. OB is $\perp$ to AB. § 185

Also OA is $\perp$ to BC. § 430

Since $OA : OB = OB : OC$, § 294, 3

$\therefore \overline{OB}^2 = OA \times OC$ § 261

$= (AD + OD) \times OC.$ Ax. 11

$\therefore r^2 = (h + r) \times OC.$ Ax. 9

$\therefore OC = \dfrac{r^2}{h + r}.$ Ax. 4

Since $CD = OD - OC,$

$\therefore CD = r - \dfrac{r^2}{h + r}$ Ax. 9

$= \dfrac{hr + r^2 - r^2}{h + r}$

$= \dfrac{hr}{h + r}.$

Now $z = 2\pi ra.$ § 691

Here $a = CD = \dfrac{hr}{h + r}.$ Proved

$\therefore z = 2\pi \left(\dfrac{hr}{h + r}\right) r$ Ax. 9

$= \dfrac{2\pi hr^2}{h + r}.$ Q. E. F.

2. *Find the volume v of a sphere in terms of c, the circumference.*

$$v = \tfrac{1}{6}\pi d^3. \qquad \S\ 706$$

But $$d = \frac{c}{\pi}. \qquad \S\ 384$$

$$\therefore v = \tfrac{1}{6}\pi\left(\frac{c}{\pi}\right)^3 \quad \text{Ax. 9}$$

$$= \frac{c^3}{6\pi^2}.$$

3. *Find the radius r of a sphere in terms of v, the volume.*

$$v = \tfrac{4}{3}\pi r^3. \qquad \S\ 706$$

$$\therefore r^3 = \frac{v}{\tfrac{4}{3}\pi} \qquad \text{Ax. 4}$$

$$= \frac{3v}{4\pi}.$$

$$\therefore r = \sqrt[3]{\frac{3v}{4\pi}} \qquad \text{Ax. 5}$$

$$= \sqrt[3]{\frac{6\pi^2 v}{8\pi^3}}$$

$$= \frac{1}{2\pi}\sqrt[3]{6\pi^2 v}.$$

4. *Find the diameter d of a sphere in terms of s, the area of the surface.*

$$s = 2\pi r d. \qquad \S\ 688$$

But $$r = \tfrac{1}{2}d.$$

$$\therefore 2\pi d\left(\tfrac{1}{2}d\right) = s. \qquad \text{Ax. 9}$$

$$\therefore \pi d^2 = s.$$

$$\therefore d^2 = \frac{s}{\pi}. \qquad \text{Ax. 4}$$

$$\therefore d = \sqrt{\frac{s}{\pi}} \qquad \text{Ax. 5}$$

$$= \sqrt{\frac{\pi s}{\pi^2}}$$

$$= \frac{1}{\pi}\sqrt{\pi s}.$$

5. *Find the circumference c of a sphere in terms of s, the area of the surface.*

$$s = 2\pi r d. \qquad \S\ 688$$

But $$d = \frac{c}{\pi}. \qquad \S\ 384$$

Whence $$r = \frac{c}{2\pi}. \qquad \text{Ax. 4}$$

$$\therefore 2\pi\left(\frac{c}{2\pi}\right)\left(\frac{c}{\pi}\right) = s. \qquad \text{Ax. 9}$$

$$\therefore \frac{c^2}{\pi} = s.$$

$$\therefore c^2 = \pi s. \qquad \text{Ax. 3}$$

$$\therefore c = \sqrt{\pi s}. \qquad \text{Ax. 5}$$

6. *What is the altitude a of a zone, if its area is z and the volume of the sphere is v?*

$$z = ac. \qquad \S\ 691$$

But $$c = \pi d. \qquad \S\ 384$$

$$\therefore \pi a d = z. \qquad \text{Ax. 9}$$

Now $$v = \tfrac{1}{6}\pi d^3. \qquad \S\ 706$$

$$\therefore d^3 = \frac{v}{\tfrac{1}{6}\pi} \qquad \text{Ax. 4}$$

$$= \frac{6v}{\pi}.$$

$$\therefore d = \sqrt[3]{\frac{6v}{\pi}} \qquad \text{Ax. 5}$$

$$= \sqrt[3]{\frac{6\pi^2 v}{\pi^3}}$$

$$= \frac{1}{\pi}\sqrt[3]{6\pi^2 v}.$$

Substitute this value of d in the equation $\pi a d = z$.

Then

$$\pi a\left(\frac{1}{\pi}\sqrt[3]{6\pi^2 v}\right) = z. \qquad \text{Ax. 9}$$

$$\therefore a\sqrt[3]{6\pi^2 v} = z.$$

$$\therefore a = \frac{z}{\sqrt[3]{6\pi^2 v}} \qquad \text{Ax. 4}$$

$$= \sqrt[3]{\frac{z^3}{6\pi^2 v}}$$

$$= \sqrt[3]{\frac{36\pi v^2 z^3}{216\pi^3 v^3}}$$

$$= \frac{z}{6\pi v}\sqrt[3]{36\pi v^2}.$$

7. *Show that in a spherical pyramid* $v = \frac{1}{3}br$. *Find* r *in terms of* v *and* b; *also* b *in terms of* v *and* r.

Solution. (1) For the proof that $v = \frac{1}{3}br$, see Ex. 43, p. 425.

(2)

$$v = \frac{1}{3}br. \qquad (1)$$

$$\therefore r = \frac{v}{\frac{1}{3}b} \qquad \text{Ax. 4}$$

$$= \frac{3v}{b}. \qquad \text{Q.E.F.}$$

(3)

$$v = \frac{1}{3}br.$$

That is,

$$\frac{1}{3}br = v.$$

$$\therefore b = \frac{v}{\frac{1}{3}r} \qquad \text{Ax. 4}$$

$$= \frac{3v}{r}. \qquad \text{Q.E.F.}$$

8. *Find a formula for the volume of the metal in a spherical iron shell, the inside radius being* r *and the thickness of the metal being* t.

Solution. Let v denote the volume of the metal in the given spherical iron shell.

Since the volume of the metal in the shell is the difference between the volumes of two spheres, the one with radius $r + t$, the other with radius r,

$$\therefore v = \frac{4}{3}\pi(r+t)^3 - \frac{4}{3}\pi r^3 \qquad \S\ 706$$

$$= \frac{4}{3}\pi[(r+t)^3 - r^3]$$

$$= \frac{4}{3}\pi(r^3 + 3r^2t + 3rt^2 + t^3 - r^3)$$

$$= \frac{4}{3}\pi t(t^2 + 3rt + 3r^2). \qquad \text{Q.E.F.}$$

9. *Find a formula for the weight of a spherical shell, the inside radius being* r, *the thickness of the metal being* t, *and the weight of a cubic unit of metal being* w.

Solution. Let v denote the volume, and W the weight, of the metal in the given spherical shell.

Since the volume of the metal in the shell is the difference between the volumes of two spheres, the one with radius $r + t$, the other with radius r,

$$\therefore v = \frac{4}{3}\pi(r+t)^3 - \frac{4}{3}\pi r^3 \qquad \S\ 706$$

$$= \frac{4}{3}\pi[(r+t)^3 - r^3]$$

$$= \frac{4}{3}\pi(r^3 + 3r^2t + 3rt^2 + t^3 - r^3)$$

$$= \frac{4}{3}\pi t(t^2 + 3rt + 3r^2).$$

$$\therefore W = \frac{4}{3}\pi wt(t^2 + 3rt + 3r^2). \qquad \text{Q.E.F.}$$

10. *If the area of a zone z equals $2\pi ra$ (§ 691), find a formula for a in terms of z and r.*

Solution. $$z = 2\pi ra. \qquad \text{§ 691}$$

$$\therefore a = \frac{z}{2\pi r}, \text{ by Ax. 4.} \qquad \text{Q.E.F.}$$

11. *If the area of a zone is expressd by the formula $z = 2\pi ra$, what is the diameter of the sphere upon which a zone z has an altitude a?*

Solution. $$z = 2\pi ra. \qquad \text{§ 691}$$

$$\therefore 2r = \frac{z}{\pi a}. \qquad \text{Ax. 4}$$

That is, $$d = \frac{z}{\pi a}. \qquad \text{Q.E.F.}$$

12. *Find the area z of a zone of altitude a on a sphere whose area of surface is s.*

Solution. $s = 4\pi r^2$, where r is the radius of the given sphere. § 689

$$\therefore r^2 = \frac{s}{4\pi}. \qquad \text{Ax. 4}$$

$$\therefore r = \sqrt{\frac{s}{4\pi}} \qquad \text{Ax. 5}$$

$$= \sqrt{\frac{\pi s}{4\pi^2}}$$

$$= \frac{1}{2\pi}\sqrt{\pi s}.$$

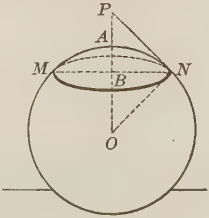
But $$z = 2\pi ra. \qquad \text{§ 691}$$

$$\therefore z = 2\pi a \times \frac{1}{2\pi}\sqrt{\pi s} \qquad \text{Ax. 9}$$

$$= a\sqrt{\pi s}. \qquad \text{Q.E.F.}$$

13. *Find a formula for the area a of that part of the surface of a sphere of radius r seen from a point at a distance d above the surface.*

Given the zone AMN of area a of the sphere whose center is O and radius r, seen from P, a point at the distance d above the surface.



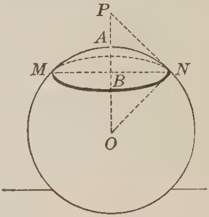
Required to find a in terms of r and d.

Solution. Draw PO intersecting the spherical surface in A and the circular base MN of the zone AMN in B.

Also draw PN tangent to the sphere at N in the ⊙ MN, and the radius ON.

Now the △ PNO is a rt. △. § 185

Furthermore, since P and O are each equidistant from all points in the $\odot MN$,



$\therefore PO$ is $\perp$ to the plane of the $\odot MN$ at its center B. §§ 151, 431

$\therefore PO$ is $\perp$ to BN. § 430

Then in the rt. $\triangle PNO$,

$PO : ON = ON : OB$. § 294, 3

$\therefore \overline{ON}^2 = PO \times OB$. § 261

But $ON = r$, and $PO = r + d$.

$\therefore r^2 = (r + d) \times OB$. Ax. 9

$\therefore OB = \dfrac{r^2}{r+d}$. Ax. 4

Now $AB = OA - OB$.

$\therefore AB = r - \dfrac{r^2}{r+d}$ Ax. 9

$= \dfrac{r^2 + rd - r^2}{r+d}$

$= \dfrac{rd}{r+d}$.

Now the area of any zone is equal to the product of the circumference of a great $\odot$ by the altitude. § 691

$\therefore a = 2\,\pi r \times AB$ Ax. 9

$= 2\,\pi r \times \dfrac{rd}{r+d}$ Ax. 9

$= \dfrac{2\,\pi r^2 d}{r+d}$. Q.E.F.

Exercise 112. Page 427

1. *Find the locus of a point at a given distance from a given point.*

Given d a fixed distance and A a fixed point.

Required to find the locus of a point at the distance d from the point A.

Construction. With A as a center and d as a radius, describe a spherical surface.

Then this spherical surface is the locus required. Q.E.F.

Proof. Any point on the spherical surface is at the distance d from the point A.

Any point either within or without the sphere is at a distance less, or greater, than d from the point A.

Therefore the spherical surface constructed is the locus of a point at the distance d from the point A. Q.E.D.

2. *Find the locus of a point at a given distance from a given straight line.*

Given AB a straight line, and d a fixed distance.

Required to find the locus of a point at the distance d from the line AB.

Construction. On AB as a side construct a rectangle $ABCD$, making $BC = d$.

Revolve the rectangle $ABCD$ about AB as an axis, thereby generating a cylinder of revolution.

Then this cylindric surface is the locus required. Q. E. F.

Proof. Any point on the cylindric surface is at the distance d from AB.

Any point either within or without the cylinder of revolution is at a distance less, or greater, than d from AB.

Therefore the cylindric surface constructed is the locus of a point at the distance d from the line AB. Q. E. D.

3. *Find the locus of a point at a given distance from a given plane.*

Given MN any plane, and d a fixed distance.

Required to find the locus of a point at the distance d from the plane MN.

Construction. At any point P in the plane MN erect a $\perp PF = d$.

Produce FP, making $PF' = d$.

Through F and F' pass the planes RS and TV respectively, $\parallel$ to the plane MN.

Then the planes RS and TV constitute the locus required. Q. E. F.

Proof. Any point in the plane RS or in the plane TV is at the distance d from the plane MN.

Any point not in the plane RS or in the plane TV is at a distance less, or greater, than d from the plane MN.

Therefore the planes RS and TV constitute the locus of a point at the distance d from the plane MN. Q. E. D.

4. *Find the locus of a point at a given distance from a given cylindric surface.*

Given AB any cylindric surface, and d a fixed distance.

Required to find the locus of a point at the distance d from the cylindric surface AB.

Construction. Let CD be the directrix of the given cylindric surface AB.

On opposite sides of the surface AB, at the distance d from AB, draw the lines EF and GH $\parallel$ to the line CD.

Generate two new cylindric surfaces, KL and MN, by moving the lines EF and GH so as to be always at the distance d from the cylindric surface AB.

Then the cylindric surfaces KL and MN constitute the locus required. Q.E.F.

Proof. Any point in the cylindric surface KL or in the cylindric surface MN is at the distance d from the given cylindric surface AB.

Any point not in the cylindric surface KL or in the cylindric surface MN is at a distance less, or greater, than the distance d from the given cylindric surface AB.

Therefore the cylindric surfaces KL and MN constitute the locus of a point at the distance d from the cylindric surface AB. Q.E.D.

5. *Find the locus of a point at a given distance from a given spherical surface.*

Given a spherical surface having r as a radius; also a fixed distance d.

Required to find the locus of a point at the distance d from the given spherical surface.

Construction. Let O be the center of the given spherical surface. With O as a center, and with the radii $r + d$ and $r - d$, describe two new spherical surfaces.

Then these two spherical surfaces constitute the locus required. Q.E.F.

Proof. Any point on either of the two spherical surfaces constructed is at the distance d from the given spherical surface.

Any point not in either of the spherical surfaces constructed is at a distance less, or greater, than the distance d from the given spherical surface.

Therefore the two spherical surfaces constructed constitute the locus of a point at the distance d from the given spherical surface. Q.E.D.

Discussion. For the case where $r - d = 0$, the center of the given spherical surface is a point fulfilling the required conditions; for the case where $r - d$ is a negative number there is obviously but one spherical surface fulfilling the required conditions.

6. *Find the locus of a point equidistant from two given points.*

The locus of a point equidistant from two given points is the plane $\perp$ to the line joining the given points at its mid-point.

7. *Find the locus of a point equidistant from two given planes.*

Given the planes MN and PQ.

Required to find the locus of a point equidistant from the planes MN and PQ.

(1) When the given planes are ∥.

Construction. Erect a ⊥ to the plane MN at any point H in MN.

Produce this ⊥ to meet the plane PQ in K.

Bisect the line HK in A.

Through A pass the plane RS ∥ to the plane MN.

Then the plane RS is the locus required. Q.E.F.

Proof. The plane RS is ∥ to the plane PQ.

Then any point in RS is equidistant from the planes MN and PQ, and any point without the plane RS is unequally distant from the planes MN and PQ.

Therefore the plane RS is the locus of points equidistant from the ∥ planes MN and PQ. Q.E.D.

(2) When the given planes are not ∥.

Construction. Produce the given planes MN and PQ to intersect in the line EF.

Draw the plane TV bisecting the dihedral ∠ EF.

Then the plane TV is the locus required. Q.E.F.

Proof. Any point in the plane TV is equidistant from the planes MN and PQ, and any point without the plane TV is unequally distant from the planes MN and PQ.

Therefore the plane TV is the locus of a point equidistant from the nonparallel planes MN and PQ. Q.E.D.

8. *Find the locus of a point at a given distance from a given point and at another given distance from a given straight line.*

Given the point A, the straight line BC, and the fixed distances d and d'.

Required to find the locus of a point at the distance d from the point A and at the distance d' from the line BC.

Construction. With A as a center and a radius equal to d describe a spherical surface.

On BC as a side construct a rectangle $BCDE$, making $BE = d'$.

Revolve the rectangle $BCDE$ about BC as an axis, thereby generating a cylinder of revolution.

Then the intersection of the spherical and cylindric surfaces is the locus required. Q.E.F.

Proof. Any point in the intersection of the spherical and cylindric surfaces is at the distance d from the point A and at the distance d' from the line BC.

Any point not in the intersection of the spherical and cylindric surfaces is at a distance less, or greater, than d from the point A, or at a distance less, or greater, than d' from the line BC.

Therefore the intersection of the spherical and cylindric surfaces is the locus of a point at the distance d from the point A and at the distance d' from the line BC. Q.E.D.

9. *Find the locus of a point at a given distance from a given point and at another given distance from a given plane.*

Given the point A, the plane MN, and the fixed distances d and d'.

Required to find the locus of a point at the distance d from the point A and at the distance d' from the plane MN.

Construction. With A as a center and d as a radius describe a spherical surface.

At any point P in the plane MN erect $PF \perp$ to the plane MN, making $PF = d'$.

Produce FP to F', making $PF' = d'$.

Through F and F' pass the planes RS and TV respectively, ∥ to the plane MN.

Let the planes RS and TV intersect the spherical surface in the ⓢ CD and GH.

Then the ⓢ CD and GH constitute the locus required. Q.E.F.

Proof. Any point in the ⊙ CD or in the ⊙ GH is at the distance d from the point A and at the distance d' from the plane MN.

Any point not in the ⊙ CD or in the ⊙ GH is at a distance less, or greater, than d from the point A, or at a distance less, or greater, than d' from the plane MN.

Therefore the ⓢ CD and GH constitute the locus of a point at the distance d from the point A and at the distance d' from the plane MN. Q.E.D.

10. *Find the locus of a point at a given distance from a given point and equidistant from two other given points.*

Given the points A, B, and C; also the fixed distance d.

Required to find the locus of a point at the distance d from the point A and equidistant from the points B and C.

Construction. With A as a center and a radius equal to d describe a spherical surface.

Draw the line BC.

Bisect the line BC in H.

Through H pass the plane $MN \perp$ to the line BC.

Let the plane MN intersect the spherical surface in the $\odot$ DE.

Then the $\odot$ DE is the locus required. Q.E.F.

Proof. Any point in the $\odot$ DE is at the distance d from the point A and equidistant from the points B and C.

Any point not in the $\odot$ DE is at a distance less, or greater, than d from the point A, or unequally distant from the points B and C.

Therefore the $\odot$ DE is the locus of a point at the distance d from the point A and equidistant from the points B and C. Q.E.D.

11. *Find the locus of a point at a given distance from a given point and equidistant from two given planes.*

Given the point A, the planes MN and PQ, and the fixed distance d.

Required to find the locus of a point at the distance d from the point A and equidistant from the planes MN and PQ.

(1) When the two given planes are ∥.

Construction. With A as a center and a radius equal to d describe a spherical surface.

Erect a $\perp$ at any point B in the plane MN, and produce it to meet the plane PQ in C.

Bisect the line BC in D.

Through D pass the plane RS ∥ to the plane MN.

Let the plane RS intersect the spherical surface in the $\odot$ EF.

Then the $\odot$ EF is the locus required. Q.E.F.

Proof. Any point in the $\odot$ EF is at the distance d from the point A and equidistant from the planes MN and PQ.

Any point not in the $\odot$ EF is at a distance less, or greater, than d from the point A, or unequally distant from the planes MN and PQ.

Therefore the $\odot$ EF is the locus of a point at the distance d from the point A and equidistant from the ∥ planes MN and PQ. Q.E.D.

(2) When the two given planes are not ∥.

Construction. With A as a center and a radius equal to d describe a spherical surface.

Produce the planes MN and PQ to intersect in the line BC.

Bisect the dihedral $\angle BC$ by the plane TV.

Produce the plane TV to intersect the spherical surface in the $\odot$ DE.

Then the $\odot$ DE is the locus required. Q.E.F.

Proof. Any point in the ⊙ DE is at the distance d from the point A and equidistant from the planes MN and PQ.

Any point not in the ⊙ DE is at a distance less, or greater, than d from the point A, or unequally distant from the planes MN and PQ.

Therefore the ⊙ DE is the locus of a point at the distance d from the point A and equidistant from the nonparallel planes MN and PQ. Q. E. D.

12. *Find one or more points at a distance d_1 from a given point, at a distance d_2 from a given straight line, and at a distance d_3 from a given plane.*

Given the point A, the straight line BC, and the plane MN; also the fixed distances d_1, d_2, and d_3.

Required to find the points which are at the distance d_1 from the point A, at the distance d_2 from the line BC, and at the distance d_3 from the plane MN.

Construction. With A as a center and with a radius equal to d_1 describe a spherical surface.

On the line BC as a side construct the rectangle $BCDE$, making $BE = d_2$.

Revolve the rectangle $BCDE$ about the line BC as an axis, thereby generating a cylindric surface.

Let this cylindric surface intersect the spherical surface in the closed-curved line KL.

Erect a ⊥ PF to the plane MN at any point P in MN, making $PF = d_3$. Produce FP to F', making $PF' = d_3$.

Through F and F' pass the planes RS and TV respectively, ∥ to the plane MN.

Let the planes RS and TV intersect the closed curve KL in the points U and W, and X and Y respectively.

Then the points U, W, X, and Y fulfill the required conditions. Q. E. F.

Proof. The points U, W, X, and Y lie in the surface of the sphere whose center is A; they lie in the cylindric surface generated by the rectangle $BCDE$ revolving about BC as an axis; and they lie in the planes RS and TV, the first two being in RS, the last two in TV.

Therefore the points U, W, X, and Y are at the distance d_1 from the point A, at the distance d_2 from the line BC, and at the distance d_3 from the plane MN. Q. E. D.

13. *Find one or more points at a distance d_1 from a given point, at a distance d_2 from a given plane, and equidistant from two other given planes.*

Given the point A, the planes MN, PQ, and RS, and the fixed distances d_1 and d_2.

Required to find the points which are at the distance d_1 from the point A, at the distance d_2 from the plane MN, and equidistant from the planes PQ and RS.

Construction. With A as a center and with a radius equal to d_1 describe a spherical surface.

Erect the $\perp$ GF to the plane MN at any point G in MN, making $GF = d_2$.

Produce FG to F', making $GF' = d_2$.

Through F and F' pass the planes TV and WX respectively, $\parallel$ to the plane MN.

Let the planes TV and WX intersect the spherical surface in the ⓢ BC and DE respectively.

Next construct a plane YZ equidistant from the given planes PQ and RS as follows:

For the case where PQ and RS are $\parallel$ planes, erect a $\perp$ to the plane PQ at H and produce it to intersect the plane RS in K.

Bisect the line HK in L.

Through L pass the plane YZ $\parallel$ to the plane PQ.

For the case where PQ and RS are nonparallel planes produce these planes to intersect in the line $A'B'$.

Bisect the dihedral $\angle A'B'$ by the plane YZ.

Let the plane YZ intersect the ⓢ BC and DE in the points B and C, and D and E respectively.

Then the points B, C, D, and E fulfill the required conditions. Q.E.F.

Proof. The points B, C, D, and E lie in the surface of the sphere whose center is A; they lie in the planes TV and WX, the first two being in TV, the last two in WX; and they lie in the plane YZ.

Therefore the points B, C, D, and E are at the distance d_1 from the point A, at the distance d_2 from the plane MN, and equidistant from the planes PQ and RS. Q.E.D.

14. *Find one or more points equidistant from two given points, equidistant from two given planes, and at a distance r from a given point.*

Given the points A, B, and C, the planes MN and PQ, and the fixed distance r.

Required to find the points which are equidistant from the points A and B, equidistant from the planes MN and PQ, and at the distance r from the point C.

Construction. Draw the line AB.

Bisect AB in E.

Through E pass the plane RS $\perp$ to AB.

Then construct a plane TV equidistant from the given planes MN and PQ as follows:

For the case where MN and PQ are ∥ planes, erect a ⊥ to the plane MN at D and produce it to intersect the plane PQ in F.

Bisect the line DF in G.

Through G pass the plane TV ∥ to the plane MN.

For the case where MN and PQ are nonparallel planes, produce the planes MN and PQ to intersect in the line $A'B'$.

Bisect the dihedral $\angle A'B'$ by the plane TV.

Let the plane TV intersect the plane RS in the line KL.

About the given point C, with r as a radius, describe a spherical surface.

Let this spherical surface be intersected by the line KL in the points K and L.

Then the points K and L fulfill the required conditions. Q.E.F.

Proof. The points K and L lie in the plane RS, also in the plane TV, and in the surface of the sphere whose center is C.

Therefore the points K and L are equidistant from the points A and B, they are equidistant from the planes MN and PQ, and they are at the distance r from the point C. Q.E.D.

15. *Find the locus of the center of a sphere whose surface touches two given planes and passes through two given points that lie between the planes.*

Given the points A and B, which lie between the planes MN and PQ.

Required to find the locus of the center of a sphere whose surface touches the planes MN and PQ and passes through the points A and B.

Construction. Construct a plane RS equidistant from the given planes MN and PQ as follows:

For the case where MN and PQ are ∥ planes, erect a ⊥ to the plane MN at C and produce it to intersect the plane PQ in D.

Bisect the line CD in E.

Through E pass the plane RS ∥ to the plane MN.

For the case where MN and PQ are nonparallel planes, produce the planes MN and PQ to intersect in the line GH.

Bisect the dihedral $\angle GH$ by the plane RS.

Draw the line AB and bisect AB in K.

Through K pass the plane TV ⊥ to AB.

Let the plane TV intersect the plane RS in the line XY.

Then the line XY is the locus required. Q.E.F.

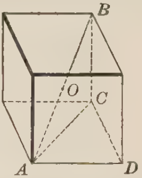
Proof. Any point in the line XY is equidistant from the planes MN and PQ and the points A and B.

Any point not in the line XY is unequally distant from the planes MN and PQ and the points A and B.

Therefore the line XY is the locus of the center of a sphere whose surface touches the planes MN and PQ and passes through the points A and B. Q.E.D.

Exercise 113. Page 428

1. *The volume of a sphere is to the volume of the inscribed cube as* π *is to* $\frac{2}{3}\sqrt{3}$.



Proof. Let O be the mid-point of the diagonal AB of the cube AB.

Since the diagonals of a cube are equal concurrent lines which bisect one another, Ex. 4, p. 325; Ex. 2, p. 327

$\therefore$ O is equidistant from the vertices of the cube.

$\therefore$ O is the center of the sphere of which AB is the inscribed cube. §§ 621, 648

Then OA is the radius of that sphere.

Hence $\frac{4}{3}\pi \cdot \overline{OA}^3 =$ the volume of the sphere. § 706

Draw AC, a diagonal of the face AC of the cube AB.

The line BC is $\perp$ to AC. § 430

In the rt. $\triangle ACB$, $\overline{AB}^2 = \overline{AC}^2 + \overline{BC}^2$. § 337

In the rt. $\triangle ADC$, $\overline{AC}^2 = \overline{AD}^2 + \overline{CD}^2$. § 337

$\therefore \overline{AB}^2 = \overline{AD}^2 + \overline{CD}^2 + \overline{BC}^2$. Ax. 9

But $AD = CD = BC$.

Whence $\overline{AD}^2 = \overline{CD}^2 = \overline{BC}^2$. Ax. 5

$\therefore \overline{AB}^2 = 3\,\overline{AD}^2$. Ax. 9

Now $OA = \frac{1}{2}AB$. Hyp.

$\therefore 2\,OA = AB$. Ax. 3

Whence $4\,\overline{OA}^2 = \overline{AB}^2$. Ax. 5

$\therefore 4\,\overline{OA}^2 = 3\,\overline{AD}^2$. Ax. 8

$\therefore \overline{AD}^2 = \frac{4}{3}\,\overline{OA}^2$. Ax. 4

$\therefore AD = \sqrt{\frac{4}{3}\,\overline{OA}^2}$ Ax. 5

$= \frac{2}{3}\sqrt{3} \cdot OA$.

$\therefore \overline{AD}^3 = \frac{8}{9}\sqrt{3} \cdot \overline{OA}^3$. Ax. 5

That is, $\frac{8}{9}\sqrt{3} \cdot \overline{OA}^3 =$ the volume of the cube. § 534

Therefore the ratio of the volumes of the sphere and the inscribed cube

$$= \frac{\frac{4}{3}\pi \cdot \overline{OA}^3}{\frac{8}{9}\sqrt{3} \cdot \overline{OA}^3}$$

$$= \frac{\pi}{\frac{2}{3}\sqrt{3}}, \text{ or } \pi : \tfrac{2}{3}\sqrt{3}.$$ Q.E.D.

2. *The volume of a sphere is to the volume of the circumscribed cube as π is to 6.*

Proof. Let r be the radius of the given sphere.

Then $2r$ is an edge of the cube circumscribed about the given sphere.

Now the volume of the sphere $= \frac{4}{3}\pi r^3$, § 706

and the volume of the circumscribed cube

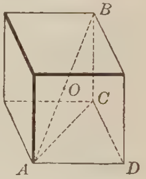
$= (2r)^3 = 8r^3$. § 534

Therefore the ratio of the volumes of the sphere and the circumscribed cube

$$= \frac{\frac{4}{3}\pi r^3}{8r^3}$$

$$= \frac{\pi}{6}, \text{ or } \pi : 6.$$ Q.E.D.

3. *Find the ratio of the volume of a cube inscribed in a sphere to that of a circumscribed cube.*



Solution. It is obvious that the diagonal of the inscribed cube of a sphere is equal to the diameter of the sphere and also to an edge of the circumscribed cube.

Let AB be a diagonal of the cube AB inscribed in a sphere.

Draw AC, a diagonal of the face AC of the cube AB.

The line BC is $\perp$ to AC. § 430

In the rt. $\triangle ACB$, $\overline{AB}^2 = \overline{AC}^2 + \overline{BC}^2$. § 337

In the rt. $\triangle ADC$, $\overline{AC}^2 = \overline{AD}^2 + \overline{CD}^2$. § 337

$\therefore \overline{AB}^2 = \overline{AD}^2 + \overline{CD}^2 + \overline{BC}^2$. Ax. 9

But $AD = CD = BC$.

Whence $\overline{AD}^2 = \overline{CD}^2 = \overline{BC}^2$. Ax. 5

$\therefore \overline{AB}^2 = 3\,\overline{AD}^2$. Ax. 9

$\therefore \overline{AD}^2 = \frac{1}{3}\overline{AB}^2$. Ax. 4

$\therefore AD = \sqrt{\frac{1}{3}\overline{AB}^2}$ Ax. 5

$= \frac{1}{3}\sqrt{3} \cdot AB$.

$\therefore \overline{AD}^3 = \frac{1}{9}\sqrt{3} \cdot \overline{AB}^3$. Ax. 5

That is, the volume of the cube AB inscribed in a sphere

$= \frac{1}{9}\sqrt{3} \cdot \overline{AB}^3$. § 534

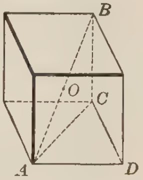
The volume of the circumscribed cube of the sphere

$= \overline{AB}^3$. § 534

Therefore the ratio of the volumes of the inscribed and circumscribed cubes of a sphere $= \dfrac{\frac{1}{9}\sqrt{3}\cdot\overline{AB}^3}{\overline{AB}^3}$

$= \dfrac{1}{3\sqrt{3}}$, or $1:3\sqrt{3}$. Q.E.F.

4. *Find the difference between the volumes of two cubes, one inscribed in a sphere of radius* 10 *in. and the other circumscribed about it.*



Solution. It is obvious that the diagonal of the inscribed cube of a sphere is equal to the diameter of the sphere and also to an edge of the circumscribed cube.

Let AB be a diagonal of the cube AB inscribed in a sphere whose radius is 10 in.

Then $AB = 2 \times 10$ in., or 20 in.

Draw AC, a diagonal of the face AC of the cube AB.

The line BC is $\perp$ to AC. § 430

In the rt. $\triangle ACB$, $\overline{AB}^2 = \overline{AC}^2 + \overline{BC}^2$. § 337

In the rt. $\triangle ADC$, $\overline{AC}^2 = \overline{AD}^2 + \overline{CD}^2$. § 337

$\therefore \overline{AB}^2 = \overline{AD}^2 + \overline{CD}^2 + \overline{BC}^2$. Ax. 9

But $AD = CD = BC$.

Whence $\overline{AD}^2 = \overline{CD}^2 = \overline{BC}^2$. Ax. 5

$\therefore \overline{AB}^2 = 3\,\overline{AD}^2$. Ax. 9

$\therefore \overline{AD}^2 = \frac{1}{3}\overline{AB}^2$. Ax. 4

$\therefore AD = \sqrt{\frac{1}{3}\overline{AB}^2}$ Ax. 5

$= \frac{1}{3}\sqrt{3}\cdot AB$.

$\therefore \overline{AD}^3 = \frac{1}{9}\sqrt{3}\cdot\overline{AB}^3$ Ax. 5

$= \frac{1}{9}\sqrt{3}\cdot 20^3$ Ax. 9

$= \frac{8000}{9}\sqrt{3}$.

That is, the volume of the cube AB inscribed in the given sphere

$= \frac{8000}{9}\sqrt{3}$ cu. in. § 534

The volume of the circumscribed cube of the given sphere

$= \overline{AB}^3$ § 534

$= 20^3$ cu. in. Ax. 9

$= 8000$ cu. in.

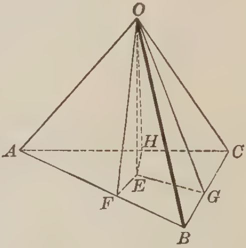
Therefore the difference between the volumes of the inscribed and circumscribed cubes of the given sphere

$= 8000$ cu. in. $- \frac{8000}{9}\sqrt{3}$ cu. in.

$= 8000$ cu. in. $- 1539.6$ cu. in.

$= 6460.4$ cu. in. Q.E.F.

5. *The planes perpendicular to the three faces of a trihedral angle, and bisecting the face angles, meet in a straight line.*



Given in the trihedral $\angle O\text{-}ABC$ the planes OFE, OGE, and $OHE \perp$ to the faces AOB, BOC, and AOC, and bisecting the face $\measuredangle$ AOB, BOC, and AOC respectively.

To prove that the planes OFE, OGE, and OHE meet in a straight line.

Proof. The plane OFE is the locus of a point equidistant from the edges OA and OB. Ex. 1, p. 314

Also the plane OGE is the locus of a point equidistant from the edges OB and OC.

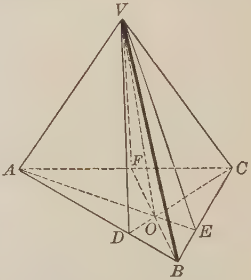
Therefore OE, the line of intersection of the planes OFE and OGE, is equidistant from the edges OA and OC.

But the plane OHE is the locus of a point equidistant from the edges OA and OC.

Therefore the plane OHE contains OE; that is, the plane OHE passes through OE.

Therefore the planes OFE, OGE, and OHE meet in the straight line OE. Q.E.D.

6. *The planes that pass through the edges of a trihedral angle, and are perpendicular to the opposite faces, meet in a straight line.*



Given, in the trihedral $\angle V\text{-}ABC$, planes passed through the edges VA, VB, and VC, $\perp$ to the opposite faces VBC, VAC, and VAB respectively.

To prove that these planes meet in a straight line.

Proof. Let the planes VAE and VBF, $\perp$ to the faces VBC and VAC respectively, intersect in the straight line VO. § 429

Through O, any point in VO, pass a plane $\perp$ to VO, and let this plane intersect the faces of the trihedral $\angle$ in the lines AB, AC, and BC, and the planes VAE and VBF in the lines AE and BF respectively. § 429

Through VC pass the plane $VCD \perp$ to AB, cutting ABC in CD. § 433

The planes VAE, VBF, and VCD are each ⊥ to the plane ABC, and the plane VCD is also ⊥ to the plane VAB. § 477

Since the planes VBC and ABC are each ⊥ to the plane VAE,

∴ BC is ⊥ to the plane VAE. § 478

∴ BC is ⊥ to the line AE. § 430

That is, AE is ⊥ to BC.

Similarly, it may be shown that the lines BF and CD are ⊥ to the lines AC and AB respectively.

Now O is the point of intersection of the lines AE and BF.

Since the lines AE, BF, and CD are concurrent, Ex. 4, p. 78

∴ the line CD passes through O.

That is, O lies in the plane VCD.

Then the line VO lies in the plane VCD. § 422

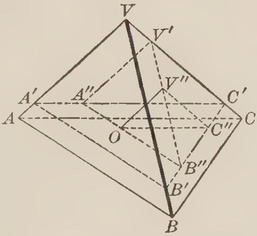
That is, the plane VCD passes through VO.

Therefore the planes VAE, VBF, and VCD meet in the straight line VO. Q. E. D.

7. *The altitude of a regular tetrahedron is equal to the sum of four perpendiculars let fall from any point within the tetrahedron upon the four faces.*

Given O, any point within the regular tetrahedron $V\text{-}ABC$; also the four ⊥s from O upon the four faces of the tetrahedron $V\text{-}ABC$, and the altitude of the tetrahedron $V\text{-}ABC$.

To prove that the sum of the four ⊥s from O upon the four faces of the tetrahedron $V\text{-}ABC$ is equal to the altitude of the tetrahedron $V\text{-}ABC$.



Proof. Through O pass the plane $A'B'C'$ ∥ to the face ABC of the tetrahedron $V\text{-}ABC$. § 457

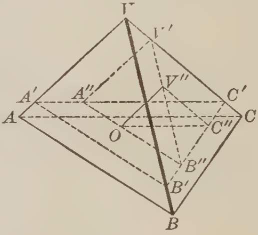
Through O pass the plane $A''V'B''$ ∥ to the face $A'VB'$ of the tetrahedron $V\text{-}A'B'C'$.

Finally, through O pass the plane $OV''C''$ ∥ to the face $A''V'C'$ of the tetrahedron $V'\text{-}A''B''C'$.

The tetrahedrons $V\text{-}A'B'C'$, $V'\text{-}A''B''C'$, and $V''\text{-}OB''C''$, formed in succession, are regular tetrahedrons. § 567

Then the altitude of the tetrahedron $V''\text{-}OB''C''$, plus the ⊥ from O upon the face VAC of the given tetrahedron, is equal to the altitude of the tetrahedron $V'\text{-}A''B''C'$.

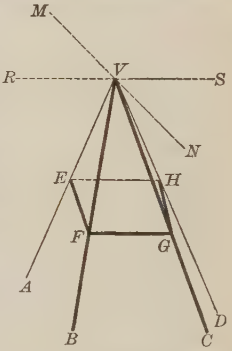
Furthermore the altitude of the tetrahedron V'-$A''B''C'$, plus the $\perp$ from O upon the face VAB of the given tetrahedron, is equal to the altitude of the tetrahedron V-$A'B'C'$.



And the altitude of the tetrahedron V-$A'B'C'$, plus the $\perp$ from O upon the face ABC of the given tetrahedron, is equal to the altitude of the given tetrahedron.

Therefore the sum of the four $\perp$s from O upon the four faces of the tetrahedron V-ABC is equal to the altitude of the tetrahedron V-ABC. Q. E. D.

8. *To cut a given tetrahedral angle by a plane so that the section shall be a parallelogram.*



Given the tetrahedral $\angle V$-$ABCD$.

Required to cut the tetrahedral $\angle V$-$ABCD$ by a plane so that the section shall be a ▱.

Solution. Let the planes of the opposite faces AVB and CVD of the given tetrahedral angle intersect in the line MN. § 429

Also let the planes of the opposite faces AVD and BVC of the given tetrahedral angle intersect in the line RS.

The two intersecting lines MN and RS determine the plane RVM. § 425

Through E, a point on the edge VA of the given tetrahedral angle, pass a plane ∥ to the plane RVM (§ 457), forming the section $EFGH$. § 429

Then $EFGH$ is the ▱ required. Q. E. F.

Proof. EF is ∥ to MN. § 453

Also HG is ∥ to MN.

$\therefore$ EF is ∥ to HG. § 446

Likewise, EH is ∥ to FG.

Therefore the section $EFGH$ is a ▱, by § 118. Q. E. D.

Discussion. The cutting plane may be made to pass through any point on the side of the plane RVM on which the given tetrahedral angle lies. Consequently there is an indefinite number of solutions to this problem.

9. *Compare the volumes of the solids generated by the revolution of a rectangle successively about two adjacent sides, the sides being a and b respectively.*

Solution. The revolution of the given rectangle about its side a as an axis generates a right circular cylinder with altitude a and radius b. § 576

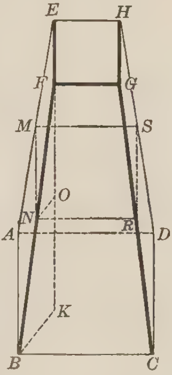
Therefore the volume of this cylinder $= \pi ab^2$. § 590

The revolution of the given rectangle about its side b as an axis generates a right circular cylinder with altitude b and radius a. § 576

Therefore the volume of this second cylinder $= \pi a^2 b$. § 590

Therefore the ratio of the volumes of the two right circular cylinders generated by the revolution of the given rectangle successively about its sides a and b as axes $= \dfrac{\pi ab^2}{\pi a^2 b}$, or $\dfrac{b}{a}$. Q.E.F.

10. *Find the difference between the volume of a frustum of a pyramid and the volume of a prism each 24 ft. high, if the bases of the frustum are squares with sides 20 ft. and 16 ft. respectively, and the base of the prism is the section of the frustum parallel to the bases and midway between them.*



Solution. The volume of a frustum of a pyramid is expressed by the formula $v = \frac{1}{3} a\left(b + b' + \sqrt{bb'}\right)$. § 565

Therefore, for the given frustum $ABCD\text{-}EFGH$,

$$\begin{aligned} v &= \tfrac{1}{3} \times 24\left(20^2 + 16^2 + \sqrt{20^2 \times 16^2}\right) \quad \text{Ax. 9} \\ &= 8\,(400 + 256 + 320) \\ &= 8 \times 976 \\ &= 7808. \end{aligned}$$

Let $MNRS$ be the section of the given frustum $ABCD\text{-}EFGH$ ∥ to the bases and midway between them.

From F let fall the ⊥ FK upon the base $ABCD$, cutting the mid-section $MNRS$ in O. § 437

Draw BK and NO.

The lines NO and BK are ∥. § 453

$\therefore FN : FB = FO : FK$. § 273

But $FO : FK = 1 : 2$. Given

$\therefore FN : FB = 1 : 2$. Ax. 8

That is, N is the mid-point of BF.

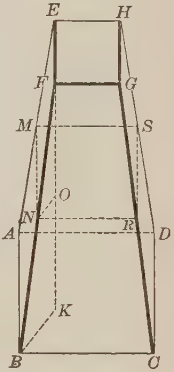
Similarly, it may be shown that R, S, and M are the mid-points of CG, DH, and AE respectively.

Then each side of the mid-section $MNRS$ equals half the sum of a side of square $ABCD$ and a side of square $EFGH$. § 137

That is, a side of $MNRS = \frac{1}{2}(AB + EF)$

$= \frac{1}{2}(20 + 16)$ Ax. 9

$= \frac{1}{2} \times 36$, or 18.



Moreover, the ∠s of $MNRS$ are rt. ∠s. § 461

∴ $MNRS$ is a square 18 ft. on a side.

Now the volume of a prism is expressed by the formula $v = ba$. § 539

Therefore, for the given prism whose base is the mid-section $MNRS$,

$v = 18^2 \times 24$ Ax. 9

$= 324 \times 24$

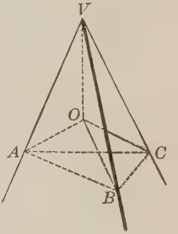
$= 7776$.

Therefore the difference between the volumes of the given frustum of a pyramid and the given prism

$= 7808$ cu. ft. $- 7776$ cu. ft.

$= 32$ cu. ft. Q.E.F.

11. *To draw a line through the vertex of any trihedral angle, making equal angles with its edges.*



Given any trihedral $\angle V\text{-}ABC$ whose edges are VA, VB, and VC.

Required to draw through V a line, making with the edges VA, VB, and VC equal angles.

Construction. Take VA, VB, and VC all equal.

Through the points A, B, and C pass a plane. § 427

From V let fall $VO \perp$ to the plane ABC. § 437

Then the line VO makes equal angles with the edges VA, VB, and VC. Q.E.F.

Proof. Draw the lines AO, BO, and CO.

The line VO is $\perp$ to the plane ABC. Const.

∴ VO is $\perp$ to the lines AO, BO, and CO. § 430

In the rt. △s VOA, VOB, and VOC,

$VO = VO = VO$, Iden.

and $VA = VB = VC$. Const.

∴ the rt. △s VOA, VOB, and VOC are congruent. § 89

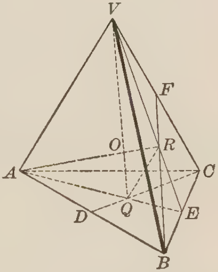
Hence the ∠s AVO, BVO, and CVO are equal, by § 67. Q.E.D.

12. *The lines drawn from each vertex of a tetrahedron to the point of intersection of the medians of the opposite face all meet in a point called the center of gravity of the tetrahedron, which divides each line so that the ratio of the shorter segment to the whole line is* 1 : 4.

Given in the tetrahedron $V\text{-}ABC$ the lines drawn from the vertices to the points of intersection of the medians of the opposite faces.

To prove that these lines are concurrent and divide each other so that the ratio of the shorter segment to the whole line is 1 : 4.

Proof. Let the medians AE and CD of the face $\triangle ABC$ intersect in Q; also let the medians BF and VE of the face $\triangle VBC$ intersect in R.



Let VQ and AR be two of the given lines, drawn to Q and R from V and A respectively.

Draw the line QR.

Now the lines AE and VE determine the plane AEV. § 425

Then QR lies in the plane AEV. § 422

Also the lines AR and VQ lie in the plane AEV. § 422

Therefore the lines AR and VQ intersect in some point O.

Since $QE = \frac{1}{3} AE$,

and $RE = \frac{1}{3} VE$, Ex. 5, p. 78

$\therefore QR$ is $\parallel$ to AV. § 276

Therefore in the $\triangle$s QOR and VOA,

$\angle OQR = \angle OVA$,

and $\angle ORQ = \angle OAV$. § 100

Therefore the $\triangle$s QOR and VOA are similar. § 286

$\therefore QR : VA = OQ : OV = OR : OA$. § 282

Since QR is $\parallel$ to AV, Proved

$\therefore \angle EQR = \angle EAV$,

and $\angle ERQ = \angle EVA$. § 102

Therefore the $\triangle$s QER and AEV are similar. § 286

$\therefore QR : AV = RE : VE$. § 282

But $RE : VE = 1 : 3$. § 264

$\therefore QR : AV = 1 : 3$. Ax. 8

$\therefore OQ : OV = 1 : 3$,

and $OR : OA = 1 : 3$. Ax. 8

Whence $OV = 3\,OQ$,

and $OA = 3\,OR$. § 261

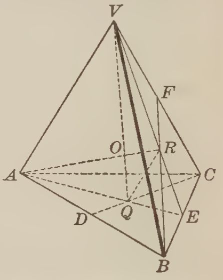
$$\therefore OV + OQ, \text{ or } VQ, = 4\,OQ,$$

and $$OA + OR, \text{ or } AR, = 4\,OR. \qquad \text{Ax. 1}$$

$$\therefore OQ : VQ = 1 : 4,$$

and $$OR : AR = 1 : 4. \qquad \S\ 264$$

In like manner it may be shown that the lines drawn to the points of intersection of the medians in the face ⧍ AVC and AVB from B and C respectively, divide AR so that the ratio of the shorter segment to the whole line is 1 : 4; that is, that these lines also pass through O.



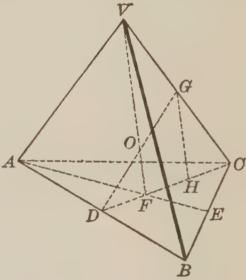
Moreover, it may be shown that AR divides each of these lines so that the ratio of the shorter segment to the whole line is 1 : 4.

Hence the lines drawn to the points of intersection of the medians of the face ⧍ of the given tetrahedron V-ABC from the opposite vertices are concurrent lines whose point of intersection, called the center of gravity of the tetrahedron, divides each of them so that the ratio of the shorter segment to the whole line is 1 : 4. Q.E.D.

13. *The lines joining the mid-points of the opposite edges of a tetrahedron all pass through the center of gravity and are bisected by it.*

Given in the tetrahedron V-ABC the lines joining the mid-points of the opposite edges.

To prove that these lines are concurrent, and that their point of intersection bisects each of them.



Proof. Let DG be one of the given lines joining the mid-points D and G of the opposite edges AB and VC respectively.

Draw the medians CD and AE of the face $\triangle ABC$, letting them intersect in F. Also draw VF.

The lines CD and CV determine the plane DCV. § 425

The lines DG and VF lie in the plane DCV. § 422

Therefore the lines DG and VF intersect in some point O.

Draw $GH \parallel$ to VF, cutting CD in H. § 233

Then $$GH \text{ bisects } CF \text{ in } H. \qquad \S\ 135$$

That is, $$CH = FH.$$

Since $$CF = \tfrac{2}{3}\,CD, \qquad \text{Ex. 5, p. 78}$$

$$\therefore FH = \tfrac{1}{3}\,CD. \qquad \text{Ax. 4}$$

But $DF = \frac{1}{3} CD$. Ex. 5, p. 78

$\therefore DF = FH$. Ax. 8

Since OF is $\parallel$ to GH and bisects DH,

$\therefore OF$ bisects DG. § 135

$\therefore OF = \frac{1}{2} GH$. § 136

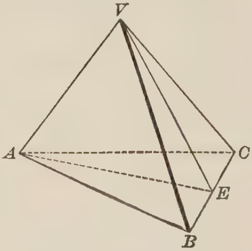
But $GH = \frac{1}{2} VF$. § 136

$\therefore OF = \frac{1}{2} \times \frac{1}{2} VF = \frac{1}{4} VF$. Ax. 9

In like manner it may be shown that the point O, which divides the line VF so that the ratio of the shorter segment to the whole line is 1 : 4, bisects each of the other given lines joining the mid-points of the opposite edges of the tetrahedron $V\text{-}ABC$.

Therefore the lines joining the mid-points of the opposite edges of the tetrahedron $V\text{-}ABC$ are concurrent, and O, their point of intersection, bisects each of them. Q. E. D.

14. *The plane which bisects a dihedral angle of a tetrahedron divides the opposite edge into segments proportional to the areas of the faces that include the dihedral angle.*



Given the plane VAE bisecting the dihedral $\angle B\text{-}VA\text{-}C$ of the tetrahedron $V\text{-}ABC$ and cutting the opposite edge BC in E.

To prove that $\frac{BE}{CE} = \frac{\triangle AVB}{\triangle AVC}$.

Proof. The tetrahedrons $V\text{-}ABE$ and $V\text{-}ACE$ have the same altitude.

Therefore $\frac{V\text{-}ABE}{V\text{-}ACE} = \frac{\triangle ABE}{\triangle ACE}$. § 562

Now the $\triangle$s ABE and ACE have the same altitude on their bases BE and CE respectively.

Therefore $\frac{\triangle ABE}{\triangle ACE} = \frac{BE}{CE}$. § 327

Therefore $\frac{V\text{-}ABE}{V\text{-}ACE} = \frac{BE}{CE}$. Ax. 8

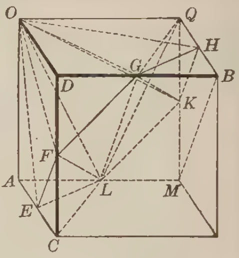
Take E as the common vertex of the tetrahedrons $V\text{-}ABE$ and $V\text{-}ACE$.

Then $V\text{-}ABE$ and $V\text{-}ACE$ have equal altitudes. § 479

Therefore $\frac{V\text{-}ABE}{V\text{-}ACE} = \frac{\triangle AVB}{\triangle AVC}$. § 562

Hence $\frac{BE}{CE} = \frac{\triangle AVB}{\triangle AVC}$, by Ax. 8. Q. E. D.

15. *To cut a given cube by a plane so that the section shall be a regular hexagon.*



Given the cube AB.

Required to cut the cube by a plane so that the section shall be a regular hexagon.

Construction. Bisect the edge AC of the cube in E. § 229

Draw OE.

About O as a center, with OE as a radius, describe a sphere.

Let the surface of this sphere cut the three edges AC, CD, and BD of the cube in E, F, and G respectively, and the edges diagonally opposite these edges in H, K, and L respectively.

Draw the lines EF, FG, GH, HK, KL, and LE.

Then the figure $EFGHKL$ is the section of the cube AB required.

Q. E. F.

Proof. Draw the radii OE, OF, OG, OH, OK, and OL.

Since $OE = OF = OG = OH = OK = OL$, § 621

and $OA = OD = OQ$, § 517

therefore the rt. ◬ OAE, ODF, ODG, OQH, OQK, and OAL are congruent. § 89

$\therefore AE = DF = DG = QH = QK = AL$. § 67

Since $AE = \frac{1}{2} AC$, Const.

$\therefore DF = \frac{1}{2} DC$,

$DG = \frac{1}{2} DB$,

$QH = \frac{1}{2} QB$,

$QK = \frac{1}{2} QM$,

and $AL = \frac{1}{2} AM$. Ax. 4

$\therefore AE = CE = CF = DF = DG = BG = BH = QH = QK = MK = ML = AL$. Ax. 8

Therefore the isosceles rt. ◬ EAL, ECF, FDG, GBH, HQK, and KML are congruent. § 69

$\therefore EL = EF = FG = GH = HK = KL$. § 67

Therefore the figure $EFGHKL$ is equilateral.

Draw the lines CL, FL, QG, and KG.

The rt. ◬ CAL and QBG are congruent. § 69

$\therefore CL = QG$. § 67

Now CF and QK are $\perp$ to CL and QG respectively. § 430

Therefore the rt. ◬ FCL and KQG are congruent. § 69

$\therefore FL = GK$. § 67

∴ $FGKL$ is a ▱. § 129

∴ the points F, G, K, and L lie in the same plane. § 117

Draw GL and BM.

Since BG and ML are equal and $\parallel$,

∴ $BGLM$ is a ▱. § 130

∴ GL is $\parallel$ to BM. § 118

Moreover HK is $\parallel$ to BM. § 136

∴ GL is $\parallel$ to HK. § 446

∴ the points G, L, H, and K lie in the same plane. § 428

Similarly, it may be shown that the points F, K, E, and L lie in the same plane.

∴ the vertices of the figure $EFGHKL$ lie in the same plane.

∴ $EFGHKL$ is an equilateral hexagon.

Now a certain point within the hexagon $EFGHKL$ is equidistant from its vertices. § 440

Therefore $EFGHKL$ may be inscribed in a ⊙. § 162

Therefore $EFGHKL$ is a regular hexagon, by § 365. Q. E. D.

16. *The volume of a right circular cylinder is equal to the product of the lateral area by half the radius.*

Given any right circular cylinder whose radius is r, altitude a, lateral area l, and volume v.

To prove that $v = l \times \frac{1}{2} r$.

Proof. $v = \pi r^2 a$. § 590

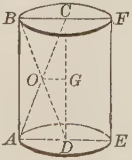
Now $l = 2\pi r a$. § 588

Then $a = \frac{l}{2\pi r}$. Ax. 4

Therefore $v = \pi r^2 \times \frac{l}{2\pi r}$ Ax. 9

$= l \times \frac{1}{2} r$. Q. E. D.

17. *The volume of a right circular cylinder is equal to the product of the area of the rectangle which generates it, by the length of the circumference generated by the point of intersection of the diagonals of the rectangle.*



Given the right circular cylinder $ABFE$ generated by the revolution of the rectangle $ABCD$ about CD as an axis; $CD = a$, $AD = r$, $OG \perp$ to CD and $= r'$, and the volume of the cylinder $ABFE = v$.

To prove that $v = ar \times 2\pi r'$.

Proof. OG is $\parallel$ to AD. § 95

O is the mid-point of CA. § 131

$\therefore OG$ bisects CD. § 135

$\therefore OG = \frac{1}{2} AD$. § 136

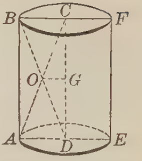
That is, $r' = \frac{1}{2} r$. Ax. 9

$\therefore r = 2r'$. Ax. 3

Now $v = \pi r^2 a$ § 590

$= ar \times \pi r$.

$\therefore v = ar \times 2\pi r'$, by Ax. 9. Q.E.D.



18. *If the altitude of a right circular cylinder is equal to the diameter of the base, the volume is equal to the total area multiplied by a third of the radius.*

Given a right circular cylinder with its altitude a equal to its diameter d, with the radius r, total area t, and volume v.

To prove that $v = t \times \frac{1}{3} r$.

Proof. $t = 2\pi r(a + r)$. § 588

Since $a = d$ Given

$= 2r$,

$\therefore t = 2\pi r(2r + r)$ Ax. 9

$= 2\pi r \times 3r$

$= 6\pi r^2$.

$\therefore t \times \frac{1}{3} r = 6\pi r^2 \times \frac{1}{3} r$ Ax. 3

$= 2\pi r^3$.

Now $v = \pi r^2 a$. § 590

But $a = 2r$. Proved

$\therefore v = \pi r^2 \times 2r$ Ax. 9

$= 2\pi r^3$.

Therefore $v = t \times \frac{1}{3} r$, by Ax. 8. Q.E.D.

19. *The surface of a sphere is two thirds the total surface of the circumscribed cylinder.*

Given a sphere and its circumscribed cylinder.

To prove that the surface of the given sphere is two thirds the total surface of the circumscribed cylinder.

Proof. Denote by r the radius of the given sphere.

Then the radius of the circumscribed cylinder is r and its height is $2r$.

The surface of the given sphere $= 4\pi r^2$. § 689

The total surface of the circumscribed cylinder

$= 2\pi r \cdot 2r + 2\pi r^2$ § 588

$= 4\pi r^2 + 2\pi r^2$

$= 6\pi r^2$.

Since $4\pi r^2 \equiv \frac{2}{3} \times 6\pi r^2,$ Iden.

therefore the surface of the given sphere is two thirds the total surface of the circumscribed cylinder. Q. E. D.

20. *The volume of a sphere is two thirds the volume of the circumscribed cylinder.*

Given v and v', the volumes of a sphere of radius r and its circumscribed cylinder respectively.

To prove that $v = \frac{2}{3} v'.$

Proof. $v = \frac{4}{3}\pi r^3.$ § 706

Also $v' = \pi r^2 a.$ § 590

But $a = 2r.$

$\therefore\ v' = \pi r^2 \cdot 2r$ Ax. 9

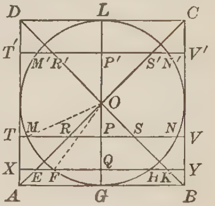
$= 2\pi r^3.$

$\therefore\ v : v' = \frac{4}{3}\pi r^3 : 2\pi r^3$ Ax. 4

$= \frac{2}{3} : 1.$

$\therefore\ v = \frac{2}{3} v'$, by § 261. Q. E. D.

21. *Given a sphere, a cylinder circumscribed about the sphere, and a cone of two nappes inscribed in the cylinder. If any two planes are drawn perpendicular to the axis of the three figures, the spherical segment between the planes is equivalent to the difference between the corresponding cylindric and conic segments.*



Given the sphere whose center is O, the right circular cylinder $ABCD$ circumscribed about the given sphere, and the cone of two nappes DOC-AOB inscribed in the cylinder $ABCD$. Let the accompanying figure represent a section of the given solids made by passing the plane $ABCD$ through their common axis LG; let TV and XY be the lines of intersection of the plane $ABCD$ with the two planes each ⊥ to the axis LG.

To prove that the spherical segment $MFHN$ is equivalent to the difference between the cylinder $TXYV$ and the frustum $REKS$ of the cone.

Proof. Let $OG = r$, $OQ = a$, and $OP = b$.

Then PQ, the altitude of the segment $MFHN$ of the given sphere,

$= OQ - OP$

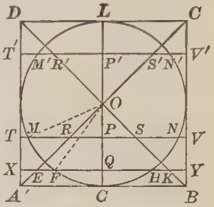
$= a - b.$ Ax. 9

Draw OF and OM.

$OF = r$; also $OM = r.$ § 621

In the rt. △ OQF, $\overline{QF}^2 = \overline{OF}^2 - \overline{OQ}^2$ § 338

$= r^2 - a^2.$ Ax. 9



$\therefore\ QF = \sqrt{r^2 - a^2}.$ Ax. 5

In the rt. △ OPM,

$\overline{PM}^2 = \overline{OM}^2 - \overline{OP}^2$ § 338

$= r^2 - b^2.$ Ax. 9

$\therefore\ PM = \sqrt{r^2 - b^2}.$ Ax. 5

Now the volume of any spherical segment of two bases is expressed by the formula

$v = \frac{1}{2} a\,(\pi r_1^2 + \pi r_2^2) + \frac{1}{6}\pi a^3.$ § 726

Therefore the volume of the spherical segment $MFHN$

$= \frac{1}{2}(a - b)\,[\pi(r^2 - a^2) + \pi(r^2 - b^2)] + \frac{1}{6}\pi(a - b)^3$ Ax. 9

$= \frac{1}{2}\pi(a - b)\,[(2r^2 - a^2 - b^2) + \frac{1}{3}(a - b)^2]$

$= \frac{1}{2}\pi(a - b)\,(2r^2 - a^2 - b^2 + \frac{1}{3}a^2 + \frac{1}{3}b^2 - \frac{2}{3}ab)$

$= \frac{1}{2}\pi(a - b)\,(2r^2 - \frac{2}{3}a^2 - \frac{2}{3}b^2 - \frac{2}{3}ab)$

$= \pi(a - b)\,[r^2 - \frac{1}{3}(a^2 + ab + b^2)].$

The volume of any right circular cylinder is expressed by the formula $v = \pi r^2 a.$ § 590

Therefore the volume of the right circular cylinder $TXYV$

$= \pi r^2(a - b).$ Ax. 9

Furthermore, $\angle OAG = 45°.$ § 363

$\therefore\ \angle OEQ = 45°.$ § 102

$\therefore\ \angle EOQ = 45°.$ § 107

$\therefore\ QE = OQ.$ § 76

But $OQ = a.$ Given

$\therefore\ QE = a.$ Ax. 8

Similarly, $PR = b.$

Now the volume of any frustum of a right circular cone is expressed by the formula $v = \frac{1}{3}\pi a\,(r^2 + r'^2 + rr').$ § 618

Therefore the volume of the frustum $REKS$ of the given cone

$= \frac{1}{3}\pi(a - b)\,(a^2 + ab + b^2)$ Ax. 9

$= \pi(a - b) \times \frac{1}{3}(a^2 + ab + b^2).$

Now $\pi(a - b)\ [r^2 - \frac{1}{3}(a^2 + ab + b^2)]$

$= \pi r^2(a - b) - \pi(a - b) \times \frac{1}{3}(a^2 + ab + b^2),$

since $r^2 - \frac{1}{3}(a^2 + ab + b^2) = r^2 - \frac{1}{3}(a^2 + ab + b^2).$ Iden.

Therefore the spherical segment $MFHN$ is equivalent to the difference between the cylinder $TXYV$ and the frustum $REKS$ of the cone.

When the given planes are on opposite sides of the center O, as TV' and XY, it may be proved in a similar manner that the spherical segment $M'FHN'$ is equivalent to the difference between the cylinder $T'XYV'$ and the sum of the cones OEK and $OR'S'$. Q.E.D.

APPENDIX TO SOLID GEOMETRY

Exercise 115. Page 437

1. *Find the volume of a truncated triangular prism, given the base* $b = 8$ *sq. in., and the distances of the three vertices* p, q, *and* r *from the plane of the base* = 3 *in.*, 4 *in., and* 5 *in. respectively.*

Solution. Let v be the volume of the given truncated triangular prism.

Then $$v = \tfrac{1}{3}b(p+q+r). \qquad \text{§ 711}$$
$$\therefore v = \tfrac{1}{3} \times 8(3+4+5) \qquad \text{Ax. 9}$$
$$= \tfrac{1}{3} \times 8 \times 12, \text{ or } 32.$$

Therefore the volume of the given truncated triangular prism is 32 cu. in. Q.E.F.

2. *Find the volume of a truncated triangular prism, given the base* $b = 9$ *sq. in., and the distances of the three vertices* p, q, *and* r *from the plane of the base* = 6 *in.*, 3 *in., and* $4\frac{1}{2}$ *in. respectively.*

Solution. Let v be the volume of the given truncated triangular prism.

Then $$v = \tfrac{1}{3}b(p+q+r). \qquad \text{§ 711}$$
$$\therefore v = \tfrac{1}{3} \times 9(6+3+4\tfrac{1}{2}) \qquad \text{Ax. 9}$$
$$= 3 \times 13\tfrac{1}{2}, \text{ or } 40\tfrac{1}{2}.$$

Therefore the volume of the given truncated triangular prism is $40\frac{1}{2}$ cu. in. Q.E.F.

3. *Find the volume of a truncated triangular prism, given the base* $b = 15$ *sq. in., and the distances of the three vertices* p, q, *and* r *from the plane of the base* = 7 *in.*, 9 *in., and* 8.1 *in. respectively.*

Solution. Let v be the volume of the given truncated triangular prism.

Then $$v = \tfrac{1}{3}b(p+q+r). \qquad \text{§ 711}$$
$$\therefore v = \tfrac{1}{3} \times 15(7+9+8.1) \qquad \text{Ax. 9}$$
$$= 5 \times 24.1, \text{ or } 120.5.$$

Therefore the volume of the given truncated triangular prism is 120.5 cu. in. Q.E.F.

4. *Find the volume of a truncated triangular prism, given the base* $b = 32$ *sq. in., and the distances of the three vertices* p, q, *and* r *from the plane of the base* = 9 *in.*, 12 *in.*, *and* 9.3 *in. respectively.*

Solution. Let v be the volume of the given truncated triangular prism.

Then $v = \frac{1}{3} b(p + q + r)$. § 711

$\therefore v = \frac{1}{3} \times 32(9 + 12 + 9.3)$ Ax. 9

$= \frac{1}{3} \times 32 \times 30.3$

$= 323.2.$

Therefore the volume of the given truncated triangular prism is 323.2 cu. in. Q.E.F.

5. *Find the volume of a truncated triangular prism, given the base* $b = 48$ *sq. in., and the distances of the three vertices* p, q, *and* r *from the plane of the base* = 16 *in.*, 15 *in.*, *and* 18 *in. respectively.*

Solution. Let v be the volume of the given truncated triangular prism.

Then $v = \frac{1}{3} b(p + q + r)$. § 711

$\therefore v = \frac{1}{3} \times 48(16 + 15 + 18)$ Ax. 9

$= 16 \times 49$, or 784.

Therefore the volume of the given truncated triangular prism is 784 cu. in. Q.E.F.

6. *A triangular rod of iron is cut square off* (*i.e. in right section*) *at one end, and slanting at the other end. The right section is an equilateral triangle* $1\frac{1}{2}$ *in. on a side. The edges of the rod are* 3 *ft.* 2 *in.*, 3 *ft.* 3 *in.*, *and* 3 *ft.* 3 *in. Find the weight of the rod, allowing* 0.28 *lb. per cubic inch.*

Solution. Let v denote the volume of the given rod, $p = 3$ ft. 2 in. = 38 in., $q = 3$ ft. 3 in. = 39 in., and $r = 3$ ft. 3 in. = 39 in.

Now $v = \frac{1}{3} b(p + q + r)$. § 712

But $b = \sqrt{s(s-a)(s-b)(s-c)}$ Ex. 1, p. 211

$= \sqrt{s(s-a)^3}$

$= \sqrt{\frac{9}{4} \times 1\frac{1}{2} \times (\frac{3}{4})^3}$ Ax. 9

$= \sqrt{\frac{9}{4} \times \frac{27}{64}}$

$= \frac{9}{16}\sqrt{3}.$

Therefore $v = \frac{1}{3} \times \frac{9}{16}\sqrt{3} \times (38 + 39 + 39)$ Ax. 9

$= \frac{3}{16}\sqrt{3} \times 116$

$= \frac{87}{4}\sqrt{3}.$

Therefore the weight of the given rod

$= \frac{87}{4}\sqrt{3} \times 0.28$ lb.

$= 6.09\sqrt{3}$ lb., or 10.548 lb. Q.E.F.

7. *Two triangular pyramids with a trihedral angle of the one equal to a trihedral angle of the other have the edges of these angles 3 in., 4 in., $3\frac{1}{2}$ in., and 5 in., $5\frac{1}{2}$ in., 6 in. respectively. Find the ratio of the volumes.*

Solution. Let v and v' denote the volumes of the two given triangular pyramids respectively.

Then
$$\frac{v}{v'} = \frac{3 \times 4 \times 3\frac{1}{2}}{5 \times 5\frac{1}{2} \times 6} \qquad \S\ 714$$
$$= \tfrac{14}{55}.$$

Therefore the ratio required is $\frac{14}{55}$. Q.E.F.

8. *Make a table giving the number of edges, vertices, and faces of each of the five regular polyhedrons, showing that in every case the number conforms to Euler's theorem* (§ 715).

	Number of edges	Number of vertices	Number of faces
Tetrahedron	6	4	4
Hexahedron	12	8	6
Octahedron	12	6	8
Dodecahedron	30	20	12
Icosahedron	30	12	20

In accord with Euler's theorem (§ 715) we find that:

$$6 + 2 = 4 + 4.$$
$$12 + 2 = 8 + 6.$$
$$12 + 2 = 6 + 8.$$
$$30 + 2 = 20 + 12.$$
$$30 + 2 = 12 + 20.$$

9. *Make a table similar to that of Ex. 8, giving the sum of the face angles in each of the five regular polyhedrons, showing that in every case* $s = (v - 2)\ 4$ *rt.* ∠s (§ 716).

	Sum of the ∠s in one face	Sum of the face ∠s
Tetrahedron	2 rt. ∠s	4 × 2 rt. ∠s, or 8 rt. ∠s
Hexahedron	4 rt. ∠s	6 × 4 rt. ∠s, or 24 rt. ∠s
Octahedron	2 rt. ∠s	8 × 2 rt. ∠s, or 16 rt. ∠s
Dodecahedron	6 rt. ∠s	12 × 6 rt. ∠s, or 72 rt. ∠s
Icosahedron	2 rt. ∠s	20 × 2 rt. ∠s, or 40 rt. ∠s

In accord with the theorem of § 716 we find that:

$$8 \text{ rt. } \angle s = (4 - 2) \times 4 \text{ rt. } \angle s.$$
$$24 \text{ rt. } \angle s = (8 - 2) \times 4 \text{ rt. } \angle s.$$
$$16 \text{ rt. } \angle s = (6 - 2) \times 4 \text{ rt. } \angle s.$$
$$72 \text{ rt. } \angle s = (20 - 2) \times 4 \text{ rt. } \angle s.$$
$$40 \text{ rt. } \angle s = (12 - 2) \times 4 \text{ rt. } \angle s.$$

10. *There can be no seven-edged polyhedron.*

Proof. We know that $e + 2 = v + f$. § 715

A polyhedron must have at least four faces, and hence four vertices.

In such a polyhedron, $e + 2 = 4 + 4$ Ax. 9

$= 8.$

$\therefore\ e = 6.$ Ax. 2

Consider a polyhedron with five faces, and hence with at least five vertices.

In such a polyhedron, $e + 2 = 5 + 5 = 10.$ Ax. 9

$\therefore\ e = 8.$ Ax. 2

Therefore there can be no seven-edged polyhedron. Q.E.D

11. *Can there be a nine-edged polyhedron?*

Taking the discussion in Ex. 10 a step further, we see that e cannot have the value 9 in the formula

$$e + 2 = v + f. \qquad \text{§ 715}$$

Therefore there can be no nine-edged polyhedron.

12. *What is the sum of the face angles of a six-edged polyhedron?*

Solution. A six-edged polyhedron is a tetrahedron. § 715

Therefore the sum of the face ∠s of a six-edged polyhedron equals $(4 - 2) \times 4$ rt. ∠s, or 8 rt. ∠s, by § 716. Q.E.F.

13. *What is the sum of the face angles of a polyhedron with five vertices? with four vertices? Consider the possibility of a polyhedron with three vertices.*

Solution. $s = (v - 2)\ 4$ rt. ∠s. § 716

$\therefore\ s = (5 - 2) \times 4$ rt. ∠s Ax. 9

$= 3 \times 4$ rt. ∠s

$= 12$ rt. ∠s.

Also $s = (4 - 2) \times 4$ rt. ∠s Ax. 9

$= 2 \times 4$ rt. ∠s

$= 8$ rt. ∠s. Q.E.F.

Clearly there can be no polyhedron with three vertices, for three points determine only one plane. § 427

Exercise 116. Page 443

1. *Deduce from the formula for the volume of a prismatoid, $v = \frac{1}{6} a (b + b' + 4m)$, the formula for the volume of a cube, $v = a^3$.*

Solution. $v = \frac{1}{6} a (b + b' + 4m).$ Given

In any cube, $b = b' = m.$

Substitute for b' and m their equal b in the given formula.

Then $v = \frac{1}{6} a \times 6b$, or ab. Ax. 9
But $b = a^2$. § 320
$\therefore v = a \cdot a^2$ Ax. 9
$= a^3$. Q.E.F.

2. *Deduce from the formula for the volume of a prismatoid,* $v = \frac{1}{6} a (b + b' + 4m)$, *the formula for the volume of a prism,* $v = ba$.

Solution. $v = \frac{1}{6} a (b + b' + 4m)$. Given
In any prism, $b = b' = m$.

Substitute for b' and m their equal b in the given formula.

Then $v = \frac{1}{6} a \times 6b$ Ax. 9
$= ba$. Q.E.F.

3. *Deduce from the formula for the volume of a prismatoid,* $v = \frac{1}{6} a (b + b' + 4m)$, *the formula for the volume of a pyramid,* $v = \frac{1}{3} ba$.

Solution. $v = \frac{1}{6} a (b + b' + 4m)$. Given
In any pyramid, b', the upper base, is zero.
Also $m : b = 1^2 : 2^2$ § 556
$= 1 : 4$.
$\therefore 4m = b$. § 261

Substitute for b' its value zero, and for $4m$ its value b, in the given formula.

Then $v = \frac{1}{6} a \times 2b$ Ax. 9
$= \frac{1}{3} ba$. Q.E.F.

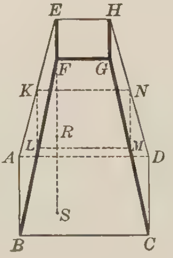
4. *Deduce from the formula for the volume of a prismatoid,* $v = \frac{1}{6} a (b + b' + 4m)$, *the formula for the volume of a parallelepiped,* $v = ba$.

Solution. $v = \frac{1}{6} a (b + b' + 4m)$. Given
In any parallelepiped, $b = b' = m$.

Substitute for b' and m their equal b in the given formula.

Then $v = \frac{1}{6} a \times 6b$ Ax. 9
$= ba$. Q.E.F.

5. *Deduce from the formula for the volume of a prismatoid,* $v = \frac{1}{6} a (b + b' + 4m)$, *the formula for the volume of a frustum of a pyramid,* $v = \frac{1}{3} a \left(b + b' + \sqrt{bb'}\right)$.



Solution. Let $ABCD$-$EFGH$ be a frustum of any pyramid. Let fall the ⊥ FS to the base $ABCD$, meeting $ABCD$ in S. § 437

Bisect FS in R. § 229

Through R pass the plane $KLMN \parallel$ to $ABCD$, cutting the edges of the given frustum in K, L, M, N. § 457

Then $FR : FS = FL : FB.$ § 462

But $FR : FS = 1 : 2.$ Const.

$\therefore FL : FB = 1 : 2.$ Ax. 8

$\therefore 2\ FL = FB.$ § 261

$\therefore FL = \frac{1}{2} FB.$ Ax. 4

That is, L is the mid-point of the edge FB.

Likewise K is the mid-point of the edge EA.

$\therefore KL = \frac{1}{2}(AB + EF).$ § 137

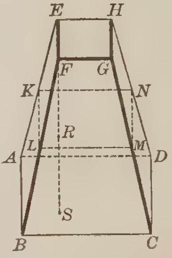
Similarly, $LM = \frac{1}{2}(BC + FG)$, etc.

Moreover KL is $\parallel$ to AB,

and LM is $\parallel$ to BC. § 453

$\therefore \angle KLM = \angle ABC.$ § 461

Likewise $\angle LMN = \angle BCD$, etc.



$\therefore$ the polygons $KLMN$ and $ABCD$ are similar. § 282

$\therefore$ the polygons $KLMN$ and $EFGH$ are similar.

Now let the bases $ABCD$ and $EFGH$ and the mid-section $KLMN$ be denoted by b, b', and m respectively, and the corresponding sides AB and EF of the base polygons by s and s' respectively, and FS by a.

Then $KL = \frac{1}{2}(s + s').$ Ax. 9

$$\therefore \frac{b}{m} = \frac{s^2}{\frac{1}{4}(s+s')^2},$$

and
$$\frac{b'}{m} = \frac{s'^2}{\frac{1}{4}(s+s')^2}.$$ § 334

Whence
$$\frac{\sqrt{b}}{\sqrt{m}} = \frac{s}{\frac{1}{2}(s+s')},$$

and
$$\frac{\sqrt{b'}}{\sqrt{m}} = \frac{s'}{\frac{1}{2}(s+s')}.$$ Ax. 5

$$\therefore \frac{\sqrt{b}+\sqrt{b'}}{\sqrt{m}} = \frac{s+s'}{\frac{1}{2}(s+s')}$$ Ax. 1

$$= 2.$$

$$\therefore \sqrt{b}+\sqrt{b'} = 2\sqrt{m}.$$ Ax. 3

$$\therefore b + b' + 2\sqrt{bb'} = 4\,m.$$ Ax. 5

Add $b + b'$ to both members of this equation.

Then $2\,b + 2\,b' + 2\sqrt{bb'} = b + b' + 4\,m.$ Ax. 1

$\therefore 2\,(b + b' + \sqrt{bb'}) = b + b' + 4\,m.$

Multiply both members of this equation by $\frac{1}{6}a$.

Then $\frac{1}{3}a\,(b + b' + \sqrt{bb'}) = \frac{1}{6}a\,(b + b' + 4\,m).$ Ax. 3

In any prismatoid $v = \frac{1}{6}a\,(b + b' + 4\,m).$ Given

Therefore $v = \frac{1}{3}a\,(b + b' + \sqrt{bb'})$, by Ax. 8. Q.E.F.

6. *A prismatoid has an upper base 3 sq. in., a lower base 7 sq. in., an altitude 3 in., and a mid-section 4 sq. in. What is the volume?*

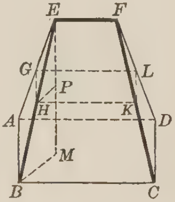
Solution. $v = \frac{1}{6}a(b + b' + 4m)$. § 725

$\therefore v = \frac{1}{6} \times 3(7 + 3 + 16)$ Ax. 9

$= \frac{1}{2} \times 26$, or 13.

Therefore the volume of the given prismatoid is 13 cu. in. Q. E. F.

7. *A wedge has for its base a rectangle l in. long and w in. wide. The cutting edge is e in. long, and is parallel to the base. The distance from e to the base is d in. Deduce a formula for the volume of the wedge. Apply this formula to the case in which* $l = 6$, $w = 1$, $e = 5$, $d = 3$.



Given the wedge EF-$ABCD$ whose base $ABCD$ is a rectangle l in. long and w in. wide, and whose cutting edge EF, ∥ to the base $ABCD$, is e in. long, the ⊥ distance EM of EF from $ABCD$ being d in.

Required to find the volume v of EF-$ABCD$ in terms of l, w, e, and d.

Solution. Bisect EM in P. § 229

Through P pass the plane $GHKL$ ∥ to the base $ABCD$, cutting the faces in the lines GH, HK, KL, and LG. § 457

Then GH is ∥ to AB,

and KL is ∥ to CD. § 453

But AB is ∥ to CD. § 118

$\therefore GH$ is ∥ to KL. § 446

For like reason HK is ∥ to LG.

$\therefore GHKL$ is a ▱. § 118

Moreover $\angle GHK = \angle ABC$. § 461

Since $\angle ABC =$ a rt. $\angle$, § 119

$\therefore \angle GHK =$ a rt. $\angle$. Ax. 8

For like reason the remaining angles of the ▱ $GHKL$ are rt. $\angle$s.

$\therefore$ the ▱ $GHKL$ is a rectangle. § 119

Draw HP and BM.

HP is ∥ to BM. § 453

$\therefore \angle EHP = \angle EBM$,

and $\angle EPH = \angle EMB$. § 102

$\therefore \triangle$s EPH and EMB are similar § 286

$\therefore EH : EB = EP : EM$. § 282

But $EP : EM = 1 : 2$. Const.

$\therefore EH : EB = 1 : 2$. Ax. 8

$\therefore 2\,EH = EB$. § 261

Whence $EH = \frac{1}{2} EB.$ Ax. 4

$\therefore$ H is the mid-point of EB.

Likewise G and K are the mid-points of EA and FC respectively.

Now EF is $\parallel$ to $ABCD$. Given

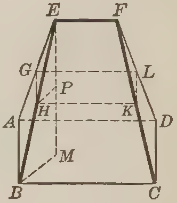
$\therefore$ EF is $\parallel$ to BC.

$\therefore HK = \frac{1}{2}(EF + BC)$ § 137

$= \frac{1}{2}(e + l),$ Ax. 9

and $GH = \frac{1}{2} AB$ § 136

$= \frac{1}{2} w.$ Ax. 9



$\therefore$ the area of the rectangle $GHKL$

$= \frac{1}{2}(e + l) \times \frac{1}{2} w$ § 320

$= \frac{1}{4} w (e + l).$

Also the area of the rectangle $ABCD$

$= l \times w.$ § 320

For any prismatoid, $v = \frac{1}{6} a (b + b' + 4m).$ § 725

Here $a = d$, $b = l \times w$, $b' = 0$, and $m = \frac{1}{4} w (e + l)$.

$\therefore v = \frac{1}{6} d [l \times w + 4 \times \frac{1}{4} w (e + l)]$ Ax. 9

$= \frac{1}{6} d [l \times w + w (e + l)]$

$= \dfrac{wd (e + 2l)}{6}.$

Therefore the volume of the given wedge is expressed by the formula

$$v = \frac{wd (e + 2l)}{6}.$$

When $l = 6$, $w = 1$, $e = 5$, and $d = 3$,

$v = \frac{1}{6} \times 1 \times 3 (5 + 2 \times 6)$ Ax. 9

$= \frac{1}{2} \times 17$

$= \frac{17}{2}$, or $8\frac{1}{2}$. Q.E.F.

Exercise 117. Page 445

1. *Find the volume of a spherical segment having the bases* $b = 4$ *and* $b' = 5$, *and the altitude* $a = 1$.

Solution. $v = \frac{1}{2} a (b + b') + \frac{1}{6} \pi a^3.$ § 726

$\therefore v = \frac{1}{2}(4 + 5) + \frac{1}{6}\pi$ Ax. 9

$= 4.5 + 0.5236$

$= 5.0236.$ Q.E.F.

2. *Find the volume of a spherical segment having the bases* $b = 4$ *and* $b' = 6$, *and the altitude* $a = 1\frac{1}{4}$.

Solution. $v = \frac{1}{2} a (b + b') + \frac{1}{6} \pi a^3.$ § 726

$\therefore v = \frac{1}{2} \times 1\frac{1}{4}(4 + 6) + \frac{1}{6} \times (1\frac{1}{4})^3 \pi$ Ax. 9

$= \frac{25}{4} + \frac{125}{384} \pi$

$= 6.25 + 1.0227$

$= 7.2727.$ Q.E.F.

3. *Find the volume of a spherical segment naving the bases* $b = 5$ *and* $b' = 7$, *and the altitude* $a = 2\frac{1}{8}$.

Solution. $v = \frac{1}{2} a (b + b') + \frac{1}{6} \pi a^3.$ § 726

$\therefore v = \frac{1}{2} \times 2\frac{1}{8}(5 + 7) + \frac{1}{6}(2\frac{1}{8})^3 \pi$ Ax. 9

$= \frac{51}{4} + \frac{4913}{512} \times 0.5236$

$= 12.75 + 5.0243$

$= 17.7743.$ Q.E.F.

4. *Find the volume of a spherical segment having the bases* $b = 6$ *and* $b' = 8$, *and the altitude* $a = 1\frac{1}{2}$.

Solution. $v = \frac{1}{2} a (b + b') + \frac{1}{6} \pi a^3.$ § 726

$\therefore v = \frac{1}{2} \times 1\frac{1}{2}(6 + 8) + \frac{1}{6}(1\frac{1}{2})^3 \pi$ Ax. 9

$= 10.5 + 1.76715$

$= 12.26715.$ Q.E.F.

5. *Find the volume of a spherical segment having the bases* $b = 8$ *and* $b' = 12$, *and the altitude* $a = 2$.

Solution. $v = \frac{1}{2} a (b + b') + \frac{1}{6} \pi a^3.$ § 726

$\therefore v = \frac{1}{2} \times 2(8 + 12) + \frac{1}{6}(2)^3 \pi$ Ax. 9

$= 20 + 4.1888$

$= 24.1888.$ Q.E.F.

6. *Find the volume of a spherical segment having the bases* $b = 12$ *and* $b' = 15$, *and the altitude* $a = 3\frac{1}{2}$.

Solution. $v = \frac{1}{2} a (b + b') + \frac{1}{6} \pi a^3.$ § 726

$\therefore v = \frac{1}{2} \times 3\frac{1}{2}(12 + 15) + \frac{1}{6}(3\frac{1}{2})^3 \pi$ Ax. 9

$= 47.25 + 22.44935$

$= 69.69935.$ Q.E.F.

7. *Find the volume of a spherical segment having the bases* $b = 27$ *sq. in. and* $b' = 32$ *sq. in., and the altitude* $a = 2.33$ *in.*

Solution. $v = \frac{1}{2} a (b + b') + \frac{1}{6} \pi a^3.$ § 726

$\therefore v = \frac{1}{2} \times 2.33(27 + 32) + \frac{1}{6} \times 2.33^3 \pi$ Ax. 9

$= 68.735 + 6.6232$

$= 75.3582.$

$\therefore$ the volume of the given spherical segment is 75.3582 cu. in. Q.E.F.

8. *Find the volume of a spherical segment having the radii of the bases* $r_1 = 3$ *and* $r_2 = 4$, *and the altitude* $a = 2$.

Solution.
$$v = \tfrac{1}{2}\,a\left(\pi r_1^2 + \pi r_2^2\right) + \tfrac{1}{6}\,\pi a^3. \qquad \text{§ 726}$$
$$\therefore\ v = \tfrac{1}{2} \times 2\,(3^2\pi + 4^2\pi) + \tfrac{1}{6} \times 2^3\pi \qquad \text{Ax. 9}$$
$$= 25\,\pi + \tfrac{4}{3}\,\pi$$
$$= 26\tfrac{1}{3}\,\pi, \text{ or } 82.7288.$$

Q.E.F.

9. *Find the volume of a spherical segment having the radii of the bases* $r_1 = 4$ *and* $r_2 = 7$, *and the altitude* $a = 3$.

Solution.
$$v = \tfrac{1}{2}\,a\left(\pi r_1^2 + \pi r_2^2\right) + \tfrac{1}{6}\,\pi a^3. \qquad \text{§ 726}$$
$$\therefore\ v = \tfrac{1}{2} \times 3\,(4^2\pi + 7^2\pi) + \tfrac{1}{6} \times 3^3\pi \qquad \text{Ax. 9}$$
$$= \tfrac{195}{2}\,\pi + \tfrac{9}{2}\,\pi$$
$$= \tfrac{204}{2}\,\pi$$
$$= 102\,\pi, \text{ or } 320.4432.$$

Q.E.F.

10. *Find the volume of a spherical segment having the radii of the bases* $r_1 = 8$ *and* $r_2 = 5$, *and the altitude* $a = 4\frac{1}{2}$.

Solution.
$$v = \tfrac{1}{2}\,a\left(\pi r_1^2 + \pi r_2^2\right) + \tfrac{1}{6}\,\pi a^3. \qquad \text{§ 726}$$
$$\therefore\ v = \tfrac{1}{2} \times 4\tfrac{1}{2}(8^2\pi + 5^2\pi) + \tfrac{1}{6}(4\tfrac{1}{2})^3\pi \qquad \text{Ax. 9}$$
$$= \tfrac{9}{4} \times 89\,\pi + \tfrac{243}{16}\,\pi$$
$$= \tfrac{3204}{16}\,\pi + \tfrac{243}{16}\,\pi$$
$$= \tfrac{3447}{16}\,\pi, \text{ or } 676.81845.$$

Q.E.F.

11. *Find the volume of a spherical segment having the radii of the bases* $r_1 = 5$ *and* $r_2 = 3$, *and the altitude* $a = 1\frac{1}{2}$.

Solution.
$$v = \tfrac{1}{2}\,a\left(\pi r_1^2 + \pi r_2^2\right) + \tfrac{1}{6}\,\pi a^3. \qquad \text{§ 726}$$
$$\therefore\ v = \tfrac{1}{2} \times 1\tfrac{1}{2}\,(5^2\pi + 3^2\pi) + \tfrac{1}{6}\,(1\tfrac{1}{2})^3\pi \qquad \text{Ax. 9}$$
$$= \tfrac{51}{2}\,\pi + \tfrac{9}{16}\,\pi$$
$$= \tfrac{417}{16}\,\pi, \text{ or } 81.87795.$$

Q.E.F.

12. *Find the volume of a spherical segment having the radii of the bases* $r_1 = 6$ *and* $r_2 = 5$, *and the altitude* $a = 1\frac{1}{4}$.

Solution.
$$v = \tfrac{1}{2}\,a\left(\pi r_1^2 + \pi r_2^2\right) + \tfrac{1}{6}\,\pi a^3. \qquad \text{§ 726}$$
$$\therefore\ v = \tfrac{1}{2} \times 1\tfrac{1}{4}\,(6^2\pi + 5^2\pi) + \tfrac{1}{6}\,(1\tfrac{1}{4})^3\pi \qquad \text{Ax. 9}$$
$$= \tfrac{305}{8}\,\pi + \tfrac{125}{384}\,\pi$$
$$= 119.7735 + 1.02266$$
$$= 120.7962.$$

Q.E.F.

13. *Find the volume of a spherical segment having the radii of the bases* $r_1 = 9$ *and* $r_2 = 10$, *and the altitude* $a = 2\frac{3}{4}$.

Solution.
$$v = \tfrac{1}{2}\,a\left(\pi r_1^2 + \pi r_2^2\right) + \tfrac{1}{6}\,\pi a^3. \qquad \text{§ 726}$$

$\therefore\ v = \frac{1}{2} \times 2\frac{3}{4}(9^2\pi + 10^2\pi) + \frac{1}{6}(2\frac{3}{4})^3\pi$ Ax. 9

$= \frac{1991}{8}\pi + \frac{1331}{384}\pi$

$= 781.8657 + 10.88924$

$= 792.7549.$ Q.E.F.

14. *Find the volume of a spherical segment having the radii of the bases* $r_1 = 9$ *in. and* $r_2 = 7$ *in., and the altitude* $a = 4.75$ *in.*

Solution. $v = \frac{1}{2}a(\pi r_1^2 + \pi r_2^2) + \frac{1}{6}\pi a^3.$ § 726

$\therefore\ v = \frac{1}{2} \times 4.75(9^2\pi + 7^2\pi) + \frac{1}{6} \times 4.75^3\pi$ Ax. 9

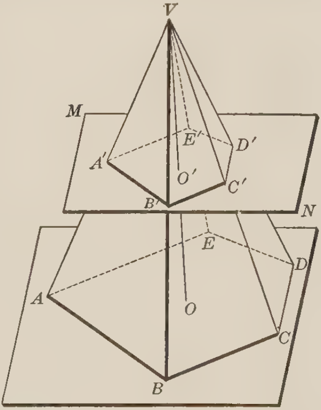
$= \frac{1235}{4}\pi + \frac{6859}{384}\pi$

$= \frac{125419}{384}\pi$, or 1026.0842.

$\therefore$ the volume of the spherical segment is 1026.0842 cu. in. Q.E.F.

Exercise 118. Page 446

1. *A pyramid 6 ft. high is cut by a plane parallel to the base, the area of the section being* $\frac{1}{9}$ *that of the base. How far from the vertex is the cutting plane?*



Given the pyramid *V-ABCDE*, whose altitude *VO* is 6 ft., cut by the plane *MN* ∥ to the base *ABCDE* so that the section *A'B'C'D'E'* is $\frac{1}{9}$ of the base *ABCDE*, and *VO'* is the altitude of the pyramid cut off.

Required to find the length of *VO'*.

Solution. $A'B'C'D'E' : ABCDE = \overline{VO'}^2 : \overline{VO}^2.$ § 556

But $A'B'C'D'E' : ABCDE = 1 : 9.$ Given

$\therefore\ \overline{VO'}^2 : \overline{VO}^2 = 1 : 9.$ Ax. 8

$\therefore\ VO' : VO = 1 : 3.$ Ax. 5

$\therefore\ VO' : 6 = 1 : 3.$ Ax. 9

$\therefore\ 3\,VO' = 6.$ § 261

$\therefore\ VO' = 2.$ Ax. 4

Therefore the length of *VO'* is 2 ft. Q.E.F.

2. *Find the area of a spherical triangle whose angles are* 100°, 120°, *and* 140°, *the diameter of the sphere being* 16 *in.*

Solution. The spherical excess of the given △

$= 100° + 120° + 140° - 180°$, or 180°. § 693

Therefore the given spherical △ is equivalent to a lune whose angle is half 180°, or 90°. § 695

But the area of such a lune $= \frac{90}{360}$ of the spherical surface § 694

$= \frac{1}{4} \times 4\pi r^2$ Ax. 9

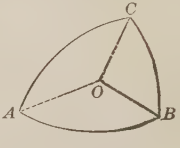
$= \frac{1}{4} \times 4\left(\frac{d}{2}\right)^2 \pi$ Ax. 9

$= (\frac{16}{2})^2 \pi$ Ax. 9

$= 64\pi$, or 201.0624.

Therefore the area of the given spherical △ is 201.0624 sq. in. Q. E. F.

3. *Two angles of a spherical triangle are 80° and 120°. Find the limits of the third angle and prove that the greatest possible area of the triangle is four times the least possible area, the sphere on which it is drawn being given.*



Given the spherical △ ABC on the sphere whose center is O, with the ∠s A and B equal to 80° and 120° respectively.

Required (1) to find the limits of the ∠ C; (2) to prove that the greatest possible area of the spherical △ ABC is four times the least possible area.

Solution. (1) Let $A'B'C'$ be the polar triangle of spherical △ ABC

In polar △ $A'B'C'$, $a' = 100°$, and $b' = 60°$. § 667

Then $c' < a' + b' < 100° + 60° < 160°$. § 663

$\therefore C > 180° - 160° > 20°$. § 667

Since $b' + c' > a'$ (§ 663), it follows that $c' > a' - b'$. Ax. 6

$c' > 100° - 60° > 40°$. Ax. 9

$\therefore C < 180° - 40° < 140°$. § 667

Therefore the limits of the ∠ C are 20° and 140°. Q. E. F.

(2) Assuming that ∠ C has its minimum value, the spherical excess of the △ $ABC = 80° + 120° + 20° - 180°$, or 40°. § 693

∴ the △ ABC is equivalent to a lune whose angle is 20°. § 695

∴ the area of △ ABC is $\frac{20}{360}$ of the spherical surface. § 694

Assuming that ∠ C has its maximum value, the spherical excess of the △ $ABC = 80° + 120° + 140° - 180°$, or 160°. § 693

∴ the △ ABC is equivalent to a lune whose angle is 80°. § 695

∴ the area of △ ABC is $\frac{80}{360}$ of the spherical surface. § 694

Therefore the greatest possible area of the spherical △ ABC is four times the least possible area. Q. E. D.

4. *An irregular portion, less than half, of a material sphere is given. Show how the radius can be found, compasses and ruler being allowed.*

Given ABC an irregular portion of a sphere less than a hemisphere.

Required to find the radius of the sphere of which ABC is an irregular portion.

Construction. With any convenient point P (Fig. 1) on the surface of the given portion ABC of the sphere as a pole, and any convenient radius arc, describe by means of the compasses a small $\odot$.

Take the points D, E, and F on the circumference of this $\odot$, and with the compasses measure the chords DE, EF, and DF which include the plane $\triangle DEF$.

Construct the $\triangle D'E'F'$ (Fig. 2) congruent to the $\triangle DEF$.

Circumscribe a $\odot$ about the $\triangle D'E'F'$. § 240

Draw $O'D'$ the radius of the $\odot D'E'F'$.

Construct the rt. $\triangle GHK$ (Fig. 3) so that the hypotenuse $GK =$ the chord PD (Fig. 1), and the side $HK = O'D'$ (Fig. 2).

At K draw the line $KM \perp$ to GK. § 228

Produce KM and GH to meet in M.

Finally, bisect GM in N. § 229

Then MN is the radius required. Q.E.F.

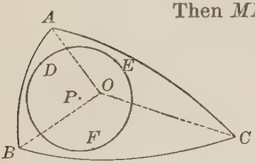


FIG. 1

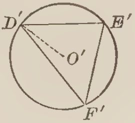


FIG. 2

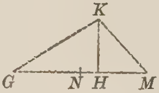


FIG. 3

Proof. Assume the diameter PP' of the sphere, of which ABC is a portion, to be drawn, intersecting the $\odot DEF$ in its center O; also assume the lines PD, $P'D$, and the radius OD to be drawn.

The diameter PP' is $\perp$ to the plane of the $\odot DEF$. § 623

$\therefore$ PP' is $\perp$ to OD. § 430

$\therefore$ $\angle POD$ is a rt. $\angle$.

In the rt. $\triangle$s GHK and POD, $GK = PD$, Const.

and $HK = O'D'$ Const.

$= OD$. Ax. 9

$\therefore$ the rt. $\triangle$s GHK and POD are congruent. § 89

$\therefore$ the $\angle$s HGK and OPD are equal. § 67

$\therefore$ in the rt. $\triangle$s GKM and PDP', $GK = PD$, Const.

and $\angle HGK = \angle OPD$. Proved

$\therefore$ the rt. $\triangle$s GKM and PDP' are congruent. § 72

$\therefore$ $GM = PP'$. § 67

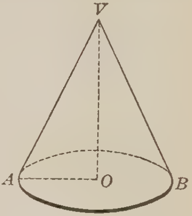
But $MN = \frac{1}{2} GM$. Const.

$\therefore$ $MN = \frac{1}{2} PP'$. Ax. 9

That is, MN is the radius of the sphere of which ABC is the given irregular portion less than a hemisphere. Q.E.D.

5. *Find the volume of a cone of revolution, the area of the total surface of which is 200π sq. ft., and the altitude of which is 16 ft.*

Given VA an element, and VO the altitude, of the cone of revolution V-AB; also the total area of the cone of revolution $V\text{-}AB = 200\pi$ sq. ft., and $VO = 16$ ft.



Required to find the volume of the cone of revolution V-AB.

Solution. Draw the radius OA.

The line VO is $\perp$ to the radius OA. § 430

$\therefore \angle VOA$ is a rt. $\angle$.

Denote OA by r, and VA by s.

In the rt. △ VOA, $\overline{VA}^2 - \overline{OA}^2 = \overline{VO}^2$. § 338

$\therefore s^2 - r^2 = 16^2$ Ax. 9

$= 256.$ (1)

Since $t = \pi rs + \pi r^2$, § 610

$\therefore 200\pi = \pi rs + \pi r^2.$ Ax. 9

Whence $r^2 + rs = 200.$ (2) Ax. 4

Now solve the simultaneous quadratic equations for r^2:

$$\begin{cases} s^2 - r^2 = 256, & (1) \\ r^2 + rs = 200. & (2) \end{cases}$$

Let $s = vr.$ Hyp.

Substitute this value of s in (1) and (2).

Then
$$\begin{cases} v^2r^2 - r^2 = 256, & (3) \\ r^2 + vr^2 = 200. & (4) \end{cases}$$ Ax. 9

From (3), $r^2 = \dfrac{256}{v^2 - 1}.$ Ax. 4

From (4), $r^2 = \dfrac{200}{1 + v}.$ (5) Ax. 4

$\therefore \dfrac{256}{v^2 - 1} = \dfrac{200}{1 + v}.$ Ax. 8

$\therefore 256 = 200\,v - 200.$ Ax. 3

$\therefore 200\,v = 256 + 200$ Ax. 1

$= 456.$

$\therefore v = 2.28.$ (6) Ax. 4

Substitute this value of v in (5).

Then $r^2 = \dfrac{200}{1 + 2.28}$ Ax. 9

$= \dfrac{200}{3.28}$

$= \frac{2500}{41}.$

The volume of a cone of revolution

$$= \tfrac{1}{3}\,\pi r^2 a. \qquad \text{§ 612}$$

Therefore the volume of the cone $V\text{-}AB$

$$= \tfrac{1}{3} \times 16 \times \tfrac{2500}{41}\,\pi \qquad \text{Ax. 9}$$

$$= \tfrac{40000}{41} \times 1.0472$$

$$= \tfrac{4}{41} \times 10{,}472, \text{ or } 1021.659.$$

Therefore the volume of the cone of revolution $V\text{-}AB$ is 1021.659 cu. ft. Q.E.F.

6. *The volumes of two similar polyhedrons are* 64 *cu. ft. and* 216 *cu. ft. respectively. If the area of the surface of the first polyhedron is* 112 *sq. ft., find the area of the surface of the second polyhedron.*

Solution. Let s and s' denote any two corresponding edges of the given similar polyhedrons, and a and a' their surface areas respectively.

Then $$s^3 : s'^3 = 64 : 216. \qquad \text{§ 723}$$

$$\therefore\ s : s' = 4 : 6 \qquad \text{Ax. 5}$$

$$= 2 : 3.$$

$$\therefore\ s^2 : s'^2 = 4 : 9. \qquad \text{§ 270}$$

But $$a : a' = s^2 : s'^2. \qquad \text{§ 721}$$

$$\therefore\ a : a' = 4 : 9. \qquad \text{Ax. 8}$$

$$\therefore\ 112 : a' = 4 : 9. \qquad \text{Ax. 9}$$

$$\therefore\ 4\,a' = 1008. \qquad \text{§ 261}$$

$$\therefore\ a' = 252. \qquad \text{Ax. 4}$$

Therefore the area of the surface of the second polyhedron is 252 sq. ft. Q.E.F.

7. *A solid sphere of metal of radius* 12 *in. is recast into a hollow sphere. If the cavity is spherical, of the same radius as the original sphere, find the thickness of the shell.*

Solution. Clearly the volume of the shell and cavity is twice the volume of the given solid sphere of metal.

Let x denote the number of inches in the thickness of the shell.

Then the radius of the shell and cavity considered as a sphere is $(x + 12)$ in. Ax. 11

Therefore the volume of the shell and cavity

$$= \tfrac{4}{3}\,(x + 12)^3\,\pi. \qquad \text{§ 706}$$

The volume of the solid sphere of metal

$$= \tfrac{4}{3} \times 12^3\,\pi, \text{ or } 2304\,\pi. \qquad \text{§ 706}$$

$$\therefore\ \tfrac{4}{3}\,(x + 12)^3\,\pi = 2 \times 2304\,\pi$$

$$= 4608\,\pi.$$

$$\therefore\ (x + 12)^3 = 3456. \qquad \text{Ax. 4}$$

$$\therefore\ x + 12 = 12\sqrt[3]{2}. \qquad \text{Ax. 5}$$

$\therefore x = 12\sqrt[3]{2} - 12$ Ax. 2

$= 12(\sqrt[3]{2} - 1)$

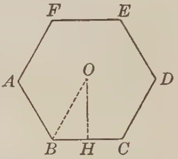
$= 12(1.2599 - 1)$

$= 12 \times 0.2599$, or 3.1188.

Therefore the thickness of the shell is 3.1188 in. Q.E.F.

8. *The stone spire of a church is a regular pyramid 50 ft. high on a hexagonal base each side of which is 10 ft. There is a hollow part which is also a regular pyramid 45 ft. high, on a hexagonal base of which each side is 9 ft. Find the number of cubic feet of stone in the spire.*

Solution. The volume of stone in the church spire equals the difference between the volumes of two regular hexagonal pyramids, one with altitude 50 ft. and side of base 10 ft., the other with altitude 45 ft. and side of base 9 ft.



Let $ABCDEF$ represent the regular hexagonal base of the larger pyramid.

Draw OB and OH, the radius and apothem of $ABCDEF$ respectively.

Now $\angle ABC = 120°$. § 145

$\therefore \angle OBH = 60°$. § 363

$\therefore \angle BOH = 30°$. § 107

Furthermore in the rt. $\triangle OHB$,

$\overline{OH}^2 = \overline{OB}^2 - \overline{BH}^2$. § 338

But $BH = \frac{1}{2} OB$. Ex. 2, p. 80

Whence $2\,BH$ or $BC = OB$. Ax. 3

$\therefore \overline{OH}^2 = \overline{OB}^2 - \frac{1}{4}\overline{OB}^2$ Ax. 9

$= \frac{3}{4}\overline{OB}^2$

$= \frac{3}{4}\overline{BC}^2$. Ax. 9

$\therefore OH = \frac{1}{2}\sqrt{3} \cdot BC$ Ax. 5

$= \frac{1}{2}\sqrt{3} \times 10$, or $5\sqrt{3}$. Ax. 9

Therefore the area of $ABCDEF$, the base of the larger pyramid,

$= \frac{1}{2} \times 60 \times 5\sqrt{3}$, or $150\sqrt{3}$. § 386

Denote by $A'B'C'D'E'F'$ the regular hexagonal base of the smaller pyramid.

Then $\dfrac{\text{area of } ABCDEF}{\text{area of } A'B'C'D'E'F'} = \dfrac{10^2}{9^2}$. § 374

$\therefore \dfrac{150\sqrt{3}}{\text{area of } A'B'C'D'E'F'} = \dfrac{10^2}{9^2}$ Ax. 9

$= \frac{100}{81}$.

$\therefore 100 \times \text{area of } A'B'C'D'E'F' = 81 \times 150\sqrt{3}$. § 261

$\therefore$ the area of $A'B'C'D'E'F' = 0.81 \times 150\sqrt{3}$ Ax. 4

$= 121.5\sqrt{3}$.

Since the volume of any pyramid $= \frac{1}{3}ba$, § 561

$\therefore$ the volume of the larger regular hexagonal pyramid

$= \frac{1}{3} \times 150\sqrt{3} \times 50$ Ax. 9

$= 2500\sqrt{3}$;

and the volume of the smaller regular hexagonal pyramid

$= \frac{1}{3} \times 121.5\sqrt{3} \times 45$ Ax. 9

$= 40.5 \times 45\sqrt{3}$

$= 1822.5\sqrt{3}$.

Therefore the number of cubic feet of stone in the church spire

$= 2500\sqrt{3} - 1822.5\sqrt{3}$

$= 677.5\sqrt{3}$, or 1173.46. Q.E.F.

9. *The volumes of a hemisphere, right circular cone, and right circular cylinder are equal. Their bases are also equal, each being a circle of radius* 10 *in. Find the altitude of each.*

Solution. Let r denote the radius of the base of the hemisphere, of the right circular cone, and of the right circular cylinder; a_1 the altitude of the right circular cone; and a_2 the altitude of the right circular cylinder; also v the volume of each.

Then $v = \frac{1}{2} \times \frac{4}{3}\pi r^3$, or $\frac{2}{3}\pi r^3$, § 706

$v = \frac{1}{3}\pi r^2 a_1$, § 612

and $v = \pi r^2 a_2$. § 590

$\therefore \frac{2}{3}\pi r^3 = \frac{1}{3}\pi r^2 a_1 = \pi r^2 a_2$. Ax. 8

$\therefore 2r = a_1 = 3a_2$. Ax. 4

But $r = 10$ in. Given

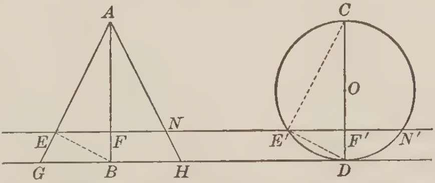
$\therefore a_1 = 2 \times 10$ in., or 20 in., Ax. 9

and $3a_2 = 2 \times 10$ in., or 20 in. Ax. 9

Whence $a_2 = 6\frac{2}{3}$ in. Ax. 4

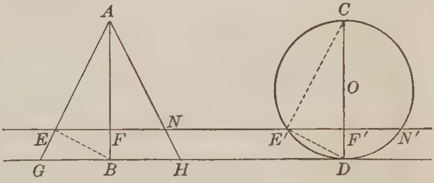
Therefore the altitudes of the given hemisphere, right circular cone, and right circular cylinder are 10 in., 20 in., and $6\frac{2}{3}$ in. respectively. Q.E.F.

10. *A sphere of radius* 5 *ft. and a right circular cone also of radius* 5 *ft. stand on a plane. If the height of the cone is equal to a diameter of the sphere, find the position of the plane that cuts the two solids in equal circular sections.*



Solution. Let $CE'D$ and AGH be sections of the given sphere and right circular cone respectively, made by passing a plane through the diameter CD of the sphere, which is $\perp$ to the plane on which it rests, and the axis AB of the right circular cone, which is $\parallel$ to CD.

Let the plane which cuts the sphere and right circular cone in equal circular sections $\parallel$ to the plane on which the given solids rest, intersect the sections $CE'D$ and AGH in the lines $E'N'$ and EN respectively, and CD and AB in F' and F respectively.



Draw DE', CE', and BE.

Now CD and AB are each $\perp$ to GD. § 430

But $E'N'$ and EN are $\parallel$ to GD. § 453

$\therefore$ CD and AB are $\perp$ to $E'N'$ and EN respectively. § 97

Moreover F' and F are the mid-points of the diameters $E'N'$ and EN of the equal circular sections respectively.

Then in the rt. $\triangle$ $E'F'D$ and EFB,

$E'F' = EF$, § 162

and $DF' = BF$. § 455

$\therefore$ the rt. $\triangle$ $E'F'D$ and EFB are congruent. § 69

$\therefore DE' = BE$, and $\angle F'DE' = \angle FBE$. § 67

Then in the $\triangle$ CDE' and ABE,

$CD = AB$, Given

$DE' = BE$, Proved

and $\angle CDE' = \angle ABE$. Proved

$\therefore$ the $\triangle$ CDE' and ABE are congruent. § 68

$\therefore \angle CE'D = \angle AEB$. § 67

But $\angle CE'D =$ a rt. $\angle$. § 215

$\therefore \angle AEB =$ a rt. $\angle$. Ax. 8

In the rt. $\triangle ABG$, $AB = CD = 2 \times 5$ ft., or 10 ft.,

and $BG = 5$ ft. Given

Since $\overline{AG}^2 = \overline{AB}^2 + \overline{BG}^2$, § 337

$\therefore \overline{AG}^2 = 10^2 + 5^2$ Ax. 9

$= 125.$

$\therefore AG = \sqrt{125}$, or $5\sqrt{5}$. Ax. 5

Now $AG : AB = AB : AE$. § 294, 3

$\therefore \overline{AB}^2 = AG \times AE$. § 261

$\therefore 100 = 5\sqrt{5} \times AE$. Ax. 9

$\therefore AE = \dfrac{100}{5\sqrt{5}}$ Ax. 4

$= 4\sqrt{5}.$

Since $EG = AG - AE,$

$\therefore\ EG = 5\sqrt{5} - 4\sqrt{5}$, or $\sqrt{5}$. Ax. 9

Furthermore $AE : BE = BE : EG.$ § 294, 2

$\therefore\ \overline{BE}^2 = AE \times EG$ § 261

$= 4\sqrt{5} \times \sqrt{5}$ Ax. 9

$= 4 \times 5$, or 20.

But in the rt. $\triangle AEB$, $AB : BE = BE : BF.$ § 294, 3

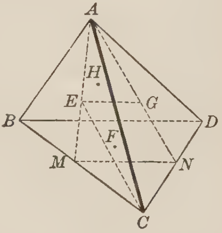
$\therefore\ \overline{BE}^2 = AB \times BF.$ § 261

$\therefore\ 20 = 10 \times BF.$ Ax. 9

$\therefore\ BF = 2.$ Ax. 4

Hence the plane cutting the given sphere and right circular cone in equal circular sections, and the plane on which these solids rest are 2 ft. distant from each other. Q.E.F.

11. *The vertices of one regular tetrahedron are at the centers of the faces of another regular tetrahedron. Find the ratio of the volumes.*



Given the regular tetrahedron $EFGH$ whose vertices E, F, G, and H are the centers of the faces ABC, BCD, ACD, and ABD respectively of the regular tetrahedron $ABCD$.

Required to determine the ratio of the volumes of the regular tetrahedrons $EFGH$ and $ABCD$.

Solution. Through the points A, E, and G pass a plane (§ 427) cutting the faces ABC, ACD, and BCD in the lines AM, AN, and MN respectively. § 429

An edge EG of the regular tetrahedron $EFGH$ lies in the plane AMN. § 422

The lines AM and AN bisect BC and CD perpendicularly in M and N respectively. § 151

Moreover AM and AN are equal, being altitudes of the congruent equilateral ⸫ ABC and ACD. § 67

Also AE and AG are equal, being radii of the congruent ⸫ ABC and ACD. § 358

Then ⸫ AEG and AMN are similar. § 288

$\therefore\ EG : MN = AE : AM.$ (1) § 282

But $MN = \frac{1}{2} BD.$ § 136

Denote an edge of the regular tetrahedron $ABCD$ by e.

Then $MN = \frac{1}{2} e.$ Ax. 9

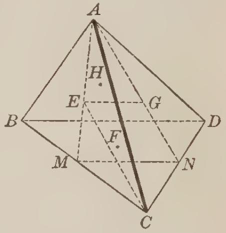
Draw the radius EC of the face $\triangle ABC$. § 358

Now $\angle ACB = 60^\circ.$ § 145

Then $\angle ECM = 30^\circ.$ § 363

$\therefore \angle CEM = 60^\circ.$ § 107

$\therefore EC = 2\,EM.$ Ex. 2, p. 80



But $AE = EC.$ § 162

$\therefore AE = 2\,EM.$ Ax. 8

$\therefore AE + EM = 3\,EM.$ Ax. 1

That is, $AM = 3\,EM.$ Ax. 11

$\therefore AE : AM = 2 : 3.$ Ax. 4

Therefore (1) becomes

$EG : \frac{1}{2}\,e = 2 : 3.$ Ax. 8

Whence $3\,EG = e.$ § 261

$\therefore EG = \frac{1}{3}\,e.$ Ax. 4

Since $\frac{\text{vol. of } EFGH}{\text{vol. of } ABCD} = \frac{\overline{EG}^3}{\overline{AC}^3},$ § 722

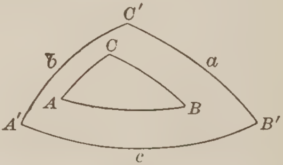
$$\therefore \frac{\text{vol. of } EFGH}{\text{vol. of } ABCD} = \frac{(\frac{1}{3}\,e)^3}{e^3} \qquad \text{Ax. 9}$$

$$= \frac{\frac{1}{27}\,e^3}{e^3}$$

$$= \frac{1}{27}.$$

Q. E. F.

12. *Find the area of a spherical triangle if the perimeter of its polar triangle is* 297° *and the radius of the sphere is* 10 *centimeters.*



Given the perimeter of the spherical $\triangle A'B'C'$, the polar $\triangle$ of the spherical $\triangle ABC$, equal to 297°, both ⚠ being drawn on a sphere whose radius is 10 centimeters.

Required to find the area of the spherical $\triangle ABC$.

Solution. Denote by a, b, and c the sides of the $\triangle A'B'C'$ opposite the $\angle$s A, B, and C, respectively, of the $\triangle ABC$.

Then $a + A = 180^\circ,$

$b + B = 180^\circ,$

and $c + C = 180^\circ.$ § 667

Adding, $a + b + c + A + B + C = 540^\circ.$ Ax. 1

But $a + b + c = 297^\circ.$ Given

$\therefore A + B + C = 540^\circ - 297^\circ$, or $243^\circ.$ Ax. 2

Therefore the spherical excess of the $\triangle ABC$

$= 243^\circ - 180^\circ$, or $63^\circ.$ § 693

Therefore the spherical △ ABC is equivalent to a lune whose angle is half 63°, or $31\frac{1}{2}$°. § 695

But the area of such a lune $= \frac{31\frac{1}{2}}{360}$ of the spherical surface § 694

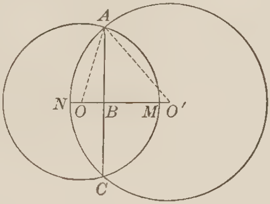
$= \frac{7}{80} \times 4\pi r^2$ § 689

$= \frac{7}{20} \times 10^2\pi$ Ax. 9

$= 35\pi$, or 109.956.

Therefore the area of the spherical △ ABC is 109.956 sq. cm. Q. E. F.

13. *The radii of two spheres are* 13 *in. and* 15 *in. respectively, and the distance between the centers is* 14 *in. Find the volume of the solid common to both spheres,—a spherical lens.*



Given the two spheres whose centers are O and O' intersecting each other, thus forming the spherical lens $AMCN$; also $OO' = 14$ in., $OM = 13$ in., and $O'N = 15$ in.

Required to find the volume of the spherical lens $AMCN$.

Solution. Draw AO and AO'.

Also draw AC, cutting OO' in B.

The line OO' is ⊥ to the circular section AC at its center B. § 652

$\therefore OO'$ is ⊥ to AB. § 430

In the rt. △ ABO, $\overline{AO}^2 - \overline{OB}^2 = \overline{AB}^2$. § 338

And in the rt. △ ABO',

$\overline{AO'}^2 - \overline{O'B}^2 = \overline{AB}^2$. § 338

$\therefore \overline{AO}^2 - \overline{OB}^2 = \overline{AO'}^2 - \overline{O'B}^2$. Ax. 8

$\therefore \overline{O'B}^2 - \overline{OB}^2 = \overline{AO'}^2 - \overline{AO}^2$. Axs. 1, 2

$\therefore (O'B + OB)(O'B - OB) = \overline{AO'}^2 - \overline{AO}^2$.

That is, $OO'(O'B - OB) = \overline{AO'}^2 - \overline{AO}^2$.

But $AO' = O'N = 15$ in., and $AO = OM = 13$ in. § 162

$\therefore 14(O'B - OB) = 15^2 - 13^2$ Ax. 9

$= 225 - 169$

$= 56$.

$\therefore O'B - OB = 4$. Ax. 4

But $O'B + OB = 14$. Given

Adding the last two equations, we have

$2\,O'B = 18$. Ax. 1

Whence $O'B = 9$. Ax. 4

Then $OB = 5$. Ax. 2

Now $ON = O'N - OO'$

$= 15 - 14$, or 1. Ax 9

Also $O'M = OO' - OM$

$= 14 - 13$, or 1. Ax. 9

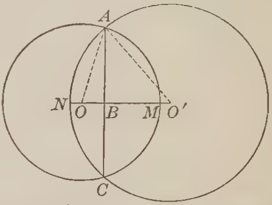
Since $BN = OB + ON$, Ax. 11

$\therefore BN = 5 + 1$, or 6. Ax. 9

And since $BM = O'B - O'M$,

$\therefore BM = 9 - 1$, or 8. Ax. 9

The spherical lens $AMCN$ is composed of two segments of the given spheres, having the common circular base AC whose radius is AB.



Since, in the rt. △ ABO, $\overline{AB}^2 = \overline{AO}^2 - \overline{OB}^2$, § 338

$\therefore \overline{AB}^2 = 13^2 - 5^2$ Ax. 9

$= 169 - 25$, or 144.

The volume of a spherical segment of one base is expressed by the formula $v = \frac{1}{2}\pi r^2 a + \frac{1}{6}\pi a^3$. § 726

Therefore the volume of the spherical segment ANC

$= \frac{1}{2} \times 144 \times 6\,\pi + \frac{1}{6} \times 6^3\,\pi$ Ax. 9

$= 432\,\pi + 36\,\pi$

$= 468\,\pi$;

and the volume of the spherical segment AMC

$= \frac{1}{2} \times 144 \times 8\,\pi + \frac{1}{6} \times 8^3\,\pi$ Ax. 9

$= 576\,\pi + 85\frac{1}{3}\,\pi$

$= 661\frac{1}{3}\,\pi$.

Therefore the volume of the spherical lens $AMCN$

$= 468\,\pi + 661\frac{1}{3}\,\pi$ Ax. 11

$= 1129\frac{1}{3}\,\pi$, or 3547.9136.

Therefore the volume of the spherical lens is 3547.9136 cu. in.

Q.E.F.

14. *The radius of the base of a right circular cylinder is r and the altitude of the cylinder is a. Find the radius and the volume of a sphere whose surface is equivalent to the lateral surface of the cylinder.*

Solution. Let x be the length of the radius of the sphere whose surface is equivalent to the lateral surface of the given cylinder.

Then the surface of the sphere $= 4\,\pi x^2$. § 689

Now the lateral surface of the given cylinder $= 2\,\pi r a$. § 588

$\therefore 4\,\pi x^2 = 2\,\pi r a$.

$\therefore x^2 = \frac{1}{2} r a$. Ax. 4

$\therefore x = \sqrt{\frac{1}{2} r a}$ Ax. 5

$= \frac{1}{2}\sqrt{2 r a}$.

The volume of the given sphere $= \frac{4}{3}\pi x^3$. § 706

Therefore the volume of the sphere

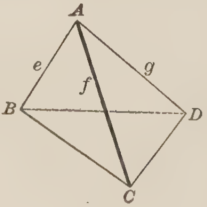
$$= \tfrac{4}{3} \times \left(\tfrac{1}{2}\sqrt{2\,ra}\right)^3 \cdot \pi \qquad \text{Ax. 9}$$

$$= \tfrac{4}{3} \times \tfrac{1}{4}\, ra\sqrt{2\,ra} \cdot \pi$$

$$= \tfrac{1}{3}\, \pi ra\sqrt{2\,ra}.$$

Therefore the sphere whose surface is equivalent to the lateral surface of the given cylinder has its radius equal to $\frac{1}{2}\sqrt{2\,ra}$, and its volume equal to $\frac{1}{3}\pi ra\sqrt{2\,ra}$. Q.E.F.

15. *If the polyhedral angle at the vertex of a triangular pyramid is trirectangular, and the areas of the lateral faces are a, b, and c respectively, and the area of the base is d, then* $a^2 + b^2 + c^2 = d^2$.



Given in the pyramid $A\text{-}BCD$ the $\angle A$, a trirectangular $\angle$; also the areas of the lateral faces ABC, ACD, and ABD equal to a, b, and c respectively, and the area of the base BCD equal to d.

To prove that $a^2 + b^2 + c^2 = d^2$.

Proof. Let e, f, and g denote the lateral edges AB, AC, and AD respectively.

Then $a = \frac{1}{2}\,ef$,

$b = \frac{1}{2}\,fg$,

and $c = \frac{1}{2}\,eg$. § 325

$\therefore\ a^2 = \frac{1}{4}\,e^2f^2$,

$b^2 = \frac{1}{4}\,f^2g^2$,

and $c^2 = \frac{1}{4}\,e^2g^2$. Ax. 5

Adding, $a^2 + b^2 + c^2 = \frac{1}{4}\,(e^2f^2 + f^2g^2 + e^2g^2)$. Ax. 1

Furthermore $\overline{BC}^2 = e^2 + f^2$,

$\overline{CD}^2 = f^2 + g^2$,

and $\overline{BD}^2 = e^2 + g^2$. § 337

Therefore $BC = \sqrt{e^2 + f^2}$,

$CD = \sqrt{f^2 + g^2}$,

and $BD = \sqrt{e^2 + g^2}$. Ax. 5

Now let s denote the semiperimeter of the $\triangle BCD$.

Then $d = \sqrt{s(s - BC)(s - CD)(s - BD)}$. Ex. 1, p. 211

$\therefore\ d^2 = s(s - BC)(s - CD)(s - BD)$. Ax. 5

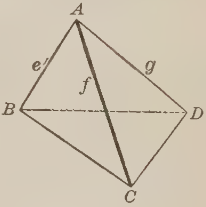
But $s = \frac{1}{2}\left[\sqrt{e^2 + f^2} + \sqrt{f^2 + g^2} + \sqrt{e^2 + g^2}\right]$, Ax. 9

$(s - BC) = \frac{1}{2}\left[-\sqrt{e^2 + f^2} + \sqrt{f^2 + g^2} + \sqrt{e^2 + g^2}\right]$,

$(s - CD) = \frac{1}{2}\left[\sqrt{e^2 + f^2} - \left(\sqrt{f^2 + g^2} + \sqrt{e^2 + g^2}\right)\right]$,

and $(s - BD) = \frac{1}{2}\left[\sqrt{e^2 + f^2} + \left(\sqrt{f^2 + g^2} - \sqrt{e^2 + g^2}\right)\right]$. Ax. 2

For convenience find first the product of the values of s and $(s-BC)$; then find the product of the values of $(s-CD)$ and $(s-BD)$; finally, multiply the first product obtained by the second. The resultant product is the value of d^2.

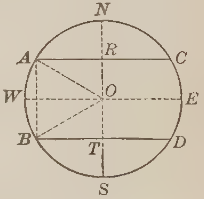


Therefore

$$\begin{aligned} d^2 &= \tfrac{1}{4}\left[\left(\sqrt{f^2+g^2}+\sqrt{e^2+g^2}\right)^2-\left(\sqrt{e^2+f^2}\right)^2\right] \\ &\quad \times\tfrac{1}{4}\left[\left(\sqrt{e^2+f^2}\right)^2-\left(\sqrt{f^2+g^2}-\sqrt{e^2+g^2}\right)^2\right] \\ &= \tfrac{1}{2}\left[g^2+\sqrt{(f^2+g^2)(e^2+g^2)}\right] \\ &\quad \times\tfrac{1}{2}\left[-g^2+\sqrt{(f^2+g^2)(e^2+g^2)}\right] \\ &= \tfrac{1}{4}(-g^4+e^2f^2+f^2g^2+e^2g^2+g^4) \\ &= \tfrac{1}{4}(e^2f^2+f^2g^2+e^2g^2). \end{aligned}$$

Therefore $a^2+b^2+c^2=d^2$, by Ax. 8. Q.E.D.

16. *If the earth is a sphere with a diameter of* 8000 *mi., find the area of the zone bounded by the parallels* 30° *north latitude and* 30° *south latitude. Show that this zone and the planes of the circles include* $\frac{11}{16}$ *of the volume of the earth.*



Solution. Let $ABDC$ represent a section of the earth made by a plane passed through its axis NS. Let the section $ABDC$ cut the plane of the equator in the line WE, and the planes of the small Ⓢ at 30° north latitude and 30° south latitude in the lines AC and BD respectively. Join A and B by the lines OA and OB to the center of the great ⊙ $ABDC$, and draw the chord AB.

Arc WA = arc $WB = 30°$. Given

$\therefore \angle WOA = \angle WOB = 30°$. § 213

$\therefore \angle AOB = 60°$. Ax. 11

Now in the $\triangle AOB$, $OA = OB$. § 162

$\therefore \angle BAO = \angle ABO$. § 74

But $\angle BAO + \angle ABO + \angle AOB = 180°$. § 107

Whence $\angle BAO + \angle ABO = 180° - \angle AOB$. Ax. 2

$\therefore 2\angle BAO = 180° - 60°$, or $120°$. Ax. 9

$\therefore \angle BAO = 60°$. Ax. 4

$\therefore \angle ABO = 60°$. Ax. 8

Therefore the $\triangle AOB$ is equilateral. § 77

But $OA = \frac{1}{2} \times 8000$ mi., or 4000 mi.

$\therefore AB = 4000$ mi. Ax. 9

Since the area of any zone is equal to $2\pi ra$, § 691

$\therefore$ the area of the zone $ABDC = 2 \times 4000 \times 4000\,\pi$ Ax. 9

$= 32{,}000{,}000\,\pi$, or 100,531,200.

That is, the area of the zone $ABCD = 100{,}531{,}200$ sq. mi.

Furthermore, since NS is $\perp$ to the plane of the equator at O,

NS is $\perp$ to WE. § 430

For like reason NS is $\perp$ to AR and BT.

Arc $AN =$ arc $BS = 90° - 30°$, or $60°$.

$\therefore \angle AOR = \angle BOT = 60°$. § 213

Then in the rt. $\triangle$ ARO and BTO,

$OA = OB$, § 162

and $\angle AOR = \angle BOT$. Proved

$\therefore$ the rt. $\triangle$ ARO and BTO are congruent. § 91

$\therefore AR = BT$. § 67

Therefore the circular bases of the spherical segment $ABDC$ of the earth have equal radii.

In the rt. $\triangle ARO$,

$\angle OAR = 30°$, since $\angle AOR = 60°$. § 107

$\therefore OR = \frac{1}{2} OA$. Ex. 2, p. 80

$\therefore OR = \frac{1}{2} \times 4000$ mi. Ax. 9

$= 2000$ mi.

But $\overline{AR}^2 = \overline{OA}^2 - \overline{OR}^2$ § 338

$= 4000^2 - 2000^2$ Ax. 9

$= 16{,}000{,}000 - 4{,}000{,}000$

$= 12{,}000{,}000$.

Now the volume of a spherical segment of two bases is expressed by the formula $v = \frac{1}{2} a\left(\pi r_1^2 + \pi r_2^2\right) + \frac{1}{6}\pi a^3$. § 726

In the spherical segment $ABDC$,

$r_1 = r_2$. Proved

Hence, for the spherical segment $ABDC$,

$v = \frac{1}{2} a\left(2\pi r_1^2\right) + \frac{1}{6}\pi a^3$ Ax. 9

$= \pi r_1^2 a + \frac{1}{6}\pi a^3$

$= 12{,}000{,}000 \times 4000\,\pi + \frac{1}{6} \times 4000^3\,\pi$ Ax. 9

$= 48{,}000{,}000{,}000\,\pi + 10{,}666{,}666{,}666\frac{2}{3}\,\pi$

$= 58{,}666{,}666{,}666\frac{2}{3}\,\pi$.

But the volume of the earth is expressed by the formula

$v = \frac{1}{6}\pi d^3$. § 706

Hence, for the earth,

$v = \frac{1}{6} \times 8000^3\,\pi$ Ax. 9

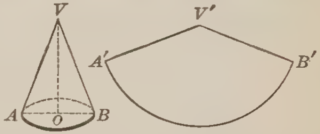
$= \frac{1}{6} \times 512{,}000{,}000{,}000\,\pi$

$= 85{,}333{,}333{,}333\frac{1}{3}\,\pi$.

Therefore $\dfrac{\text{vol. of segment } ABDC}{\text{vol. of earth}} = \dfrac{58{,}666{,}666{,}666\frac{2}{3}\,\pi}{85{,}333{,}333{,}333\frac{1}{3}\,\pi} = \dfrac{11}{16}$.

Q. E. F.

17. *The altitude of a cone of revolution is* 12 *centimeters and the radius of its base is* 5 *centimeters. Compute the radius of the sector of paper which, when rolled up, will just cover the convex surface of the cone, and compute the size of the central angle of this sector in degrees, minutes, and seconds.*



Given the cone of revolution $V\text{-}AB$ with the altitude $VO =$ 12 centimeters, and the radius OA of the base $=$ 5 centimeters; also the circular sector $V'A'B'$, whose radius $V'A'$ equals the slant height VA of the cone $V\text{-}AB$, and whose arc $A'B'$ equals the circumference of the base AB of the cone $V\text{-}AB$.

Required to find (1) the length of $V'A'$; (2) the size of the $\angle V'$.

Solution. (1) VO is $\perp$ to the base AB of the given cone.

$\therefore VO$ is $\perp$ to OA. § 430

In the rt. $\triangle VOA$, $\overline{VA}^2 = \overline{VO}^2 + \overline{OA}^2$. § 337

$\therefore \overline{VA}^2 = 12^2 + 5^2$ Ax. 9

$= 144 + 25$, or 169.

$\therefore VA = 13$. Ax. 5

That is, the length of $V'A'$ is 13 centimeters, by Ax. 8. Q.E.F.

(2) The circumference of the base of the given cone

$= 2\pi r$ centimeters § 385

$= 10\pi$ centimeters. Ax. 9

$\therefore$ the arc $A'B' = 10\pi$ centimeters. Ax. 8

But the circumference of the $\odot$ of which $V'A'$ is a radius

$= 26\pi$ centimeters. § 385

$$\therefore \frac{\text{arc } A'B'}{\text{entire circumference}} = \frac{10\pi}{26\pi} = \frac{5}{13}.$$

$$\therefore \frac{\angle V'}{360^\circ} = \frac{5}{13}.$$

$\therefore 13 \times \angle V' = 5 \times 360^\circ = 1800^\circ$. § 261

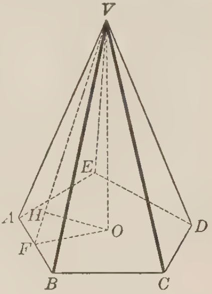
$\therefore \angle V' = 138\frac{6}{13}^\circ$, by Ax. 4. Q.E.F.

18. *The volume of any regular pyramid is equal to one third of its lateral area multiplied by the perpendicular distance from the center of its base to any lateral face.*

Given $V\text{-}ABCDE$ any regular pyramid with altitude $VO = a$; with the line $OH \perp$ to the lateral face AVB and $= h$; with slant height $= s$, perimeter of base $= p$, apothem of base $= m$, area of lateral surface $= l$, area of base $= b$, and volume $= v$.

To prove that $v = \frac{1}{3}hl$.

Proof. Through the lines VO and OH pass the plane VOH, cutting the face $\triangle AVB$ in VF and the base in OF. §§ 425, 429



Then the plane VOH is $\perp$ to the planes AVB and $ABCDE$. § 477

$\therefore AB$ is $\perp$ to the plane VOH. § 478

$\therefore AB$ is $\perp$ to the lines VF and OF. § 430

$\therefore VF = s$, and $OF = m$.

The lines VO and OH are $\perp$ to the lines OF and VF respectively. § 430

The $\angle FVO$ is common to the rt. $\triangle$ VOF and VHO.

$\therefore$ the rt. $\triangle$ VOF and VHO are similar. § 287

$\therefore s : a = m : h$. § 282

$\therefore hs = am$. § 261

$\therefore m = \dfrac{hs}{a}$. Ax. 4

Moreover $l = \frac{1}{2} sp$. § 553

$\therefore p = \dfrac{l}{\frac{1}{2} s}$ Ax. 4

$= \dfrac{2l}{s}$.

Now $v = \frac{1}{3} ba$. § 561

But $b = \frac{1}{2} mp$. § 386

$\therefore v = \frac{1}{3} \times \frac{1}{2} mp \cdot a$ Ax. 9

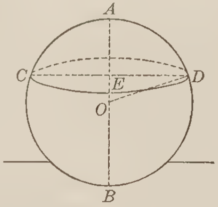
$= \frac{1}{6} mpa$.

Substitute in this equation the values of m and p already found.

Then $v = \dfrac{1}{6} \times \dfrac{hs}{a} \times \dfrac{2l}{s} \times a$ Ax. 9

$= \frac{1}{3} hl$. Q. E. D.

19. *If the area of a zone of one base is n times the area of the circle which forms its base, the altitude of the zone is $\dfrac{1}{n}(n-1)$ times the diameter of the sphere. Discuss the special case when $n = 1$.*



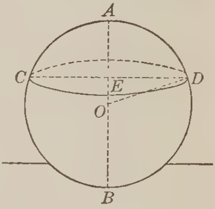
Given in the sphere whose center is O the zone CAD, whose area is n times the area of its circular base CD, and whose altitude is AE. Let $AE = a$, $CE = r'$, $OA = r$, and $AB = d$.

To prove that $a = \dfrac{1}{n}(n-1) \cdot d$.

Proof. The area of the zone $CAD = 2\pi ra$. § 691

The area of the small ⊙ $CD = \pi r'^2$. § 389

$$\therefore n \cdot \pi r'^2 = 2\pi ra.$$



Whence $r'^2 = \frac{2ra}{n}$. Ax. 4

Moreover, since $a : r' = r' : 2r - a$, § 294, 2

$\therefore r'^2 = a(2r - a)$. § 261

$\therefore a(2r - a) = \frac{2ra}{n}$. Ax. 8

$\therefore 2r - a = \frac{2r}{n}$. Ax. 4

$\therefore a = 2r - \frac{2r}{n}$ Axs. 1, 2

$= d - \frac{d}{n}$

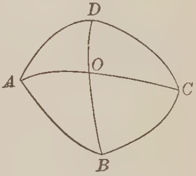
$= \frac{nd - d}{n}$

$= \frac{1}{n}(n - 1) \cdot d.$ Q. E. D.

Discussion. For the case in which $n = 1$, the formula $a = \frac{1}{n}(n - 1) \cdot d$ clearly reduces to the formula $a = 0$.

That is, there is no zone of one base whose area is equal to the area of its base.

20. *If the four sides of a spherical quadrilateral are equal, its diagonals are perpendicular to each other.*



Given AC and BD the diagonals of the spherical quadrilateral $ABCD$ whose sides are equal.

To prove that AC and BD are ⊥ to each other.

Proof. The spherical ⨻ ABC and ADC are isosceles.

In the isosceles spherical ⨻ ABC and ADC,

$AB = AD$,

$BC = DC$, Given

and $AC = AC$. Iden.

Therefore the isosceles spherical ⨻ ABC and ADC are congruent. § 673

$\therefore \angle BAC = \angle DAC$. § 67

In the spherical ⨻ AOB and AOD,

$AB = AD$, Given

$AO = AO$, Iden.

and $\angle BAC = \angle DAC$. Proved

But the equal parts occur in the reverse order.

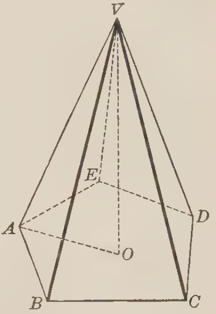
Therefore the spherical $\triangle$ AOB and AOD are symmetric. § 675

$\therefore \angle AOB = \angle AOD.$ § 67

$\therefore \angle AOB =$ a rt. $\angle$.

Therefore AC and BD are $\perp$ to each other. Q.E.D.

21. *Find the volume of a pyramid whose base contains* 30 *square centimeters if one lateral edge is* 5 *centimeters and the angle formed by this edge and the plane of the base is* 45°.



Given the pyramid V-$ABCDE$ whose base $ABCDE$ has an area of 30 square centimeters and whose lateral edge VA, which makes with the plane $ABCDE$ an $\angle$ of 45°, has a length of 5 centimeters.

Required to find the volume of the pyramid V-$ABCDE$.

Solution. Draw VO, the altitude of the pyramid V-$ABCDE$.

Through VA and VO pass a plane, § 425 cutting the base $ABCDE$ in the line OA. § 429

Since VO is $\perp$ to the plane $ABCDE$, Const.

$\therefore VO$ is $\perp$ to OA. § 430

$\therefore \angle VOA = 90°.$

But $\angle VAO = 45°.$ § 485

$\therefore$ in the rt. $\triangle VOA$, $\angle AVO = 45°.$ § 107

$\therefore VO = OA.$ § 76

Moreover $\overline{VO}^2 + \overline{OA}^2 = \overline{VA}^2.$ § 337

$\therefore 2\,\overline{VO}^2 = \overline{VA}^2.$ Ax. 9

$\therefore \overline{VO}^2 = \frac{1}{2}\,\overline{VA}^2$ Ax. 4

$= \frac{1}{2} \times 5^2$ Ax. 9

$= \frac{25}{2}.$

$\therefore VO = \sqrt{\frac{25}{2}}$ Ax. 5

$= \frac{5}{2}\sqrt{2}.$

Now the volume of any pyramid $= \frac{1}{3}\,ba.$ § 561

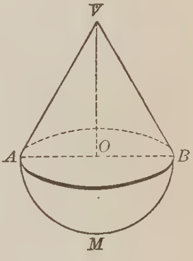
Therefore the volume of the pyramid V-$ABCDE$

$= \frac{1}{3} \times 30 \times \frac{5}{2}\sqrt{2}$ Ax. 9

$= 25\sqrt{2}.$

Therefore the volume of the pyramid V-$ABCDE$ is $25\sqrt{2}$ cubic centimeters. Q.E.F.

22. *On the base of a right circular cone a hemisphere is constructed outside the cone. The surface of the hemisphere equals the surface of the cone. If r is the radius of the hemisphere, find the slant height of the cone, the inclination of the slant height to the base, and the volume of the entire solid.*



Given the right circular cone V-AB and the hemisphere AMB, having equal surfaces and constructed on opposite sides of the base AB, whose radius $OA = r$.

Required to find (1) the length of VA, the slant height of the cone V-AB; (2) the $\angle$ of inclination of VA with the base AB; and (3) the volume of the entire solid V-AMB.

Solution. (1) The lateral surface of the cone V-AB

$= \frac{1}{2} sc$ § 609

$= \frac{1}{2} \times 2\pi rs$ Ax. 9

$= \pi rs$

$= \pi r \times VA.$ Ax. 9

But the lateral surface of the hemisphere AMB

$= \frac{1}{2} \times 4\pi r^2$ § 689

$= 2\pi r^2.$

$\therefore \pi r \times VA = 2\pi r^2.$

$\therefore VA = 2r$, by Ax. 4. Q.E.F.

(2) Draw VO, the axis of the cone V-AB.

Then VO is $\perp$ to OA. § 430

In the rt. $\triangle VOA$, $VA = 2r = 2\,OA.$ Ax. 9

$\therefore \angle VAO = 2\angle AVO.$ Ex. 2, p. 80, Converse

But $\angle VAO + \angle AVO = 90°.$ § 107

$\therefore 3\angle AVO = 90°.$ Ax. 9

$\therefore \angle AVO = 30°.$ Ax. 4

Hence $\angle VAO = 60°.$ Ax. 8

That is, the $\angle$ of inclination of VA with the base AB is 60°, by § 485. Q.E.F.

(3) In the rt. $\triangle VOA$, $\overline{VO}^2 = \overline{VA}^2 - \overline{OA}^2.$ § 338

$\therefore \overline{VO}^2 = (2r)^2 - r^2$ Ax. 9

$= 4r^2 - r^2$, or $3r^2$.

$\therefore VO = r\sqrt{3}.$ Ax. 5

Since the volume of any circular cone

$= \frac{1}{3}\pi r^2 a,$ § 612

therefore the volume of the cone V-$AB = \frac{1}{3} \times \pi r^2 \times r\sqrt{3}$

$= \frac{1}{3}\sqrt{3}\,\pi r^3.$

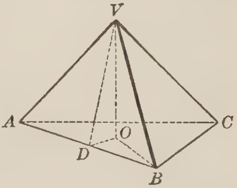
Furthermore the volume of the hemisphere AMB

$$= \tfrac{1}{2} \times \tfrac{4}{3}\pi r^3 \qquad \text{§ 706}$$
$$= \tfrac{2}{3}\pi r^3.$$

Therefore the volume of the entire solid V-AMB

$$= \tfrac{1}{3}\sqrt{3}\,\pi r^3 + \tfrac{2}{3}\pi r^3 \qquad \text{Ax. 11}$$
$$= \tfrac{1}{3}\pi r^3(\sqrt{3} + 2). \qquad \text{Q.E.F.}$$

23. *Find the total surface and the volume of a regular tetrahedron whose edge equals 8 centimeters.*



Given the regular tetrahedron V-ABC with its edge equal to 8 centimeters.

Required to find (1) the total area of V-ABC; (2) the volume of V-ABC.

Solution. (1) Draw VD, the slant height of V-ABC, to the edge AB.

Then $DA = DB$. § 85

That is, $DB = \tfrac{1}{2}AB$.

$\therefore DB = \tfrac{1}{2} \times 8$, or 4. Ax. 9

In the rt. $\triangle VDB$, $\overline{VD}^2 = \overline{VB}^2 - \overline{DB}^2$ § 338

$= 8^2 - 4^2$ Ax. 9

$= 64 - 16$, or 48.

$\therefore VD = \sqrt{48}$ Ax. 5

$= 4\sqrt{3}$.

Now the area of the $\triangle AVB = \tfrac{1}{2}AB \times VD$. § 325

$\therefore$ the area of the $\triangle AVB = \tfrac{1}{2} \times 8 \times 4\sqrt{3}$ Ax. 9

$= 16\sqrt{3}$.

But the four faces of the given tetrahedron V-ABC are congruent. § 567

Therefore the total area of V-ABC

$= 4 \times 16\sqrt{3}$, or $64\sqrt{3}$. Ax. 11

Hence the total area of V-ABC is $64\sqrt{3}$ sq. cm. Q.E.F.

(2) Draw VO, the altitude of V-ABC, to the base ABC.

Also draw OD and OB.

The point O is the center of the equilateral $\triangle ABC$. § 546

$\therefore$ in the $\triangle ABC$, $OB = \tfrac{2}{3}h$. Ex. 5, p. 78

$h = \tfrac{1}{2}s\sqrt{3}$. Ex. 14, p. 206

$\therefore h = 4\sqrt{3}$. Ax. 9

$\therefore OB = \tfrac{2}{3} \times 4\sqrt{3}$ Ax. 9

$= \tfrac{8}{3}\sqrt{3}$.

In the rt. $\triangle ABC$, $\overline{VO}^2 = \overline{VB}^2 - \overline{OB}^2$ § 337

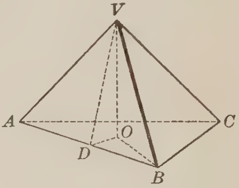
$= 8^2 - \left(\frac{8}{3}\sqrt{3}\right)^2$ Ax. 9

$= 64 - \frac{64}{3}$

$= \frac{128}{3}.$

$\therefore VO = \sqrt{\frac{128}{3}}$ Ax. 5

$= \frac{8}{3}\sqrt{6}.$



Now the volume of $V\text{-}ABC$

$= \frac{1}{3}ba.$ § 561

But the $\triangle ABC$ and AVB are congruent. § 567

Whence $b = 16\sqrt{3}.$

$\therefore$ the volume of $V\text{-}ABC = \frac{1}{3} \times 16\sqrt{3} \times \frac{8}{3}\sqrt{6}$ Ax. 9

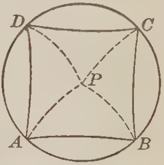
$= 42\frac{2}{3}\sqrt{2}.$

Therefore the volume of $V\text{-}ABC$ is $42\frac{2}{3}\sqrt{2}$ cu. cm. Q. E. F.

24. *If a spherical quadrilateral is inscribed in a small circle, the sum of two opposite angles is equal to the sum of the other two angles.*

Given the spherical quadrilateral $ABCD$ inscribed in the small ⊙ $ABCD$.

To prove that $\angle ADC + \angle ABC$
$= \angle BAD + \angle BCD.$



Proof. Let P be the pole of the small ⊙ $ABCD$ within the circumference of that ⊙.

Draw the arcs of great ⊙ PA, PB, PC, and PD.

The arcs PA, PB, PC, PD are all equal. § 636

Therefore the spherical $\triangle APB$, BPC, CPD, and APD are isosceles $\triangle$.

$\angle PDA = \angle PAD,$

$\angle PDC = \angle PCD,$

$\angle PBA = \angle PAB,$

and $\angle PBC = \angle PCB.$ § 679

Adding, $\angle PDA + \angle PDC + \angle PBA + \angle PBC$
$= \angle PAD + \angle PAB + \angle PCD + \angle PCB.$ Ax. 1

That is, $\angle ADC + \angle ABC = \angle BAD + \angle BCD$, by Ax. 11. Q. E. D.

25. *By what number must the dimensions of a cylinder of revolution be multiplied to obtain a similar cylinder of revolution with surface n times that of the first? with volume n times that of the first?*

Solution. (1) Let s and s' represent the surfaces of two similar cylinders, and d and d' any two corresponding dimensions, s being n times s'.

Then $s : s' = d^2 : d'^2.$ § 592

Also $s : s' = n : 1.$ Given

$\therefore\ d^2 : d'^2 = n : 1.$ Ax. 8

$\therefore\ n \cdot d'^2 = d^2.$ § 261

$\therefore\ \sqrt{n} \cdot d' = d.$ Ax. 5

Therefore the dimensions of s' must be multiplied by $\sqrt{n}$, that the resultant similar cylinder may have n times the area of s'. Q.E.F.

(2) Let v and v' represent the volumes of two similar cylinders, and d and d' any two corresponding dimensions, v being n times v'.

Then $v : v' = d^3 : d'^3.$ § 592

Also $v : v' = n : 1.$ Given

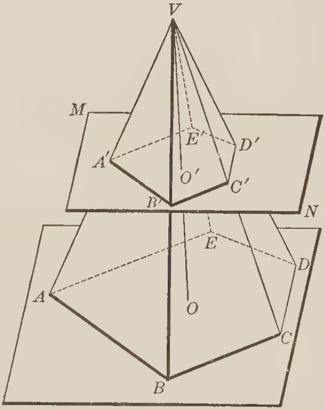
$\therefore\ d^3 : d'^3 = n : 1.$ Ax. 8

$\therefore\ n \cdot d'^3 = d^3.$ § 261

$\therefore\ \sqrt[3]{n} \cdot d' = d.$ Ax. 5

Therefore the dimensions of v' must be multiplied by $\sqrt[3]{n}$, that the resultant similar cylinder may have n times the volume of v'. Q.E.F.

26. *A pyramid is cut by a plane parallel to the base midway between the vertex and the plane of the base. Compare the volumes of the entire pyramid and the pyramid cut off.*



Given the pyramid $V\text{-}ABCDE$, whose altitude is VO, cut by the plane MN in the section $A'B'C'D'E'$, the plane MN being ∥ to the base $ABCDE$ midway between the vertex V and the base $ABCDE$.

Required to determine the ratio of the volumes of the pyramids $V\text{-}ABCDE$ and $V\text{-}A'B'C'D'E'$.

Solution. Let the plane MN cut VO in O'.

Then $VO : VO' = VA : VA'.$ § 555, 1

But $VO : VO' = 2 : 1.$ Given

$\therefore\ VA : VA' = 2 : 1.$ Ax. 8

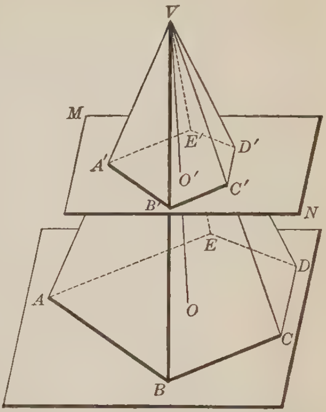
The polygons $ABCDE$ and $A'B'C'D'E'$ are similar. § 555, 2

Also, since AB and $A'B'$ are ∥, § 453

$\therefore$ the face $\triangle$ VAB and $VA'B'$ are similar. § 285

By a like course of reasoning the remaining corresponding face triangles may be proved to be similar.

Therefore the pyramids $V\text{-}ABCDE$ and $V\text{-}A'B'C'D'E'$ have their corresponding faces similar.



The polyhedral $\angle V$ is common to the pyramids $V\text{-}ABCDE$ and $V\text{-}A'B'C'D'E'$.

Also, since $\angle EAB$

$= \angle E'A'B'$, § 461

and $\angle VAB = \angle VA'B'$,

and $\angle VAE = \angle VA'E'$, § 102

and these corresponding ∡ occur in the same order,

∴ the polyhedral ∡ $A\text{-}VEB$ and $A'\text{-}VE'B'$ are equal. § 493

For like reason the remaining corresponding polyhedral ∡ are equal.

Therefore the pyramids $V\text{-}ABCDE$ and $V\text{-}A'B'C'D'E'$ have their corresponding polyhedral ∡ equal.

Therefore the pyramids $V\text{-}ABCDE$ and $V\text{-}A'B'C'D'E'$ are similar. § 710

$$\therefore \frac{V\text{-}ABCDE}{V\text{-}A'B'C'D'E'} = \frac{\overline{VA}^3}{\overline{VA'}^3}. \qquad \text{§ 723}$$

But

$$\frac{\overline{VA}^3}{\overline{VA'}^3} = \frac{2^3}{1} = \frac{8}{1}. \qquad \text{Ax. 5}$$

$$\therefore \frac{V\text{-}ABCDE}{V\text{-}A'B'C'D'E'} = \frac{8}{1}, \text{ by Ax. 8.} \qquad \text{Q.E.F.}$$

27. *The height of a regular hexagonal pyramid is 36 ft. and one side of the base is 6 ft. What are the dimensions of a similar pyramid whose volume is $\frac{1}{20}$ that of the first?*

Given VO and $V'O'$ the altitudes of the two similar regular hexagonal pyramids $V\text{-}ABCDEF$ and $V'\text{-}A'B'C'D'E'F'$ respectively, and AB and $A'B'$ any two corresponding base edges.

$$VO = 36 \text{ ft.}, \; AB = 6 \text{ ft., and } \frac{V'\text{-}A'B'C'D'E'F'}{V\text{-}ABCDEF} = \frac{1}{20}.$$

Required to find the lengths of $V'O'$ and $A'B'$.

Solution.

$$\frac{V'\text{-}A'B'C'D'E'F'}{V\text{-}ABCDEF} = \frac{\overline{A'B'}^3}{\overline{AB}^3}. \qquad \text{§ 723}$$

But

$$\frac{VO'}{VO} = \frac{A'B'}{AB}. \qquad \text{§ 719}$$

Whence

$$\frac{\overline{V'O'}^3}{\overline{VO}^3} = \frac{\overline{A'B'}^3}{\overline{AB}^3}. \qquad \text{Ax. 5}$$

$$\therefore \frac{V'\text{-}A'B'C'D'E'F'}{V\text{-}ABCDEF} = \frac{\overline{V'O'}^3}{\overline{VO}^3}. \qquad \text{Ax. 8}$$

Now $$\frac{V'\text{-}A'B'C'D'E'F'}{V\text{-}ABCDEF} = \frac{1}{20}. \qquad \text{Given}$$

$$\therefore \frac{\overline{V'O'}^3}{\overline{VO}^3} = \frac{1}{20},$$

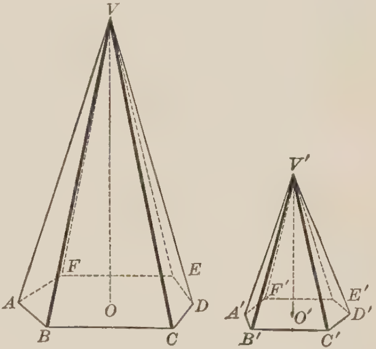
and $$\frac{\overline{A'B'}^3}{\overline{AB}^3} = \frac{1}{20}. \qquad \text{Ax. 8}$$

$$\therefore \frac{\overline{V'O'}^3}{36^3} = \frac{1}{20},$$

and $$\frac{\overline{A'B'}^3}{6^3} = \frac{1}{20}. \qquad \text{Ax. 9}$$

Whence $$\overline{V'O'}^3 = \frac{36^3}{20},$$

and $$\overline{A'B'}^3 = \frac{6^3}{20}. \qquad \text{Ax. 3}$$



$$\therefore V'O' = \sqrt[3]{\frac{36^3}{20}} \qquad \text{Ax. 5}$$

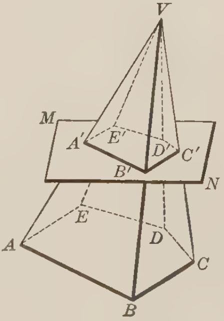
$$= \sqrt[3]{\frac{36^3 \times 50}{1000}} = 3.6\sqrt[3]{50}, \text{ or } 13.262;$$

and $$A'B' = \sqrt[3]{\frac{6^3}{20}} \qquad \text{Ax. 5}$$

$$= \sqrt[3]{\frac{6^3 \times 50}{1000}} = 0.6\sqrt[3]{50}, \text{ or } 2.210.$$

Therefore the approximate lengths of $V'O'$ and $A'B'$ are 13.262 ft. and 2.210 ft. respectively. Q.E.F.

28. *One of the lateral edges of a pyramid is 4 meters. How far from the vertex will this edge be cut by a plane parallel to the base, which divides the pyramid into two equivalent parts?*



Given the pyramid V-$ABCDE$, whose lateral edge VA equals 4 meters, cut by the plane MN ∥ to the base $ABCDE$ in the section $A'B'C'D'E'$, so that the pyramid V-$A'B'C'D'E'$ cut off is equivalent to the frustum $ABCDE$-$A'B'C'D'E'$; also A' the point of division of the lateral edge VA.

Required to find the length of VA'.

Solution. The polygons $ABCDE$ and $A'B'C'D'E'$ are similar. § 555, 2

Also, since AB and $A'B'$ are ∥, § 453

∴ the face △s VAB and $VA'B'$ are similar. § 285

For like reason the remaining corresponding face △s are similar.

Therefore the pyramids $V\text{-}ABCDE$ and $V\text{-}A'B'C'D'E'$ have their corresponding faces similar.

The polyhedral $\angle V$ is common to the pyramids $V\text{-}ABCDE$ and $V\text{-}A'B'C'D'E'$.

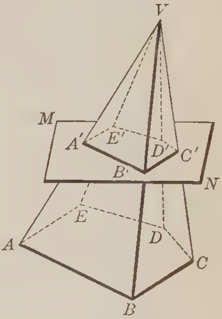
Also, since $\angle EAB = \angle E'A'B'$, § 461

and $\angle VAB = \angle VA'B'$,

and $\angle VAE = \angle VA'E'$, § 102

and these corresponding ∠s occur in the same order,

∴ the polyhedral ∠s $A\text{-}VEB$ and $A'\text{-}VE'B'$ are equal. § 493



For like reason the remaining corresponding polyhedral ∠s are equal.

Therefore the pyramids $V\text{-}ABCDE$ and $V\text{-}A'B'C'D'E'$ have their corresponding polyhedral ∠s equal.

Therefore the pyramids $V\text{-}ABCDE$ and $V\text{-}A'B'C'D'E'$ are similar. § 710

∴ $\dfrac{V\text{-}ABCDE}{V\text{-}A'B'C'D'E'} = \dfrac{\overline{VA}^3}{\overline{VA'}^3}$. § 723

But $\dfrac{V\text{-}ABCDE}{V\text{-}A'B'C'D'E'} = \dfrac{2}{1}$. Given

∴ $\dfrac{\overline{VA}^3}{\overline{VA'}^3} = \dfrac{2}{1}$. Ax. 8

∴ $2\overline{VA'}^3 = \overline{VA}^3$. § 261

∴ $\overline{VA'}^3 = \frac{1}{2}\overline{VA}^3$. Ax. 4

But $VA = 4$. Given

Whence $\overline{VA}^3 = 4^3$, or 64. Ax. 5

∴ $\overline{VA'}^3 = \frac{1}{2} \times 64$ Ax. 9

$= 32$.

∴ $VA' = \sqrt[3]{32}$ Ax. 5

$= 2\sqrt[3]{4}$.

Therefore the length of VA' is $2\sqrt[3]{4}$ meters. Q.E.F.